A new Lindelöf space with points G_δ

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Abstract. We prove that $\diamondsuit^*$ implies there is a zero-dimensional Hausdorff Lindelöf space of cardinality $2^{\aleph_1}$ which has points G_δ. In addition, this space has the property that it need not be Lindelöf after countably closed forcing.

Keywords: Lindelöf; forcing

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1. Introduction

The set-theoretic principle $\diamondsuit^*$ was formulated by Jensen ([2, p. 128] and [9, VI #16, p. 181]).

Definition 1.1. $\diamondsuit^*$ is the statement that there are countable $\mathcal{A}_\alpha \subset \mathcal{P}(\alpha)$, for $\alpha \in \omega_1$, such that for every $A \subset \omega_1$ there is a cub $C \subset \omega_1$ such that $A \cap \alpha \in \mathcal{A}_\alpha$ for all $\alpha \in C$.

Definition 1.2 ([10]). A Lindelöf space is *indestructible* if it remains Lindelöf after any countably closed forcing. A Lindelöf space is *destructible* if it is not indestructible.

Notice that $\diamondsuit^*$ implies CH but is consistent with $2^{\aleph_1}$ being arbitrarily large ([9, VII (H18)–(H20) p. 249]). As is well-known, Shelah proved, using forcing, that it is consistent with CH to have Hausdorff zero-dimensional Lindelöf spaces with points G_δ which had cardinality $\aleph_2$ (see [5]). In establishing the consistency with CH of there being no such spaces with cardinality strictly between $\aleph_1$ and $2^{\aleph_1}$, Shelah also established the relevance of the notion of a space being destructible (see [5]). I. Gorelič [4] produced another forcing construction to establish the consistency of the existence of Lindelöf spaces with points G_δ which had cardinality $2^{\aleph_1}$ while allowing $2^{\aleph_1}$ to be as large as desired. F. Tall [10] points out that each of these examples is indestructible. R. Knight [8] extended the Shelah style construction in models of GCH with special L-like combinatorial structures (flat morasses) and constructed an example of cardinality $\aleph_\omega$. Close inspection of Lemma 3.5.2 of [8] shows that this example is also indestructible. Finally, let us mention that Juhasz [6] constructed a non-Hausdorff example in ZFC which (see [10]) is destructible.

In this note we will prove

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Theorem 1.3. *$\diamondsuit^*$ implies there is a space that is zero-dimensional Hausdorff Lindelöf destructible of cardinality $2^{\aleph_1}$ and that has points G_δ.*

This is the first consistent example of a Lindelöf Hausdorff destructible space with points G_δ.

Question 1. Does every Lindelöf Hausdorff destructible space have cardinality at least $2^{\aleph_1}$?

2. A Lindelöf tree

We build our space X using the structure $2^{\leq\omega_1}$. For each $t \in 2^{\leq\omega_1}$ let $[t]$ denote the set $\{s \in 2^{\leq\omega_1} : t \subseteq s\}$. For any $t \in 2^{<\omega_1}$ such that $\mathrm{dom}(t)$ is a successor, let $t^\dagger$ be the other immediate successor of the immediate predecessor of t, i.e. t and $t^\dagger$ are the two immediate successors of $t \cap t^\dagger$. For distinct functions ρ, ψ in the tree $2^{\leq\omega_1}$, we will let $\rho \wedge \psi$ denote the maximal element of $2^{<\omega_1}$ which is an initial segment of each of them. Let σ denote the standard topology on $2^{\leq\omega_1}$ that has the family

$$\{\emptyset\} \cup \{[\rho \restriction \xi+1] : \xi \in \omega_1, \rho \in 2^{\omega_1}\} \cup \\ \{[t \restriction \xi+1] \setminus ([t^\frown 0] \cup [t^\frown 1]) : \xi \in \mathrm{dom}(t), t \in 2^{<\omega_1}\}$$

as a subbase. Of course t is isolated and $[t]$ is clopen for all t such that $\mathrm{dom}(t) \in \omega_1$ is not a limit.

This next lemma is very well-known but since it is crucial to our construction, we include a proof.

Lemma 2.1. *The topology σ on $2^{\leq\omega_1}$ is compact zero-dimensional and Hausdorff. Also, for each $\alpha \in \omega_1$, $2^{\leq\alpha}$ is a compact first-countable subspace.*

Proof: One standard method of proof is to construct a canonical embedding of $2^{\leq\omega_1}$ into $2^{2^{<\omega_1}}$ and show that the range is closed in the product topology. However, we will give a more direct proof. Certainly σ is zero-dimensional since the members of the generating subbase are easily shown to also be closed. If s, t are distinct elements of $2^{\leq\omega_1}$, we show they have disjoint neighborhoods. If $t \subset s$, then, for any $\xi \in \mathrm{dom}(t)$, $t \in [t \restriction \xi+1] \setminus ([t^\frown 0] \cup [t^\frown 1])$ and $s \in ([t^\frown 0] \cup [t^\frown 1])$. Otherwise, we may assume that $y = s \wedge t$ is strictly below each of s and t, and note that $[y^\frown 0]$ and $[y^\frown 1]$ are disjoint and each contains one of s, t.

Now assume that $\mathcal{U}$ is a cover by basic open sets. Let $T_{\mathcal{U}}$ denote the set of all $t \in 2^{<\omega_1}$ such that there is no finite subcollection of $\mathcal{U}$ whose union contains $[t]$. If $\emptyset \notin T_{\mathcal{U}}$ then $\mathcal{U}$ has a finite subcover. So assume that $T_{\mathcal{U}}$ is not empty. Observe that if $t \in T_{\mathcal{U}}$, then $t \restriction \xi \in T_{\mathcal{U}}$ for all $\xi \in \mathrm{dom}(t)$. For each $\rho \in 2^{\omega_1}$, there is a $\xi \in \omega_1$ such that $[\rho \restriction \xi+1] \in \mathcal{U}$, so we have that $T_{\mathcal{U}}$ is a subtree of $2^{<\omega_1}$ with no uncountable branch. Similarly, $T_{\mathcal{U}}$ has no maximal elements, since if each of $[t^\frown 0]$ and $[t^\frown 1]$ are covered by a finite union from $\mathcal{U}$, then certainly, $[t] = \{t\} \cup [t^\frown 0] \cup [t^\frown 1]$ is as well. Choose any maximal chain $\{t_\xi : \xi \in \alpha\} \subset T_{\mathcal{U}}$ and let $t = \bigcup\{t_\xi : \xi \in \alpha\}$. Since T has no maximal elements, t is on a limit level and $\mathcal{U}$ contains a finite cover of $[t]$. But in addition, there is some $\xi < \alpha$ such that $[t_\xi] \setminus ([t^\frown 0] \cup [t^\frown 1])$ is in $\mathcal{U}$. This is a contradiction, since it shows that $\mathcal{U}$ has a finite cover of $[t_\xi]$ – contradicting that $t_\xi \in T_{\mathcal{U}}$.

It is obvious that $2^{\leq\alpha}$ is a closed subset of $2^{\leq\omega_1}$, and, for each non-isolated $t \in 2^{\leq\alpha}$, the collection $\{[t \restriction \xi+1] \setminus ([t^\frown 0] \cup [t^\frown 1]) : \xi \in \mathrm{dom}(t)\}$ is a neighborhood base at t. □

Next we consider Lindelöf subspaces.

Lemma 2.2. *If $Y \subset 2^{<\omega_1}$ satisfies that $Y \cap 2^\alpha$ is countable for all $\alpha \in \omega_1$, then the complement of Y in $2^{\leq\omega_1}$ is Lindelöf in the topology induced by σ.*

Proof: Assume that $\mathcal{U}$ is a cover of $2^{\leq\omega_1} \setminus Y$ by basic clopen sets. Let us again set $T_{\mathcal{U}}$ to be the set of $t \in 2^{<\omega_1}$ such that $\mathcal{U}$ contains a countable cover of $[t] \setminus Y$. As in the proof of Lemma 2.1, $T_{\mathcal{U}}$ (if non-empty) is downwards closed, has no maximal elements, and no uncountable branches. Now let us show that $T_{\mathcal{U}}$ is branching. Suppose that $T_{\mathcal{U}} \cap [t]$ is a chain. Then it is a countable chain (with supremum in Y), and let $\{t_\gamma : \gamma \in \alpha\}$ be an enumeration in increasing order and let t_α denote the union. For each $\gamma \in \alpha$, we have that $t^\dagger_{\gamma+1}$ is not in $T_{\mathcal{U}}$, and so there is a countable $\mathcal{U}_\gamma \subset \mathcal{U}$ whose union covers $(\{t_\gamma\} \cup [t^\dagger_{\gamma+1}]) \setminus Y$. Furthermore there is a countable $\mathcal{U}_\alpha \subset \mathcal{U}$ that covers $[t_\alpha] \setminus Y$. It should be clear that $\bigcup\bigcup\{\mathcal{U}_\gamma : \gamma \leq \alpha\}$ covers $[t]$.

Now we have established that $T_{\mathcal{U}}$ is branching and has no maximal elements. Set $t_\emptyset = \emptyset$ and by recursion on $s \in 2^{<\omega}$, choose $t_s \in T_{\mathcal{U}}$ so that for $s \in 2^{<\omega}$, $t_s \subset (t_{s^\frown 0} \wedge t_{s^\frown 1})$ and $t_{s^\frown 0} \perp t_{s^\frown 1}$. Let $\delta \in \omega_1$ so that $\{t_s : s \in 2^{<\omega}\} \subset 2^{<\delta}$. Choose any $x \in 2^\omega$ so that $t_x = \bigcup_n t_{x\restriction n} \in 2^{\leq\delta} \setminus Y$. By construction, $\mathrm{dom}(t_x)$ is a limit ordinal. Choose any $\xi \in \mathrm{dom}(t_x)$ so that $[t_x \restriction \xi+1] \setminus ([t_x^\frown 0] \cup [t_x^\frown 1])$ is contained in some $U \in \mathcal{U}$. Fix n so that $\xi < \mathrm{dom}(t_{x\restriction n})$, and choose any $s \in 2^{<\omega}$ so that $x \restriction n \subset s$ and $s \not\subset x$. Finally we can conclude that $T_{\mathcal{U}}$ must be empty, since we have that $[t_s] \subset U$. □

3. Points G_δ

Let $\{\mathcal{A}_\alpha : \alpha \in \omega_1\}$ be a sequence as in Definition 1.1 witnessing the statement $\diamondsuit^*$.

Definition 3.1. For each limit $\alpha \in \omega_1$ let $S_\alpha = \{t \in 2^\alpha : t^{-1}(1) \in \mathcal{A}_\alpha\}$. For $0 < \alpha$ not a limit, let S_α be the empty set, and let $S_0 = \{\emptyset\}$.

Lemma 3.2. *For each $\rho \in 2^{\omega_1}$, there is a cub $C_\rho \subset \omega_1$ such that $C_\rho \subset \{\alpha : \rho \restriction \alpha \in S_\alpha\}$.*

Proof: This is just a restatement of the fact that the sequence $\{\mathcal{A}_\alpha : \alpha \in \omega_1\}$ is a $\diamondsuit^*$ sequence. □

For each $\rho \in 2^{\omega_1}$ fix a cub C_ρ as in Lemma 3.2.

Proposition 3.3. *For each $\rho \in 2^{\omega_1}$, there is a countable-to-one function $f_\rho : \omega_1 \to 2^\omega$ so that for each $x \in 2^\omega$, there is a $\delta_x \in C_\rho \cup \{0\}$ and $\delta_x < \gamma_x \in C_\rho$ so that $f_\rho^{-1}(x)$ is equal to the interval $[\delta_x, \gamma_x)$.*

Proof: First let $\{\delta_x : x \in 2^\omega\}$ be any enumeration of $C_\rho \cup \{0\}$. For each $x \in 2^\omega$, define γ_x to be $\min(C_\rho \setminus [0, \delta_x])$. Assume that $\delta_x < \delta_y$. Then it is obvious that $\gamma_x \leq \delta_y$. Now define f_ρ so that $f_\rho([\delta_x, \gamma_x)) = \{x\}$ for all $x \in 2^\omega$. □

Now we are ready to prove our main theorem.

Proof of Theorem 1.3: Fix the sequence $\{S_\alpha : \alpha \in \omega_1\}$ as in Definition 3.1, and let Y equal the union of this family. Our space X will have as its base set $(2^{\omega_1} \times 2^\omega) \cup 2^{<\omega_1} \setminus Y$. We will use the fact (Lemma 2.2) that $2^{\leq\omega_1} \setminus Y$ is Lindelöf when using the topology σ. Recall that for each $\rho \in 2^{\omega_1}$ and $\xi \in \omega_1$, $[\rho \restriction \xi + 1] \setminus Y$ is a clopen set. In this proof, for any $s \in 2^{<\omega}$, we will use $[s]_{2^\omega}$ to denote the set $\{x \in 2^\omega : s \subset x\}$.

We define a clopen base for the topology τ. For each $t \in 2^{<\omega_1}$, we use the notation $[t]_X$ to denote

$$[t]_X = [t] \cap (2^{<\omega_1} \setminus Y) \ \cup \ ([t] \cap 2^{\omega_1}) \times 2^\omega.$$

Again, for each $\rho \in 2^{\omega_1}$ and each $\xi \in \omega_1$, the set $[\rho \restriction \xi + 1]_X$ is declared to be a clopen set in τ (i.e. $[\rho \restriction \xi + 1]_X$ and its complement are in τ). Let us observe that for $t \in Y$, $[t]_X$ is equal to $[t^\frown 0]_X \cup [t^\frown 1]_X$ and so is also clopen.

Next, for each $\rho \in 2^{\omega_1}$ and each $x \in 2^\omega$, let $f_\rho^{-1}(\{x\})$ be denoted as $[\delta^\rho_x, \gamma^\rho_x)$ as per Proposition 3.3. For $s \in 2^{<\omega}$, and $\gamma \in C_\rho$, we define

$$\begin{aligned} U(\rho, s, \gamma) = &(\{\rho\} \times [s]_{2^\omega}) \cup \\ &\bigcup \{[\rho \restriction \delta^\rho_x]_X \setminus [\rho \restriction \gamma^\rho_x]_X : x \in [s]_{2^\omega} \text{ and } \gamma \leq \delta^\rho_x\} \ . \end{aligned}$$

When the choice of ρ is clear from the context, we will use δ_x, γ_x as referring to $\delta^\rho_x, \gamma^\rho_x$. The topology τ will also contain each such $U(\rho, s, \gamma)$. Notice that, for each $\gamma \in C_\rho$ and each $n \in \omega$, the family $\{U(\rho, s, \gamma) : s \in 2^n\}$ is a partition of the clopen set $[\rho \restriction \gamma]_X$, and so each is clopen.

Claim 1. For each $t \in 2^{<\omega_1} \cap X$, the family

$$\{[t \restriction \xi + 1]_X \setminus ([t^\frown 0]_X \cup [t^\frown 1]_X) : \xi \in \mathrm{dom}(t)\}$$

is a neighborhood base for t.

To show this we must consider some ρ, s, γ such that $t \in U(\rho, s, \gamma)$ and $\gamma \in C_\rho$. There is a unique $x \in 2^\omega$ such that $t \in [\rho \restriction \delta_x]_X \setminus [\rho \restriction \gamma_x]_X$. Since $\rho \restriction \delta_x \in Y$, we know that $t \neq \rho \restriction \delta_x$. Since $[\rho \restriction \delta_x]_X \setminus [\rho \restriction \gamma_x]_X$ contains $[t \restriction \delta_x + 1]_X \setminus ([t^\frown 0]_X \cup [t^\frown 1]_X)$, we have proven the claim.

Claim 2. For each $\rho \in 2^{\omega_1}$ and $z \in 2^\omega$, the point (ρ, z) is the only element of the intersection of the family $\{U(\rho, z \restriction n, \gamma_z) : n \in \omega\}$.

It is clear that for any $\gamma \in C_\rho$, $U(\rho, s, \gamma) \cap (\{\rho\} \times 2^\omega)$ is equal to $\{\rho\} \times [s]_{2^\omega}$. Now suppose that $\psi \in 2^{\omega_1} \setminus \{\rho\}$ and $t \in X \cap 2^{<\omega_1}$. Let $\rho \restriction \xi_\psi = \psi \cap \rho$ and $\rho \restriction \xi_t = t \wedge \rho$. Choose any $s \in 2^{<\omega}$ so that $z \in [s]_{2^\omega}$ and neither of $f_\rho(\xi_t)$, $f_\rho(\xi_\psi)$ are in $[s]_{2^\omega} \setminus \{z\}$. But now, if $\gamma_z \le \xi$ then $f_\rho(\xi) \ne z$. Therefore, for all $x \in [s]_{2^\omega}$ with $\gamma_z \le \gamma_x$, we have that $\{\xi_t, \xi_\psi\}$ is disjoint from $[\delta_x, \gamma_x)$, and therefore $[\rho \restriction \delta_x]_X \setminus [\rho \restriction \gamma_x]_X$ is disjoint from $\{t\} \cup (\{\psi\} \times 2^\omega)$. This completes the proof of the claim.

Let Φ be the canonical map from X (with topology τ) onto $2^{\le\omega_1} \setminus Y$ (with topology σ). That is, $\Phi(t) = t$ for all $t \in X \cap 2^{<\omega_1}$, and $\Phi((\rho, x)) = \rho$ for all $\rho \in 2^{\omega_1}$ and $x \in 2^\omega$. It is evident that point preimages under Φ are compact. It is immediate that Φ is continuous since $\Phi^{-1}[t] = [t]_X$ for all $t \in 2^{<\omega_1}$. This is also useful to show that Φ is closed. By [3, 1.4.13] it is sufficient to show that if $U \subset X$ is an open set containing a fiber $\Phi^{-1}(t)$ for some $t \in 2^{\le\omega_1} \setminus Y$, then there is a neighborhood W of t such that $\Phi^{-1}(W)$ is contained in U. Let then, $t \in 2^{\le\omega_1} \setminus Y$ and suppose that $U \subset X$ is an open set containing $\Phi^{-1}(t)$. This is obvious if $t \in 2^{<\omega_1}$, so suppose that $t = \rho \in 2^{\omega_1}$. Since $\Phi^{-1}(\rho)$ is simply $\{\rho\} \times 2^\omega$, it is clear that there is $\gamma \in C_\rho$ and $n \in \omega$ such that $U(\rho, s, \gamma) \subset U$ for each $s \in 2^n$. As remarked above, this implies that $[\rho \restriction \gamma]_X$ is contained in U. Since $[\rho \restriction \gamma]$ is a neighborhood of ρ and, again, $[\rho \restriction \gamma]_X = \Phi^{-1}([\rho \restriction \gamma])$, this completes the proof that Φ is a closed mapping.

Now that we have established that there is a perfect map (continuous, closed, point-preimages compact) from X onto a Lindelöf space, we conclude [3, 3.8.8] that X is also Lindelöf.

Finally, it is immediate that the forcing notion $2^{<\omega_1}$ will introduce a new member ψ of 2^{ω_1}. Since the forcing adds no new members to $2^{<\omega_1}$, the set $\{\psi \restriction \xi + 1 : \xi \in \omega_1\}$ is a subset of X and has no complete accumulation point in X. We conclude that X is not Lindelöf in the forcing extension. $\square$

4. Remarks on consistency

Let us consider the following principle which is evidently weaker than $\diamondsuit^*$.

Definition 4.1. $w\diamondsuit^*$ is the statement that there is a subset $Y \subset 2^{<\omega_1}$ such that

(1) for each $\alpha \in \omega_1$, $Y \cap 2^{\le\alpha}$ contains no perfect set,
(2) for each $\rho \in 2^{\omega_1}$, there is a cub $C_\rho \subset \omega_1$ such that $\{\rho \restriction \gamma : \gamma \in C_\rho\}$ is contained in Y.

Say that the set Y is a $w\diamondsuit^*$ sequence.

The hypothesis "CH and $w\diamondsuit^*$" is sufficient to prove Theorem 1.3. It is probable that this is a weaker statement than $\diamondsuit^*$ but, just as a $\diamondsuit^*$ sequence is destroyed by forcing with $2^{<\omega_1}$ (see [9, p. 300 J5]), so too is a $w\diamondsuit^*$-sequence. This implies that $w\diamondsuit^*$ fails in the models in which it has been shown that any Lindelöf points G_δ space of cardinality greater than ω_1 must be destructible. In particular, such a model (see [10]) is obtained by countably closed forcing that collapses a supercompact cardinal to $\aleph_2$. It is reasonable to conjecture that in that model Lindelöf spaces with points G_δ will have cardinality at most $\aleph_1$, and the approach till now has focused on trying to show that there are (in ZFC) no destructible Lindelöf spaces with points G_δ. However there is a stronger property that any ZFC example of such space must have which we now define. A space with character at most ω_1 would have to have this first property.

Definition 4.2. Say that a regular Lindelöf space with points G_δ is *reconstructible* if it is destructible and there is a countably closed poset so that in the forcing extension, it is no longer Lindelöf but it can be embedded into a regular Lindelöf space with points G_δ.

It may not be as natural, but there is a similar, but weaker, property which is the property we are really after. We use the word elementarily in reference to the set-theoretic notion of elementary extensions of models.

Definition 4.3. Say that a regular Lindelöf space X with points G_δ is *elementarily reconstructible* if there is a countably closed poset so that in the forcing extension, it is no longer Lindelöf and there is a regular Lindelöf space Y with points G_δ that has a dense subspace Z and a continuous mapping f from Z onto X and satisfies that f is a homeomorphism on the pre-image of the points with character at most ω_1.

Clearly an elementarily reconstructible space that has character at most ω_1 will be reconstructible. A reader of Tall's paper [10] will realize that in the forcing extension mentioned above, if there is a Lindelöf space X with points G_δ and character at most ω_1 which has cardinality greater than ω_1 then this will imply the consistency of there being regular Lindelöf spaces that are elementarily reconstructible. It may possibly be true that X itself will be elementarily reconstructible, but we do not know[1] if a supercompact cardinal is sufficient for this claim. However, we can prove, sketched below in Proposition 4.6, that a 2-huge cardinal (see [7, p. 331]) is sufficient.

On the other hand, not only does the poset $2^{<\omega_1}$ render our space to be non-Lindelöf, it also creates a subspace which cannot be embedded into a Lindelöf space with points G_δ.

Proposition 4.4. *If $Y \subset 2^{<\omega_1}$ is a $w\diamondsuit^*$-sequence, then in the forcing extension by $2^{<\omega_1}$, there is a $\psi \in 2^{\omega_1}$ such that $T_\psi(Y) = \{\alpha : \psi \restriction \alpha \in Y\}$ is stationary.*

Since $\{\psi \restriction \alpha : \alpha \in T_\psi(Y)\}$, as a subspace of $2^{<\omega_1}$, is homeomorphic to $T_\psi(Y)$ as a subspace of the ordinal ω_1, this next proposition shows that our space X is not reconstructible.

Proposition 4.5. *If S is a stationary subset of ω_1, then S cannot be embedded in a Lindelöf space with points G_δ.*

Proof: Assume that Z is a Lindelöf space with S as a subspace. Since S cannot equal a union of non-stationary sets, and Z is Lindelöf, there is a point z of Z

[1] the excellent referee noted the difficulty and suggested huge cardinals

with the property that every neighborhood of z meets S in a non-stationary set. Let us show that z is not a G_δ-point. Let $\{U_n : n \in \omega\}$ be a family of open subsets of Z, each meeting S in a non-stationary set. Since S is a subspace, $S \setminus U_n$ is a closed subset of S that misses the stationary set U_n. Of course this implies that $S \setminus U_n$ is countable. This shows that each G_δ of Z that contains z will also contain many points of S. $\square$

Following Kunen [9, VII.3.1], let $Lv'(\kappa)$ denote the standard Silver variant of the Levy collapse of a strongly inaccessible cardinal κ to ω_2 with countable conditions. If κ is strongly inaccessible, then $Lv'(\kappa)$ has cardinality κ and satisfies the κ-chain condition. We will need that if $\lambda < \kappa$ is also strongly inaccessible, then $Lv'(\kappa)$ is isomorphic to the iteration $Lv'(\lambda) * Lv'(\kappa)$ (see [9, VII.3.5]). A cardinal κ is 2-huge if there is an elementary embedding j from V into a submodel M such that κ is the critical point of j and M has the property that every subset of M with cardinality at most $j(j(\kappa))$ is also a member of M. Let us note that $j(\kappa)$ is a measurable cardinal (see [7, p. 331]). We recall that Arhangelskii [1] showed that every Lindelöf space with points G_δ has cardinality less than the first measurable cardinal.

Lemma 4.6. *Suppose that κ is a 2-huge cardinal and let G be $Lv'(\kappa)$-generic. In the forcing extension $V[G]$, every Lindelöf, points G_δ, regular space of cardinality greater than $\aleph_1$ is reconstructibly Lindelöf.*

PROOF: We work with forcing terminology rather than in the extension $V[G]$. Suppose that $\lambda \geq \kappa$ is a cardinal and that there is a $Lv'(\kappa)$-name $\dot{\tau}$ of a topology on λ that is forced to be Lindelöf, regular, and with points G_δ. By Arhangelskii's result and the fact that $j(\kappa)$ is measurable in $V[G]$, we have that λ is smaller than $j(\kappa)$. Now we apply the elementary embedding j and work briefly in the model M. We have that $j(\dot{\tau})$ is a $Lv'(j(\kappa))$-name of a Lindelöf, points G_δ topology on the set $j(\lambda)$. Following Tall [10], it can be shown that it is forced (in M) that the closure, Y, of the set $Z = j[\lambda] = \{j(\alpha) : \alpha \in \lambda\}$ in the space $(j(\lambda), j(\dot{\tau}))$ is Lindelöf and that j^{-1} maps Z continuously onto the space $(\lambda, \dot{\tau})$ as per the requirements of Definition 4.3. Finally, since $\lambda < j(\kappa)$, we have that $j(\lambda)$ is less than the strongly inaccessible cardinal $j(j(\kappa))$, and so it follows that the $Lv'(j(\kappa))$-name $j(\dot{\tau})$ is forced to be Lindelöf even in the model V. Finally, from the point of view of the forcing extension by $Lv'(\kappa)$, and the fact that $Lv'(j(\kappa))$ is isomorphic to $Lv'(\kappa) * Lv'(j(\kappa))$, we have that $X = (\lambda, \dot{\tau})$ is forced by $Lv'(\kappa)$ to be reconstructibly Lindelöf. $\square$

We close with the obvious question.

Question 2. Does CH imply there is a regular Lindelöf space with points G_δ that is elementarily reconstructible?

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