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Antiflexible Latin directed triple systems

Andrew R. Kozlik

Abstract. It is well known that given a Steiner triple system one can define a quasigroup operation $\cdot$ upon its base set by assigning $x \cdot x = x$ for all x and $x \cdot y = z$, where z is the third point in the block containing the pair $\{x, y\}$. The same can be done for Mendelsohn triple systems, where (x, y) is considered to be ordered. But this is not necessarily the case for directed triple systems. However there do exist directed triple systems, which induce a quasigroup under this operation and these are called Latin directed triple systems. The quasigroups associated with Steiner and Mendelsohn triple systems satisfy the flexible law $y \cdot (x \cdot y) = (y \cdot x) \cdot y$ but those associated with Latin directed triple systems need not. In this paper we study the Latin directed triple systems where the flexible identity holds for the least possible number of ordered pairs (x, y). We describe their geometry, present a surprisingly simple cyclic construction and prove that they exist if and only if the order n is $n \equiv 0$ or $1 \pmod 3$ and $n \geq 13$.

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1. Introduction

A *Steiner triple system* of order n, STS(n), is a pair $(V, \mathcal{B})$ where V is a set of n points and $\mathcal{B}$ is a collection of triples of distinct points taken from V such that every pair of distinct points from V appears in precisely one triple. Given an STS $(V, \mathcal{B})$ one can define a binary operation $\cdot$ on the set V by assigning $x \cdot x = x$ for all $x \in V$ and $x \cdot y = z$ whenever $\{x, y, z\} \in \mathcal{B}$. The induced operation satisfies the identities

$$x \cdot x = x, \quad y \cdot (x \cdot y) = x, \quad x \cdot y = y \cdot x$$

for all x and y in V. Any binary operation satisfying these three identities is called an *idempotent totally symmetric quasigroup*. The process described above is reversible. Given an idempotent totally symmetric quasigroup one can obtain an STS by assigning $\{x, y, x \cdot y\} \in \mathcal{B}$ for all x, $y \in V$, $x \neq y$. Thus there is a one-to-one correspondence between Steiner triple systems and idempotent totally symmetric quasigroups or *Steiner quasigroups*, as they are commonly known. All Steiner quasigroups satisfy the *flexible law* $y \cdot (x \cdot y) = (y \cdot x) \cdot y$.

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If we consider oriented triples then there are two possibilities. A *cyclic triple* (x, y, z) contains the ordered pair (x, y), (y, z) and (z, x). A *transitive triple* $\langle x, y, z\rangle$ contains the ordered pair (x, y), (y, z) and (x, z).

A *Mendelsohn triple system* of order n, MTS(n), is a pair $(V, \mathcal{B})$ where V is a set of n points and $\mathcal{B}$ is a collection of cyclic triples of distinct points taken from V such that every ordered pair of distinct points from V appears in precisely one triple. Quasigroups can be obtained from Mendelsohn triple systems by defining $x \cdot x = x$ and $x \cdot y = z$ for all $x, y \in V$, $x \neq y$, where z is the third element in the transitive triple containing the ordered pair (x, y). These so called *Mendelsohn quasigroups* satisfy the same algebraic properties as their Steiner counterparts with the exception of commutativity. Similarly there is a one-to-one correspondence between Mendelsohn triple systems and Mendelsohn quasigroups. Again, all Mendelsohn quasigroups satisfy the flexible law.

A *directed triple system* of order n, DTS(n), is a pair $(V, \mathcal{B})$ where V is a set of n points and $\mathcal{B}$ is a collection of transitive triples of distinct points taken from V such that every ordered pair of distinct points from V appears in precisely one triple. Given a DTS(n), an algebraic structure $(V, \cdot)$ can be obtained as above by defining $x \cdot x = x$ and $x \cdot y = z$ for all $x, y \in V$, $x \neq y$, where z is the third element in the transitive triple containing the ordered pair (x, y). However the structure obtained need not necessarily be a quasigroup. If for example $\langle u, x, y\rangle$ and $\langle y, v, x\rangle \in \mathcal{B}$, then $u \cdot x = v \cdot x = y$, but $u \neq v$. There do however exist DTSs that yield quasigroups. Such a DTS(n) is called a *Latin directed triple system*, denoted by LDTS(n), to reflect the fact that in this case the operation table forms a Latin square. We call the quasigroup so obtained a *DTS-quasigroup*. The binary operation will sometimes be replaced with juxtaposition, for example $x \cdot yz$ meaning $x \cdot (y \cdot z)$.

Latin directed triple systems were introduced in [3], where it was shown that an LDTS(n) exists if and only if $n \equiv 0$ or $1 \pmod 3$ and $n \neq 4$, 6 or 10. The algebraic and geometrical aspects of LDTSs were studied in [4]. Together these two papers also give enumeration results for all orders less than or equal to 13.

The following theorem was proved in [4].

Theorem 1.1. *Let $(V, \mathcal{B})$ be a directed triple system. Define a binary operation $\cdot$ on V in such a way that $x \cdot y = z$, $y \cdot z = x$ and $x \cdot z = y$ whenever $\langle x, y, z\rangle \in \mathcal{B}$, and that $x \cdot x = x$ for all $x \in V$. Then $(V, \cdot)$ is a quasigroup if and only if for all $\langle x, y, z\rangle \in \mathcal{B}$ there exist $x', y', z' \in V$ such that*

$$\langle z', y, x\rangle,\ \langle z, y', x\rangle,\ \langle z, y, x'\rangle \in \mathcal{B}.$$

In such a case $z' = y \cdot x$, $y' = z \cdot x$ and $x' = z \cdot y$.

It is now easy to see that in an LDTS, $(V, \mathcal{B})$,

$$\langle x, y, x \cdot y\rangle \in \mathcal{B} \quad \Rightarrow \quad y \cdot (x \cdot y) = (y \cdot x) \cdot y, \tag{1}$$

since if $\langle x, y, z\rangle \in \mathcal{B}$ then $\langle z', y, x\rangle \in \mathcal{B}$ for some z' and the ordered pair (x, y) satisfies the flexible identity $y \cdot (x \cdot y) = y \cdot z = x = z' \cdot y = (y \cdot x) \cdot y$. However, the flexible identity need not be satisfied for all ordered pairs of points from V. The following theorem proved in [3] gives the necessary and sufficient condition for an LDTS to be flexible.

Theorem 1.2. *A DTS-quasigroup obtained from an LDTS(n), $(V, \mathcal{B})$, satisfies the flexible law if and only if* $\langle x, y, z\rangle \in \mathcal{B} \Rightarrow \langle x, z\cdot x, y\cdot x\rangle \in \mathcal{B}$.

In [5] it was shown that a flexible LDTS(n) exists for all $n \equiv 0$ or $1 \pmod 3$, $n \neq 4, 6, 10, 12$.

In this paper we study the LDTSs whose binary operation satisfies the reverse of (1), i.e. for all $x, y \in V$, $x \neq y$,

$$y\cdot(x\cdot y) = (y\cdot x)\cdot y \quad \Rightarrow \quad \langle x, y, x\cdot y\rangle \in \mathcal{B}.$$

An LDTS satisfying this condition is called *antiflexible*. In other words an antiflexible DTS-quasigroup is one where the flexible identity $(y\cdot x)\cdot y = y\cdot(x\cdot y)$ holds for the least possible number of ordered pairs $(x, y) \in V \times V$. Thus, in a sense, antiflexible LDTSs are the LDTSs which are as distant from STSs as possible.

At first glance antiflexible LDTSs may appear to be a very artificial construct. However, there exists a surprisingly simple cyclic construction of LDTSs which naturally produces antiflexible LDTSs, see Theorem 3.1.

2. Properties

Let $(V, \mathcal{B})$ be a DTS and denote by $\mathcal{F}$ the set of all $\{x, y, z\}$ such that $\langle x, y, z\rangle \in \mathcal{B}$. This set is known as the *underlying twofold triple system* of $(V, \mathcal{B})$. Consider now $\mathcal{F}$ as a set of faces. Each edge $\{a, b\}$ is incident to two faces, hence the faces can be sewn together along common edges to form a pseudosurface. Note that we can orient a face $\{x, y, z\} \in \mathcal{F}$ as a cycle (x, y, z) whenever $\langle x, y, z\rangle \in \mathcal{B}$. It follows from Theorem 1.1 that this defines a coherent orientation. Hence $\mathcal{F}$ is an orientable pseudosurface.

A DTS is said to be *pure* if its underlying twofold triple system contains no repeated blocks. It is easy to see that every antiflexible LDTS is pure. If for some antiflexible LDTS, $(V, \mathcal{B})$, the triples $\langle x, y, z\rangle$ and $\langle z, y, x\rangle$ both belonged to $\mathcal{B}$, then $x\cdot(y\cdot x) = x\cdot z = y = z\cdot x = (x\cdot y)\cdot x$, which would imply that $\langle y, x, y\cdot x\rangle \in \mathcal{B}$. But this is a contradiction since $\langle z, y, x\rangle$ and $\langle y, x, y\cdot x\rangle$ cannot both belong to $\mathcal{B}$.

With each point $x \in V$ we can associate a partition of $V\setminus\{x\}$ into a set of cycles $(y_{1,1}, \ldots, y_{1,k_1})(y_{2,1}, \ldots, y_{2,k_2})\cdots(y_{m,1}, \ldots, y_{m,k_m})$, such that $(x, y_{i,j}, y_{i,j+1})$ and $(x, y_{i,k_i}, y_{i,1})$ are oriented faces of $\mathcal{F}$ for all $1 \leq j \leq k_i - 1$ and $1 \leq i \leq m$. If $m > 1$ then x is said to be a *pinch point*. A pseudosurface can be turned into a surface by separating each pinch point into several new points, called *vertices*, such that every vertex is associated with a single cycle. The length of the associated cycle is called the *degree* of the vertex. Thus we obtain an orientable surface. It follows from Theorem 1.1 that there are two types of vertices in this surface. A vertex may be associated with a point x and a cycle $(y_1, \dots, y_k)$ such that

$$\langle y_2, x, y_1\rangle, \langle y_3, x, y_2\rangle, \dots, \langle y_1, x, y_k\rangle \in \mathcal{B}.$$

This type of vertex is called a *middle vertex* to reflect the fact that x appears in the middle position of each of the k transitive triples. Alternatively, a vertex may be associated with a point x and a cycle $(y_1, z_1, y_2, z_2, \dots, y_k, z_k)$ such that

$$\langle x, y_1, z_1\rangle, \langle z_1, y_2, x\rangle, \langle x, y_2, z_2\rangle, \langle z_2, y_3, x\rangle, \dots, \langle x, y_k, z_k\rangle, \langle z_k, y_1, x\rangle \in \mathcal{B}.$$

This type of vertex is called a *residual vertex* in accordance with [4]. The degree of a residual vertex is always even.

Example 2.1. Let $V = \mathbb{Z}_{13}$ and define the set of starter triples $\mathcal{S} = \{\langle 1,4,0\rangle, \langle 0,6,1\rangle, \langle 2,6,0\rangle, \langle 0,5,2\rangle\}$. Let $\mathcal{B} = \{\, \langle x+i, y+i, z+i\rangle : \langle x,y,z\rangle \in \mathcal{S},\ i \in \mathbb{Z}_n \,\}$. Then $(V, \mathcal{B})$ is an antiflexible LDTS(13). As one can see from the triples listed below, the set of cycles associated with the point 0 is $(7,9,10,8)(5,2,6,1,4,11,3,12)$. Thus the point 0 separates into two vertices. The vertex associated with the cycle $(7,9,10,8)$ is a middle vertex and it is formed by the triples $\langle 9,0,7\rangle$, $\langle 10,0,9\rangle$, $\langle 8,0,10\rangle$, $\langle 7,0,8\rangle$ in $\mathcal{B}$. The vertex associated with the cycle $(5,2,6,1,4,11,3,12)$ is a residual vertex and it is formed by the triples $\langle 0,5,2\rangle$, $\langle 2,6,0\rangle$, $\langle 0,6,1\rangle$, $\langle 1,4,0\rangle$, $\langle 0,4,11\rangle$, $\langle 11,3,0\rangle$, $\langle 0,3,12\rangle$, $\langle 12,5,0\rangle$ in $\mathcal{B}$.

Theorem 2.2. *Let $(V, \mathcal{B})$ be an LDTS. Then the following conditions are equivalent:*

(i) *$(V, \mathcal{B})$ is antiflexible;*
(ii) *$\langle x,y,z\rangle \in \mathcal{B} \Rightarrow \langle x, zx, yx\rangle \notin \mathcal{B}$;*
(iii) *every residual vertex has degree at least 6.*

Proof: First assume that $(V, \mathcal{B})$ is antiflexible and let $\langle x,y,z\rangle \in \mathcal{B}$. Then using Theorem 1.1 the triple $\langle yx, y, x\rangle$ belongs to $\mathcal{B}$ as well. If it were the case that $\langle x, zx, yx\rangle \in \mathcal{B}$, then we would have $x \cdot yx = zx = xy \cdot x$. Then by assumption $\langle y, x, yx\rangle \in \mathcal{B}$. But this is a contradiction since $\langle y, x, yx\rangle$ and $\langle yx, y, x\rangle$ cannot both belong to $\mathcal{B}$. Thus $\langle x, zx, yx\rangle \notin \mathcal{B}$. We see that (i) implies (ii).

Assume that condition (ii) holds. If the cycle about a residual vertex corresponding to a point x were of length 2, say (y_1, z_1), then we would have $\langle x, y_1, z_1\rangle$, $\langle z_1, y_1, x\rangle \in \mathcal{B}$. But then $\mathcal{B}$ would contain $\langle x, z_1 \cdot x, y_1 \cdot x\rangle$, since this is the triple $\langle x, y_1, z_1\rangle$. Similarly if the cycle were of length 4, say (y_1, z_1, y_2, z_2), then we would have $\langle x, y_1, z_1\rangle$, $\langle z_1, y_2, x\rangle$, $\langle x, y_2, z_2\rangle$, $\langle z_2, y_1, x\rangle \in \mathcal{B}$. But then $\mathcal{B}$ would again contain $\langle x, z_1 \cdot x, y_1 \cdot x\rangle$, since this is the triple $\langle x, y_2, z_2\rangle$. Thus (ii) implies (iii).

Finally assume that condition (iii) holds. Let $x, y \in V$ such that $x \neq y$ and $y \cdot xy = yx \cdot y$. Now either $\langle xy, x, y\rangle$, $\langle x, xy, y\rangle$ or $\langle x, y, xy\rangle$ lies in $\mathcal{B}$. However, the first two of these possibilities violate the assumption. If $\langle xy, x, y\rangle \in \mathcal{B}$, then $\langle y, x, yx\rangle$, $\langle yx, yx \cdot y, y\rangle$, $\langle y, y \cdot xy, xy\rangle \in \mathcal{B}$, i.e. there exists a residual vertex associated with the point y and the cycle $(x, yx, y \cdot xy, xy)$. If $\langle x, xy, y\rangle \in \mathcal{B}$, then $\langle y, yx, x\rangle$, $\langle y, xy, y \cdot xy\rangle$, $\langle yx \cdot y, yx, y\rangle \in \mathcal{B}$, i.e. there exists a residual vertex associated with the point y and the cycle $(yx, x, xy, y \cdot xy)$. Thus (iii) implies (i). $\square$

3. Existence

In this section we investigate the existence spectrum of antiflexible LDTS(n). It was shown in [3] that there is no pure LDTS(n) for $3 \leq n \leq 12$. We start with a cyclic construction. An LDTS(n) is said to be *cyclic* if it admits an automorphism which permutes its points in a single cycle of length n. In [11] it was shown that a pure cyclic LDTS(n) exists if and only if $n \equiv 1 \pmod 6$ and $n \geq 13$. The following theorem shows that the construction used in [11] can always be used to produce antiflexible LDTSs. It is interesting to note, however, that for certain orders the construction can also be used to produce flexible LDTSs.

Theorem 3.1. *A cyclic antiflexible LDTS(n) exists if and only if $n \equiv 1 \pmod 6$ and $n \geq 13$.*

Proof: Let $n = 6k+1$ and $k \geq 2$. Set $V = \mathbb{Z}_n$ and define the set of starter triples

$$\mathcal{S} = \big\{ \langle r, k+2r, 0\rangle,\ \langle 0, 3k-r+1, r\rangle : r = 1, 2, \ldots, k \big\}.$$

If $k \equiv 1 \pmod 3$, then replace the starter triples

$$\langle \tfrac{2k+1}{3},\ k+2\tfrac{2k+1}{3},\ 0\rangle, \quad \langle 0,\ 3k-\tfrac{2k+1}{3}+1,\ \tfrac{2k+1}{3}\rangle \quad \text{and} \quad \langle k,\ 3k,\ 0\rangle$$

in $\mathcal{S}$ with the starter triples

$$\begin{aligned} B_1 &= \langle 4k+1,\ 0,\ \tfrac{1}{3}(5k+1)\rangle, \\ B_2 &= \langle \tfrac{1}{3}(5k+1),\ 0,\ \tfrac{1}{3}(2k+1)\rangle \text{ and} \\ B_3 &= \langle \tfrac{1}{3}(2k+1),\ 0,\ 3k+1\rangle. \end{aligned}$$

Let $\mathcal{B} = \{ \langle x+i, y+i, z+i\rangle : \langle x,y,z\rangle \in \mathcal{S},\ i \in \mathbb{Z}_n \}$. Then $(V, \mathcal{B})$ is an LDTS(n).

We check that condition (ii) of Theorem 2.2 is satisfied for each starter triple. To begin with, let us assume that $k \not\equiv 1 \pmod 3$. Let $1 \leq s \leq k$ and consider the starter triple $\langle x,y,z\rangle = \langle s, k+2s, 0\rangle$. We have $zx = 0\cdot s = 3k-s+1$. If s is even, then $\langle \frac{3}{2}s, k+2s, s\rangle \in \mathcal{B}$ (use $r = \frac{1}{2}s$ and $i = s$) i.e. $yx = \frac{3}{2}s$, and if s is odd, then $\langle \frac{1}{2}(3s-2k-1), k+2s, s\rangle \in \mathcal{B}$ (use $r = \frac{1}{2}(2k+1-s)$ and $i = \frac{1}{2}(3s-2k-1)$), i.e. $yx = \frac{1}{2}(3s-2k-1)$. If $s \leq \frac{1}{2}k$ then $\langle s, 3k-s+1, 3s\rangle \in \mathcal{B}$ (use $r = 2s$ and $i = s$), and if $s > \frac{1}{2}k$ then $\langle s, 3k-s+1, 3s-2k-1\rangle \in \mathcal{B}$ (use $r = 2k+1-2s$ and $i = 3s-2k-1$). The first two points in these two triples are x and zx respectively, but one can check that the third point is not equal to yx for any s. Thus $\langle x, zx, yx\rangle \notin \mathcal{B}$.

Now consider the starter triple $\langle x,y,z\rangle = \langle 0, 3k-s+1, s\rangle$. We have $zx = s\cdot 0 = k+2s$. If s is odd, then $\langle k-\frac{1}{2}(s-1), 3k-s+1, 0\rangle \in \mathcal{B}$ (use $r = k-\frac{1}{2}(s-1)$ and $i = 0$), i.e. $yx = k-\frac{1}{2}(s-1)$, and if s is even, then $\langle -\frac{1}{2}s, 3k-s+1, 0\rangle \in \mathcal{B}$ (use $r = \frac{1}{2}s$ and $i = -\frac{1}{2}s$), i.e. $yx = -\frac{1}{2}s$. If $s \leq \frac{1}{2}k$, then $\langle 0, k+2s, -2s\rangle \in \mathcal{B}$ (use $r = 2s$ and $i = -2s$), and if $s > \frac{1}{2}k$, then $\langle 0, k+2s, 2k-2s+1\rangle \in \mathcal{B}$ (use $r = 2k-2s+1$ and $i = 0$). We come to the same conclusion as above.

When $k \equiv 1 \pmod 3$ the statements above remain valid for all starter triples except for those that took part in the replacement, the case $s = \frac{1}{2}(k+1)$ discussed in the second paragraph and the cases $s \in \{1,\ \frac{1}{2}k,\ k\}$ discussed in the third paragraph. We prove that condition (ii) of Theorem 2.2 is satisfied for these triples as well:

For $\langle x, y, z\rangle = \langle 4k+1, 0, \frac{1}{3}(5k+1)\rangle$ we have $\langle \frac{1}{3}(5k+1), k, 4k+1\rangle \in \mathcal{B}$ (use B_3 and $i = k$), i.e. $zx = k$. If k is odd, then $\langle \frac{1}{2}(3k+1), 0, 4k+1\rangle \in \mathcal{B}$ (use $r = \frac{1}{2}(7k+1)$ and $i = 4k+1$), if k is even, then $\langle \frac{1}{2}k, 0, 4k+1\rangle \in \mathcal{B}$ (use $r = \frac{1}{2}(7k+2)$ and $i = \frac{1}{2}k$). Thus $yx \in \{\frac{1}{2}(3k+1), \frac{1}{2}k\}$ but $\langle 4k+1, k, 4k+2\rangle \in \mathcal{B}$ (use $r = 1$ and $i = 4k+1$).

For $\langle x, y, z\rangle = \langle \frac{1}{3}(5k+1), 0, \frac{1}{3}(2k+1)\rangle$ we have $\langle \frac{1}{3}(2k+1), \frac{4}{3}(2k+1), \frac{1}{3}(5k+1)\rangle \in \mathcal{B}$ (use $r = k$ and $i = \frac{1}{3}(2k+1)$), i.e. $zx = \frac{4}{3}(2k+1)$, and from B_1 we have $yx = 4k+1$. But $\langle \frac{1}{3}(5k+1), \frac{4}{3}(2k+1), \frac{1}{3}(5k-2)\rangle \in \mathcal{B}$ (use $r = 1$ and $i = \frac{1}{3}(5k-2)$).

For $\langle x, y, z\rangle = \langle \frac{1}{3}(2k+1), 0, 3k+1\rangle$ we have $\langle 3k+1, -k, \frac{1}{3}(2k+1)\rangle \in \mathcal{B}$ (use B_1 and $i = -k$), i.e. $zx = -k$, and from B_2 we have $yx = \frac{1}{3}(5k+1)$. But $\langle \frac{1}{3}(2k+1), -k, \frac{1}{3}(1-k)\rangle \in \mathcal{B}$ (use B_2 and $i = -k$).

If k is odd, then for $\langle x, y, z\rangle = \langle \frac{1}{2}(k+1), 2k+1, 0\rangle$ we have $\langle 0, \frac{1}{2}(5k+1), \frac{1}{2}(k+1)\rangle \in \mathcal{B}$ (use $r = \frac{1}{2}(k+1)$ and $i = 0$), i.e. $zx = \frac{1}{2}(5k+1)$. If $k \equiv 1 \pmod 4$, then $\langle \frac{1}{4}(1-k), 2k+1, \frac{1}{2}(k+1)\rangle \in \mathcal{B}$ (use $r = \frac{1}{4}(3k+1)$ and $i = \frac{1}{4}(1-k)$), if $k \equiv 3 \pmod 4$, then $\langle \frac{3}{4}(k+1), 2k+1, \frac{1}{2}(k+1)\rangle \in \mathcal{B}$ (use $r = \frac{1}{4}(k+1)$ and $i = \frac{1}{2}(k+1)$). Thus $yx \in \{\frac{1}{4}(1-k), \frac{3}{4}(k+1)\}$ but $\langle \frac{1}{2}(k+1), \frac{1}{2}(5k+1), \frac{5}{6}(5k+1)\rangle \in \mathcal{B}$ (use B_1 and $i = \frac{1}{2}(5k+1)$).

For $\langle x, y, z\rangle = \langle 0, 3k, 1\rangle$ we have $zx = k+2$ as before and $\langle \frac{1}{3}(11k+1), 3k, 0\rangle \in \mathcal{B}$ (use B_3 and $i = 3k$), i.e. $yx = \frac{1}{3}(11k+1)$. But $\langle 0, k+2, 6k-1\rangle \in \mathcal{B}$ (use $r = 2$ and $i = -2$).

For $\langle x, y, z\rangle = \langle 0, \frac{5}{2}k+1, \frac{1}{2}k\rangle$ we have $zx = 2k$ as before. If $k \equiv 0 \pmod 4$, then $\langle -\frac{1}{4}k, \frac{1}{2}(5k+2), 0\rangle \in \mathcal{B}$ (use $r = \frac{1}{4}k$ and $i = -\frac{1}{4}k$), and if $k \equiv 2 \pmod 4$, then $\langle \frac{1}{4}(3k+2), \frac{1}{2}(5k+2), 0\rangle \in \mathcal{B}$ (use $r = \frac{1}{4}(3k+2)$ and $i = 0$). Thus $yx \in \{-\frac{1}{4}k, \frac{1}{4}(3k+2)\}$ but $\langle 0, 2k, \frac{1}{3}(11k+1)\rangle \in \mathcal{B}$ (use B_1 and $i = 2k$).

For $\langle x, y, z\rangle = \langle 0, 2k+1, k\rangle$ we have $\langle k, -\frac{1}{3}(2k+1), 0\rangle \in \mathcal{B}$ (use B_2 and $i = -\frac{1}{3}(2k+1)$), i.e. $zx = -\frac{1}{3}(2k+1)$. If k is odd, then $\langle \frac{1}{2}(k+1), 2k+1, 0\rangle \in \mathcal{B}$ (use $r = \frac{1}{2}(k+1)$ and $i = 0$), if k is even, then $\langle -\frac{1}{2}k, 2k+1, 0\rangle \in \mathcal{B}$ (use $r = \frac{1}{2}k$ and $i = -\frac{1}{2}k$). Thus $yx \in \{\frac{1}{2}(k+1), -\frac{1}{2}k\}$ but $\langle 0, -\frac{1}{3}(2k+1), \frac{1}{3}(7k+2)\rangle \in \mathcal{B}$ (use B_3 and $i = -\frac{1}{3}(2k+1)$). □

In [4] all LDTSs of order 13 were enumerated and classified by their automorphism group. Out of the total of 1 206 969 non-isomorphic LDTS(13)s 8 444 are pure, but only two of them are antiflexible. They are the two pure cyclic systems. The starter triples for these two systems are $\langle 1, 4, 0\rangle$, $\langle 0, 6, 1\rangle$, $\langle 2, 6, 0\rangle$, $\langle 0, 5, 2\rangle$ for one and $\langle 1, 10, 0\rangle$, $\langle 0, 8, 1\rangle$, $\langle 2, 9, 0\rangle$, $\langle 0, 10, 2\rangle$ for the other. In comparison, there are 924 flexible LDTS(13)s up to isomorphism.

Next is an elementary recursive construction adapted from standard design-theoretic techniques.

Proposition 3.2. *If there exists an antiflexible LDTS(n), $n > 3$, then there exists*

(i) *an antiflexible LDTS($3n$), and*
(ii) *an antiflexible LDTS($3n-2$).*

Proof: (i) Take three copies of the LDTS(n) on point sets $\{\, i_j : i \in \mathbb{Z}_n \,\}$, $j \in \{0, 1, 2\}$ respectively, then adjoin all transitive triples

$$\langle i_0, j_1, (i+j)_2 \rangle \quad \text{and} \quad \langle (i+j-1)_2, j_1, i_0 \rangle, \quad i, j \in \mathbb{Z}_n.$$

The adjoined transitive triples create one new residual vertex of degree $2n$ for each of the points in the first and third copies of the LDTS(n). For any point i_0, where $i \in \mathbb{Z}_n$, the newly created residual vertex corresponds to the cycle

$$(0_1, i_2, 1_1, (i+1)_2, \ldots, (n-1)_1, (i-1)_2).$$

For any point i_2, where $i \in \mathbb{Z}_n$, the newly created residual vertex corresponds to the cycle

$$(0_1, (i+1)_0, (n-1)_1, (i+2)_0, \ldots, 1_1, i_0).$$

Thus the resulting system is antiflexible as long as $n > 2$.

(ii) Take three copies of the LDTS(n) on point sets $\{\, i_j : i \in \mathbb{Z}_{n-1} \,\} \cup \{\infty\}$, $j \in \{0, 1, 2\}$ respectively, then adjoin all transitive triples

$$\langle i_0, j_1, (i+j)_2 \rangle \quad \text{and} \quad \langle (i+j-1)_2, j_1, i_0 \rangle, \quad i, j \in \mathbb{Z}_{n-1}.$$

Similarly this system is antiflexible as long as $n > 3$.

□

Lemma 3.3. *If $n \equiv 3 \pmod{18}$ and $n \neq 3$, then there exists an antiflexible LDTS(n).*

Proof: It follows from Theorem 3.1 and part (i) of Proposition 3.2 that there exists an antiflexible LDTS(n) for all $n \equiv 3 \pmod{18}$, $n \geq 39$. An antiflexible LDTS(21) is given as Example A.4 in the Appendix. □

Proposition 3.4. *If there exists an antiflexible LDTS(n), $(V, \mathcal{B})$, and a quasigroup $(V \cup \{\infty\}, *)$ satisfying*

(1) $x * x = \infty$, *and*
(2) $(x * y = y * z \wedge z * y = y * x) \Rightarrow x = y = z$,

then there exists an antiflexible LDTS($2n+1$).

Proof: Let $W = V \cup \{ x' : x \in V \} \cup \{\infty'\}$. Form a set of transitive triples $\mathcal{D}$ by starting with the set $\mathcal{B}$ and adjoining all triples $\langle x', x * y, y' \rangle$, where x, $y \in V \cup \{\infty\}$, $x \neq y$. Then $(W, \mathcal{D})$ is an LDTS. We verify that $(W, \mathcal{D})$ satisfies condition (ii) of Theorem 2.2. Let $\langle x, y, z \rangle \in \mathcal{D}$. If $\langle x, y, z \rangle \in \mathcal{B}$, then $\langle x, z \cdot x, y \cdot x \rangle$ does not lie in $\mathcal{D}$, since if it did, then it would have had to come from $\mathcal{B}$, which would be a contradiction. It remains to deal with the case when $\langle x, y, z \rangle$ is of the form $\langle u', u * v, v' \rangle$, for some u, $v \in V \cup \{\infty\}$. Clearly $z \cdot x = v * u$. There exists $w \in V \cup \{\infty\}$ such that $\langle w', u * v, u' \rangle \in \mathcal{D}$, i.e. $y \cdot x = w'$. Now since $w * u = u * v$, by assumption $v * u \neq u * w$, and thus $\langle x, z \cdot x, y \cdot x \rangle = \langle u', v * u, w' \rangle$ does not lie in $\mathcal{D}$. □

A quasigroup of order n satisfying conditions (1) and (2) of Proposition 3.4 will be referred to as a *unipotent locally self-orthogonal quasigroup*, ULSOQ(n).

The remainder of the existence proof in this section uses a standard technique known as Wilson's fundamental construction for which we need the concept of a *group divisible design* (GDD). Let K be a set of positive integers. A K-GDD of type g^u is an ordered triple $(V, \mathcal{G}, \mathcal{B})$ where V is a base set of cardinality $v = gu$, $\mathcal{G}$ is a partition of V into u subsets of cardinality g called *groups* and $\mathcal{B}$ is a family of subsets called *blocks* such that (1) $|B| \in K$ for all $B \in \mathcal{B}$, and (2) every pair of distinct elements of V occurs in exacly one block or one group, but not both. We will also need K-GDDs of type $g^u m^1$. These are defined analogously, with the base set V being of cardinality $v = gu + m$ and the partition G being into u subsets of cardinality g and one set of cardinality m. If K is a singleton, then instead of $\{k\}$-GDD we write simply k-GDD. Necessary and sufficient conditions for the existence of 3-GDDs of type g^u were determined in [10] and for 3-GDDs of type $g^u m^1$ in [2]. The existence of the 4-GDDs that we will be using was determined in [1], [7], [8], [9]. A convenient reference is [6] where the existence of all the GDDs that are used can be verified.

We will assume that the reader is familiar with this construction but briefly the basic idea is as follows. Begin with a k-GDD of cardinality $v = gu$ or $gu + m$, usually called the *master GDD*. Each point is then assigned a weight, usually the same weight, say w. In effect, each point is replaced by w points. Each inflated block of the master GDD is then replaced by a k-GDD of type w^k, called a *slave GDD*. We will only need to use the value $w = 3$, and instead of slave GDDs we will use partial Latin directed triple systems. When $k = 3$ we will employ the partial LDTS(9) whose blocks are $\langle a,p,x \rangle$, $\langle b,q,y \rangle$, $\langle c,r,z \rangle$, $\langle a,q,z \rangle$, $\langle b,r,x \rangle$, $\langle c,p,y \rangle$, $\langle a,r,y \rangle$, $\langle b,p,z \rangle$, $\langle c,q,x \rangle$, $\langle x,q,a \rangle$, $\langle y,r,b \rangle$, $\langle z,p,c \rangle$, $\langle z,r,a \rangle$, $\langle x,p,b \rangle$, $\langle y,q,c \rangle$, $\langle y,p,a \rangle$, $\langle z,q,b \rangle$, $\langle x,r,c \rangle$ and the sets $\{a,b,c\}$, $\{p,q,r\}$, $\{x,y,z\}$ play the role of the groups. When $k = 4$ we will use the partial LDTS(12) whose blocks are $\langle p,a,x \rangle$, $\langle s,a,p \rangle$, $\langle x,a,s \rangle$, $\langle q,b,y \rangle$, $\langle u,b,q \rangle$, $\langle y,b,u \rangle$, $\langle r,c,z \rangle$, $\langle t,c,r \rangle$, $\langle z,c,t \rangle$, $\langle c,p,u \rangle$, $\langle u,p,y \rangle$, $\langle y,p,c \rangle$, $\langle a,q,t \rangle$, $\langle t,q,z \rangle$, $\langle z,q,a \rangle$, $\langle b,r,s \rangle$, $\langle s,r,x \rangle$, $\langle x,r,b \rangle$, $\langle c,s,y \rangle$, $\langle q,s,c \rangle$, $\langle y,s,q \rangle$, $\langle b,t,x \rangle$, $\langle p,t,b \rangle$, $\langle x,t,p \rangle$, $\langle a,u,z \rangle$, $\langle r,u,a \rangle$, $\langle z,u,r \rangle$, $\langle c,x,q \rangle$, $\langle q,x,u \rangle$, $\langle u,x,c \rangle$, $\langle a,y,r \rangle$, $\langle r,y,t \rangle$, $\langle t,y,a \rangle$, $\langle b,z,p \rangle$, $\langle p,z,s \rangle$, $\langle s,z,b \rangle$ and the sets $\{a,b,c\}$, $\{p,q,r\}$, $\{s,t,u\}$, $\{x,y,z\}$ play the role of the groups. Note that both of these partial systems induce a closed surface with all residual vertices of degree 6. To complete the construction we then "fill in" the groups of the expanded master GDD, sometimes adjoining an extra point, to all of the groups. Thus we may need antiflexible Latin directed triple systems of orders gw, mw, $gw+1$ or $mw+1$ as appropriate.

In several cases we use a $\{3,4\}$-GDD as the master GDD which requires that when we replace the inflated blocks, we employ both of the partial systems given above. Before continuing the existence proof of the antiflexible LDTSs, let us establish the existence of the $\{3,4\}$-GDDs we will be using.

Proposition 3.5. *If $g \notin \{2,6\}$ and $0 \le m \le g$, then there exists a $\{3,4\}$-GDD of type g^3m^1.*

PROOF: Take a 4-GDD of type g^4 with groups $G_i = \{1_i, \dots, g_i\}$, where $i \in \{0,1,2,3\}$. To get a $\{3,4\}$-GDD of type g^3m^1 simply remove each of the points $(m+1)_3$, $(m+2)_3$, …, g_3 from the design. In other words replace every block $\{x_0, y_1, z_2, w_3\}$ such that $m < w \le g$ with the block $\{x_0, y_1, z_2\}$ to obtain a $\{3,4\}$-GDD with groups G_1, G_2, G_3 and $G_4' = \{1_3, \dots, m_3\}$. □

Example 3.6. $\{3,4\}$-GDD of type $6^3\,5^1$.
The groups are $G_j = \{\, i_j : i \in \mathbb{Z}_6 \,\}$, where $j \in \{0,1,2\}$, and $G_3 = \{\, i_3 : i \in \mathbb{Z}_2 \,\} \cup \{\infty_0, \infty_1, \infty_2\}$.

To obtain the blocks, develop the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$, with ∞_0, ∞_1 and ∞_2 as fixed points: $\{0_0, 0_1, 0_2, \infty_0\}$, $\{0_0, 1_1, 2_2, \infty_1\}$, $\{0_0, 2_1, 4_2, \infty_2\}$, $\{0_0, 3_1, 1_2\}$, $\{0_0, 4_1, 3_2\}$, $\{0_0, 5_1, 0_3\}$, $\{0_0, 5_2, 1_3\}$, $\{0_1, 3_2, 0_3\}$.

Lemma 3.7. *If $n \equiv 0 \pmod 6$ and $n \ge 18$, then there exists an antiflexible LDTS(n).*

PROOF: Table 1 gives the schema for antiflexible LDTS(n), $n \equiv 0 \pmod 6$. No extra points are adjoined in this case. The missing antiflexible LDTSs of orders 36 and 42 as well as the systems of orders 18, 24 and 30 which are needed to construct the infinite classes are all given in the Appendix. The missing antiflexible LDTS(48) and LDTS(66) can be obtained using part (i) of Proposition 3.2 from the LDTS(16) and LDTS(22) given in the Appendix. The antiflexible LDTS(60) can be constructed by taking a master 4-GDD of type 5^4, inflating each point by a factor of 3 and using the antiflexible LDTS(15) given in the Appendix. □

Type of master 3-GDD	Orders of LDTS(n) needed	Residue classes covered modulo 18	Missing values
6^s, $s \ge 3$	18	0	36
$6^s\,8^1$, $s \ge 3$	18, 24	6	42, 60
$6^s\,10^1$, $s \ge 3$	18, 30	12	48, 66

TABLE 1. Schema for antiflexible LDTS(n), $n \equiv 0 \pmod 6$.

Lemma 3.8. *If $n \equiv 16 \pmod{18}$, then there exists an antiflexible LDTS(n).*

Proof: It follows from the previous lemma and part (ii) of Proposition 3.2 that there exists an antiflexible LDTS(n) for all $n \equiv 16 \pmod{18}$, $n \geq 52$. Antiflexible LDTSs of orders 16 and 34 are given in the Appendix. □

Lemma 3.9. *If $n \equiv 15 \pmod{18}$, then there exists an antiflexible LDTS(n).*

Proof: Table 2 gives the schema for antiflexible LDTS(n), $n \equiv 15 \pmod{18}$. Once again, no extra points are adjoined in this case. The required antiflexible LDTS(n)s of orders $n = 15$ and 27 are given in the Appendix. The antiflexible LDTS(33) can be obtained by taking an antiflexible LDTS(16) given in the Appendix together with the quasigroup given in Example A.14 and applying Proposition 3.4. Similarly the antiflexible LDTS(51) can be obtained by taking a (cyclic) antiflexible LDTS(25) together with the quasigroup given in Example A.15. The missing antiflexible LDTS(69) can be constructed using a master $\{3,4\}$-GDD of type $6^3\,5^1$ given in Example 3.6 and the antiflexible LDTS(15) and LDTS(18) given in the Appendix. The antiflexible LDTS(87) can be constructed using a master 3-GDD of type $5^4\,9^1$ together with the antiflexible LDTS(15) and LDTS(27) and the antiflexible LDTS(105) can be constructed using a master 3-GDD of type 5^7 and the LDTS(15). □

Type of master 3-GDD	Orders of LDTS(n) needed	Residue classes covered modulo 54	Missing values
$9^{2s}\,5^1$, $s \geq 2$	15, 27	15	69
$9^{2s}\,11^1$, $s \geq 2$	27, 33	33	87
$9^{2s}\,17^1$, $s \geq 2$	27, 51	51	105

Table 2. Schema for antiflexible LDTS(n), $n \equiv 15 \pmod{18}$.

Lemma 3.10. *If $n \equiv 4$, 9 or 10 $\pmod{18}$ and $n \geq 22$, then there exists an antiflexible LDTS(n).*

Proof: Table 3 gives the schema for antiflexible LDTS(n), $n \equiv 4$, 9 or 10 (mod 18). The required antiflexible LDTS(n)s of orders $n = 18$, 22, 27, 28 and 40 are given in the Appendix and the ones of orders 13 and 19 exist by Theorem 3.1. For the missing $n = 45$, 63 and 81 use part (i) of Proposition 3.2 and for $n = 46$, 64 and 82 use part (ii) of Proposition 3.2. To do this we need systems of orders 15, 21, 27, 16, 22 and 28, respectively, all of which are given in the Appendix. The missing antiflexible LDTS(58) and LDTS(76) can be constructed using master $\{3,4\}$-GDDs of types $5^3\,4^1$ and $7^3\,4^1$, respectively, adjoining an extra point and taking the antiflexible LDTSs of orders 13, 16 and 22. The missing antiflexible LDTS(112) can be constructed using a master 3-GDD of type $5^6\,7^1$, adjoining an extra point and taking the antiflexible LDTSs of orders 16 and 22. □

Type of master 4-GDD	Points adjoined	Orders of LDTS(n) needed	Residue classes covered modulo 36	Missing values
$4^{3s}\,7^1$, $s \geq 2$	1	13, 22	22	58
$4^{3s}\,13^1$, $s \geq 3$	1	13, 40	4	76, 112
$6^s\,9^1$, $s \geq 4$	0	18, 27	9, 27	45, 63, 81
$6^s\,9^1$, $s \geq 4$	1	19, 28	10, 28	46, 64, 82

TABLE 3. Schema for antiflexible LDTS(n), $n \equiv 4$, 9 or 10 (mod 18).

Theorem 3.11. *An antiflexible* $LDTS(n)$ *exists if and only if* $n \equiv 0$ *or* 1 (mod 3) *and* $n \geq 13$.

Appendix. Examples of antiflexible LDTSs

The following examples were obtained by computer with the help of the model builder Mace4 [12] using an algebraic description of a DTS-quasigroup, see [4]. We denote the elements $(i,j) \in \mathbb{Z}_m \times \mathbb{Z}_n$ as i_j. For simplicity, we omit commas from the triples.

Example A.1. Antiflexible LDTS(15).
$V = (\mathbb{Z}_7 \times \mathbb{Z}_2) \cup \{\infty\}$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$, with ∞ as a fixed point.
$\langle 2_0\,0_0\,2_1\rangle$, $\langle 2_1\,0_0\,1_1\rangle$, $\langle 1_1\,0_0\,5_1\rangle$, $\langle 5_1\,0_0\,3_1\rangle$, $\langle 3_1\,0_0\,4_1\rangle$, $\langle 4_1\,0_0\,6_1\rangle$, $\langle 6_1\,0_0\,6_0\rangle$, $\langle 6_0\,0_0\,2_0\rangle$, $\langle 0_0\,\infty\,4_0\rangle$, $\langle 0_1\,\infty\,3_1\rangle$.

Example A.2. Antiflexible LDTS(16).
$V = \mathbb{Z}_8 \times \mathbb{Z}_2$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 2_0\,0_0\,7_1\rangle$, $\langle 7_1\,0_0\,7_0\rangle$, $\langle 7_0\,0_0\,2_0\rangle$, $\langle 0_0\,2_1\,4_0\rangle$, $\langle 4_0\,2_1\,4_1\rangle$, $\langle 4_1\,2_1\,1_0\rangle$, $\langle 1_0\,2_1\,6_0\rangle$, $\langle 6_0\,2_1\,1_1\rangle$, $\langle 1_1\,2_1\,5_1\rangle$, $\langle 5_1\,2_1\,0_0\rangle$.

Example A.3. Antiflexible LDTS(18).
$V = \mathbb{Z}_3 \times \mathbb{Z}_6$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 1_0\,0_0\,0_2\rangle$, $\langle 0_2\,0_0\,0_1\rangle$, $\langle 0_1\,0_0\,1_0\rangle$, $\langle 1_0\,2_1\,1_5\rangle$, $\langle 1_5\,2_1\,2_5\rangle$, $\langle 2_5\,2_1\,0_2\rangle$, $\langle 0_2\,2_1\,1_0\rangle$, $\langle 0_1\,0_3\,2_4\rangle$, $\langle 2_4\,0_3\,1_5\rangle$, $\langle 1_5\,0_3\,0_1\rangle$, $\langle 0_1\,0_4\,2_3\rangle$, $\langle 2_3\,0_4\,1_2\rangle$, $\langle 1_2\,0_4\,2_4\rangle$, $\langle 2_4\,0_4\,0_1\rangle$, $\langle 1_0\,1_4\,0_1\rangle$, $\langle 0_1\,1_4\,1_5\rangle$, $\langle 1_5\,1_4\,1_0\rangle$, $\langle 1_2\,0_0\,1_3\rangle$, $\langle 1_3\,0_0\,1_4\rangle$, $\langle 1_4\,0_0\,2_5\rangle$, $\langle 2_5\,0_0\,1_5\rangle$, $\langle 1_5\,0_0\,2_4\rangle$, $\langle 2_4\,0_0\,2_3\rangle$, $\langle 2_3\,0_0\,0_3\rangle$, $\langle 0_3\,0_0\,1_2\rangle$, $\langle 2_1\,0_1\,2_2\rangle$, $\langle 2_2\,0_1\,1_3\rangle$, $\langle 1_3\,0_1\,2_1\rangle$, $\langle 2_2\,0_2\,1_5\rangle$, $\langle 1_5\,0_2\,0_4\rangle$, $\langle 0_4\,0_2\,2_2\rangle$, $\langle 0_2\,0_5\,1_3\rangle$, $\langle 1_3\,0_5\,0_3\rangle$, $\langle 0_3\,0_5\,0_2\rangle$.

Example A.4. Antiflexible LDTS(21).
$V = (\mathbb{Z}_{10} \times \mathbb{Z}_2) \cup \{\infty\}$.

The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$, with ∞ as a fixed point.
$\langle 2_0\,0_0\,0_1\rangle$, $\langle 0_1\,0_0\,9_1\rangle$, $\langle 9_1\,0_0\,6_1\rangle$, $\langle 6_1\,0_0\,4_1\rangle$, $\langle 4_1\,0_0\,5_1\rangle$, $\langle 5_1\,0_0\,1_1\rangle$, $\langle 1_1\,0_0\,3_1\rangle$, $\langle 3_1\,0_0\,7_1\rangle$, $\langle 7_1\,0_0\,2_1\rangle$, $\langle 2_1\,0_0\,4_0\rangle$, $\langle 4_0\,0_0\,9_0\rangle$, $\langle 9_0\,0_0\,2_0\rangle$, $\langle 0_0\,\infty\,7_0\rangle$, $\langle 0_1\,\infty\,3_1\rangle$.

Example A.5. Antiflexible LDTS(22).
$V = \mathbb{Z}_{11} \times \mathbb{Z}_2$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 1_0\,0_0\,5_0\rangle$, $\langle 5_0\,0_0\,2_1\rangle$, $\langle 2_1\,0_0\,0_1\rangle$, $\langle 0_1\,0_0\,3_0\rangle$, $\langle 3_0\,0_0\,1_0\rangle$, $\langle 0_0\,1_1\,2_0\rangle$, $\langle 2_0\,1_1\,9_0\rangle$, $\langle 9_0\,1_1\,5_1\rangle$, $\langle 5_1\,1_1\,7_0\rangle$, $\langle 7_0\,1_1\,2_1\rangle$, $\langle 2_1\,1_1\,4_1\rangle$, $\langle 4_1\,1_1\,8_0\rangle$, $\langle 8_0\,1_1\,6_1\rangle$, $\langle 6_1\,1_1\,0_0\rangle$.

Example A.6. Antiflexible LDTS(24).
$V = \mathbb{Z}_4 \times \mathbb{Z}_6$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 1_0\,0_0\,2_1\rangle$, $\langle 2_1\,0_0\,3_3\rangle$, $\langle 3_3\,0_0\,0_3\rangle$, $\langle 0_3\,0_0\,0_1\rangle$, $\langle 0_1\,0_0\,1_0\rangle$, $\langle 2_3\,1_0\,2_4\rangle$, $\langle 2_4\,1_0\,0_5\rangle$, $\langle 0_5\,1_0\,2_5\rangle$, $\langle 2_5\,1_0\,1_5\rangle$, $\langle 1_5\,1_0\,0_4\rangle$, $\langle 0_4\,1_0\,1_4\rangle$, $\langle 1_4\,1_0\,3_3\rangle$, $\langle 3_3\,1_0\,2_3\rangle$, $\langle 2_3\,0_1\,0_4\rangle$, $\langle 0_4\,0_1\,1_5\rangle$, $\langle 1_5\,0_1\,2_3\rangle$, $\langle 3_3\,0_1\,2_4\rangle$, $\langle 2_4\,0_1\,1_4\rangle$, $\langle 1_4\,0_1\,3_4\rangle$, $\langle 3_4\,0_1\,3_3\rangle$, $\langle 0_0\,0_2\,2_0\rangle$, $\langle 2_0\,0_2\,1_1\rangle$, $\langle 1_1\,0_2\,3_1\rangle$, $\langle 3_1\,0_2\,2_1\rangle$, $\langle 2_1\,0_2\,2_5\rangle$, $\langle 2_5\,0_2\,0_4\rangle$, $\langle 0_4\,0_2\,3_2\rangle$, $\langle 3_2\,0_2\,0_0\rangle$, $\langle 3_0\,0_2\,1_5\rangle$, $\langle 1_5\,0_2\,2_4\rangle$, $\langle 2_4\,0_2\,3_3\rangle$, $\langle 3_3\,0_2\,1_3\rangle$, $\langle 1_3\,0_2\,0_5\rangle$, $\langle 0_5\,0_2\,0_1\rangle$, $\langle 0_1\,0_2\,3_0\rangle$, $\langle 0_2\,0_3\,1_4\rangle$, $\langle 1_4\,0_3\,0_5\rangle$, $\langle 0_5\,0_3\,1_5\rangle$, $\langle 1_5\,0_3\,2_2\rangle$, $\langle 2_2\,0_3\,0_2\rangle$, $\langle 2_0\,0_4\,1_2\rangle$, $\langle 1_2\,0_4\,0_5\rangle$, $\langle 0_5\,0_4\,2_0\rangle$, $\langle 1_1\,0_5\,2_1\rangle$, $\langle 2_1\,0_5\,2_3\rangle$, $\langle 2_3\,0_5\,1_1\rangle$.

Example A.7. Antiflexible LDTS(27).
$V = (\mathbb{Z}_{13} \times \mathbb{Z}_2) \cup \{\infty\}$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$, with ∞ as a fixed point.
$\langle 1_0\,0_0\,5_0\rangle$, $\langle 5_0\,0_0\,0_1\rangle$, $\langle 0_1\,0_0\,3_0\rangle$, $\langle 3_0\,0_0\,1_0\rangle$, $\langle 2_0\,0_1\,11_0\rangle$, $\langle 11_0\,0_1\,8_1\rangle$, $\langle 8_1\,0_1\,2_0\rangle$, $\langle 0_0\,1_1\,2_0\rangle$, $\langle 2_0\,1_1\,9_0\rangle$, $\langle 9_0\,1_1\,0_1\rangle$, $\langle 0_1\,1_1\,5_0\rangle$, $\langle 5_0\,1_1\,11_1\rangle$, $\langle 11_1\,1_1\,7_0\rangle$, $\langle 7_0\,1_1\,10_1\rangle$, $\langle 10_1\,1_1\,3_1\rangle$, $\langle 3_1\,1_1\,0_0\rangle$, $\langle 0_0\,\infty\,6_0\rangle$, $\langle 0_1\,\infty\,7_1\rangle$.

Example A.8. Antiflexible LDTS(28).
$V = \mathbb{Z}_{14} \times \mathbb{Z}_2$.
The system is defined by the triples obtained from the following starter blocks under the action of the mappings $i_j \mapsto (i+1)_j$ and $i_j \mapsto i_{j+1}$.
$\langle 1_0\,0_0\,5_0\rangle$, $\langle 5_0\,0_0\,12_1\rangle$, $\langle 12_1\,0_0\,4_1\rangle$, $\langle 4_1\,0_0\,6_1\rangle$, $\langle 6_1\,0_0\,13_1\rangle$, $\langle 13_1\,0_0\,9_1\rangle$, $\langle 9_1\,0_0\,3_1\rangle$, $\langle 3_1\,0_0\,3_0\rangle$, $\langle 3_0\,0_0\,1_0\rangle$.

Example A.9. Antiflexible LDTS(30).
$V = \mathbb{Z}_5 \times \mathbb{Z}_6$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 0_5\,0_0\,3_5\rangle$, $\langle 3_5\,0_0\,4_5\rangle$, $\langle 4_5\,0_0\,1_5\rangle$, $\langle 1_5\,0_0\,0_5\rangle$, $\langle 0_0\,0_1\,1_0\rangle$, $\langle 1_0\,0_1\,4_0\rangle$, $\langle 4_0\,0_1\,3_0\rangle$, $\langle 3_0\,0_1\,0_0\rangle$, $\langle 3_1\,0_0\,4_2\rangle$, $\langle 4_2\,0_0\,1_2\rangle$, $\langle 1_2\,0_0\,0_2\rangle$, $\langle 0_2\,0_0\,3_2\rangle$, $\langle 3_2\,0_0\,3_1\rangle$, $\langle 2_1\,0_4\,1_5\rangle$, $\langle 1_5\,0_4\,4_2\rangle$, $\langle 4_2\,0_4\,0_2\rangle$, $\langle 0_2\,0_4\,2_1\rangle$, $\langle 3_1\,4_4\,0_2\rangle$, $\langle 0_2\,4_4\,2_5\rangle$, $\langle 2_5\,4_4\,1_3\rangle$, $\langle 1_3\,4_4\,1_5\rangle$, $\langle 1_5\,4_4\,3_1\rangle$, $\langle 0_1\,0_5\,1_3\rangle$, $\langle 1_3\,0_5\,0_2\rangle$, $\langle 0_2\,0_5\,3_1\rangle$, $\langle 3_1\,0_5\,1_2\rangle$, $\langle 1_2\,0_5\,0_1\rangle$, $\langle 2_2\,0_0\,3_3\rangle$,

$\langle 3_3\, 0_0\, 4_3\rangle$, $\langle 4_3\, 0_0\, 4_4\rangle$, $\langle 4_4\, 0_0\, 1_4\rangle$, $\langle 1_4\, 0_0\, 0_4\rangle$, $\langle 0_4\, 0_0\, 3_4\rangle$, $\langle 3_4\, 0_0\, 2_5\rangle$, $\langle 2_5\, 0_0\, 2_4\rangle$, $\langle 2_4\, 0_0\, 1_3\rangle$, $\langle 1_3\, 0_0\, 0_3\rangle$, $\langle 0_3\, 0_0\, 2_3\rangle$, $\langle 2_3\, 0_0\, 2_2\rangle$, $\langle 3_1\, 0_1\, 1_5\rangle$, $\langle 1_5\, 0_1\, 3_3\rangle$, $\langle 3_3\, 0_1\, 2_4\rangle$, $\langle 2_4\, 0_1\, 3_1\rangle$, $\langle 4_1\, 0_1\, 4_2\rangle$, $\langle 4_2\, 0_1\, 4_3\rangle$, $\langle 4_3\, 0_1\, 0_4\rangle$, $\langle 0_4\, 0_1\, 0_3\rangle$, $\langle 0_3\, 0_1\, 4_1\rangle$, $\langle 2_3\, 0_2\, 3_5\rangle$, $\langle 3_5\, 0_2\, 3_3\rangle$, $\langle 3_3\, 0_2\, 1_5\rangle$, $\langle 1_5\, 0_2\, 2_4\rangle$, $\langle 2_4\, 0_2\, 3_4\rangle$, $\langle 3_4\, 0_2\, 4_3\rangle$, $\langle 4_3\, 0_2\, 2_3\rangle$, $\langle 3_1\, 0_3\, 2_4\rangle$, $\langle 2_4\, 0_3\, 2_5\rangle$, $\langle 2_5\, 0_3\, 3_1\rangle$.

Example A.10. Antiflexible LDTS(34).
$V = \mathbb{Z}_{17} \times \mathbb{Z}_2$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$\langle 1_0\, 0_0\, 5_0\rangle$, $\langle 5_0\, 0_0\, 7_0\rangle$, $\langle 7_0\, 0_0\, 3_0\rangle$, $\langle 3_0\, 0_0\, 1_0\rangle$, $\langle 6_0\, 0_0\, 1_1\rangle$, $\langle 1_1\, 0_0\, 8_0\rangle$, $\langle 8_0\, 0_0\, 2_1\rangle$, $\langle 2_1\, 0_0\, 9_1\rangle$, $\langle 9_1\, 0_0\, 5_1\rangle$, $\langle 5_1\, 0_0\, 0_1\rangle$, $\langle 0_1\, 0_0\, 6_0\rangle$, $\langle 1_1\, 2_0\, 5_1\rangle$, $\langle 5_1\, 2_0\, 10_1\rangle$, $\langle 10_1\, 2_0\, 16_1\rangle$, $\langle 16_1\, 2_0\, 9_1\rangle$, $\langle 9_1\, 2_0\, 1_1\rangle$, $\langle 3_1\, 5_0\, 11_1\rangle$, $\langle 11_1\, 5_0\, 9_1\rangle$, $\langle 9_1\, 5_0\, 3_1\rangle$, $\langle 4_0\, 0_1\, 14_1\rangle$, $\langle 14_1\, 0_1\, 16_1\rangle$, $\langle 16_1\, 0_1\, 4_0\rangle$.

Example A.11. Antiflexible LDTS(36).
$V = (\mathbb{Z}_7 \times \mathbb{Z}_5) \cup \{\infty\}$.
The system is defined by the triples obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$, with ∞ as a fixed point.
$\langle 2_0\, 0_0\, 2_1\rangle$, $\langle 2_1\, 0_0\, 6_1\rangle$, $\langle 6_1\, 0_0\, 6_0\rangle$, $\langle 6_0\, 0_0\, 2_0\rangle$, $\langle 2_0\, 0_1\, 6_0\rangle$, $\langle 6_0\, 0_1\, 1_2\rangle$, $\langle 1_2\, 0_1\, 5_2\rangle$, $\langle 5_2\, 0_1\, 5_1\rangle$, $\langle 5_1\, 0_1\, 2_0\rangle$, $\langle 4_1\, 0_0\, 4_2\rangle$, $\langle 4_2\, 0_0\, 0_3\rangle$, $\langle 0_3\, 0_0\, 0_4\rangle$, $\langle 0_4\, 0_0\, 6_3\rangle$, $\langle 6_3\, 0_0\, 5_4\rangle$, $\langle 5_4\, 0_0\, 3_2\rangle$, $\langle 3_2\, 0_0\, 6_4\rangle$, $\langle 6_4\, 0_0\, 3_4\rangle$, $\langle 3_4\, 0_0\, 0_2\rangle$, $\langle 0_2\, 0_0\, 4_1\rangle$, $\langle 0_2\, 1_0\, 4_3\rangle$, $\langle 4_3\, 1_0\, 6_3\rangle$, $\langle 6_3\, 1_0\, 0_2\rangle$, $\langle 1_3\, 0_1\, 2_4\rangle$, $\langle 2_4\, 0_1\, 0_4\rangle$, $\langle 0_4\, 0_1\, 3_4\rangle$, $\langle 3_4\, 0_1\, 4_4\rangle$, $\langle 4_4\, 0_1\, 6_4\rangle$, $\langle 6_4\, 0_1\, 5_4\rangle$, $\langle 5_4\, 0_1\, \infty\rangle$, $\langle \infty\, 0_1\, 1_4\rangle$, $\langle 1_4\, 0_1\, 1_3\rangle$, $\langle 2_3\, 0_1\, 5_3\rangle$, $\langle 5_3\, 0_1\, 4_3\rangle$, $\langle 4_3\, 0_1\, 2_3\rangle$, $\langle 2_1\, 1_1\, 5_2\rangle$, $\langle 5_2\, 1_1\, 4_3\rangle$, $\langle 4_3\, 1_1\, 1_3\rangle$, $\langle 1_3\, 1_1\, 2_1\rangle$, $\langle 2_0\, 0_2\, 5_1\rangle$, $\langle 5_1\, 0_2\, 1_1\rangle$, $\langle 1_1\, 0_2\, 0_3\rangle$, $\langle 0_3\, 0_2\, 6_0\rangle$, $\langle 6_0\, 0_2\, 0_4\rangle$, $\langle 0_4\, 0_2\, 2_2\rangle$, $\langle 2_2\, 0_2\, 1_4\rangle$, $\langle 1_4\, 0_2\, 2_3\rangle$, $\langle 2_3\, 0_2\, 6_2\rangle$, $\langle 6_2\, 0_2\, 4_4\rangle$, $\langle 4_4\, 0_2\, 2_0\rangle$, $\langle 3_0\, 0_3\, 5_4\rangle$, $\langle 5_4\, 0_3\, 6_2\rangle$, $\langle 6_2\, 0_3\, 2_2\rangle$, $\langle 2_2\, 0_3\, 4_4\rangle$, $\langle 4_4\, 0_3\, 3_0\rangle$, $\langle 3_0\, 0_4\, 4_3\rangle$, $\langle 4_3\, 0_4\, 5_3\rangle$, $\langle 5_3\, 0_4\, 3_0\rangle$, $\langle 0_0\, \infty\, 2_3\rangle$, $\langle 2_3\, \infty\, 5_2\rangle$, $\langle 5_2\, \infty\, 3_0\rangle$.

Example A.12. Antiflexible LDTS(40).
$V = \mathbb{Z}_{20} \times \mathbb{Z}_2$.
The system is defined by the triples obtained from the following starter blocks under the action of the mappings $i_j \mapsto (i+1)_j$ and $i_j \mapsto i_{j+1}$.
$\langle 1_0\, 0_0\, 5_0\rangle$, $\langle 5_0\, 0_0\, 1_1\rangle$, $\langle 1_1\, 0_0\, 11_0\rangle$, $\langle 11_0\, 0_0\, 18_1\rangle$, $\langle 18_1\, 0_0\, 8_1\rangle$, $\langle 8_1\, 0_0\, 12_0\rangle$, $\langle 12_0\, 0_0\, 3_1\rangle$, $\langle 3_1\, 0_0\, 5_1\rangle$, $\langle 5_1\, 0_0\, 14_0\rangle$, $\langle 14_0\, 0_0\, 14_1\rangle$, $\langle 14_1\, 0_0\, 7_0\rangle$, $\langle 7_0\, 0_0\, 3_0\rangle$, $\langle 3_0\, 0_0\, 1_0\rangle$.

Example A.13. Antiflexible LDTS(42).
$V = \mathbb{Z}_7 \times \mathbb{Z}_6$.
The system is defined by the triples obtained from the following starter blocks under the action of the mappings $i_j \mapsto (i+1)_j$.
$\langle 0_1\, 0_5\, 1_5\rangle$, $\langle 1_5\, 0_5\, 5_1\rangle$, $\langle 5_1\, 0_5\, 2_5\rangle$, $\langle 2_5\, 0_5\, 1_1\rangle$, $\langle 1_1\, 0_5\, 4_5\rangle$, $\langle 4_5\, 0_5\, 0_1\rangle$, $\langle 2_0\, 0_0\, 2_1\rangle$, $\langle 2_1\, 0_0\, 6_1\rangle$, $\langle 6_1\, 0_0\, 6_0\rangle$, $\langle 6_0\, 0_0\, 2_0\rangle$, $\langle 2_0\, 0_1\, 6_0\rangle$, $\langle 6_0\, 0_1\, 1_2\rangle$, $\langle 1_2\, 0_1\, 5_2\rangle$, $\langle 5_2\, 0_1\, 5_1\rangle$, $\langle 5_1\, 0_1\, 2_0\rangle$, $\langle 5_0\, 0_4\, 1_5\rangle$, $\langle 1_5\, 0_4\, 4_3\rangle$, $\langle 4_3\, 0_4\, 5_3\rangle$, $\langle 5_3\, 0_4\, 4_2\rangle$, $\langle 4_2\, 0_4\, 5_0\rangle$, $\langle 3_0\, 0_5\, 2_4\rangle$, $\langle 2_4\, 0_5\, 4_3\rangle$, $\langle 4_3\, 0_5\, 5_2\rangle$, $\langle 5_2\, 0_5\, 3_0\rangle$, $\langle 4_1\, 0_0\, 4_2\rangle$, $\langle 4_2\, 0_0\, 5_3\rangle$, $\langle 5_3\, 0_0\, 3_4\rangle$, $\langle 3_4\, 0_0\, 0_4\rangle$, $\langle 0_4\, 0_0\, 6_3\rangle$, $\langle 6_3\, 0_0\, 0_5\rangle$, $\langle 0_5\, 0_0\, 0_3\rangle$, $\langle 0_3\, 0_0\, 2_3\rangle$, $\langle 2_3\, 0_0\, 6_5\rangle$, $\langle 6_5\, 0_0\, 4_4\rangle$, $\langle 4_4\, 0_0\, 0_2\rangle$, $\langle 0_2\, 0_0\, 4_1\rangle$, $\langle 1_2\, 5_0\, 6_3\rangle$, $\langle 6_3\, 5_0\, 6_5\rangle$, $\langle 6_5\, 5_0\, 3_4\rangle$, $\langle 3_4\, 5_0\, 0_5\rangle$, $\langle 0_5\, 5_0\, 1_2\rangle$, $\langle 1_1\, 0_1\, 4_2\rangle$,

$\langle 4_2\,0_1\,3_3\rangle$, $\langle 3_3\,0_1\,0_3\rangle$, $\langle 0_3\,0_1\,1_1\rangle$, $\langle 1_3\,0_1\,2_4\rangle$, $\langle 2_4\,0_1\,0_4\rangle$, $\langle 0_4\,0_1\,3_4\rangle$, $\langle 3_4\,0_1\,4_4\rangle$, $\langle 4_4\,0_1\,6_4\rangle$, $\langle 6_4\,0_1\,5_4\rangle$, $\langle 5_4\,0_1\,5_5\rangle$, $\langle 5_5\,0_1\,1_4\rangle$, $\langle 1_4\,0_1\,1_3\rangle$, $\langle 0_3\,2_1\,6_3\rangle$, $\langle 6_3\,2_1\,4_3\rangle$, $\langle 4_3\,2_1\,0_3\rangle$, $\langle 2_0\,0_2\,5_1\rangle$, $\langle 5_1\,0_2\,1_1\rangle$, $\langle 1_1\,0_2\,0_3\rangle$, $\langle 0_3\,0_2\,0_4\rangle$, $\langle 0_4\,0_2\,6_0\rangle$, $\langle 6_0\,0_2\,4_5\rangle$, $\langle 4_5\,0_2\,3_3\rangle$, $\langle 3_3\,0_2\,2_4\rangle$, $\langle 2_4\,0_2\,5_5\rangle$, $\langle 5_5\,0_2\,5_4\rangle$, $\langle 5_4\,0_2\,0_5\rangle$, $\langle 0_5\,0_2\,6_2\rangle$, $\langle 6_2\,0_2\,2_2\rangle$, $\langle 2_2\,0_2\,1_4\rangle$, $\langle 1_4\,0_2\,2_0\rangle$, $\langle 3_0\,0_3\,4_4\rangle$, $\langle 4_4\,0_3\,5_2\rangle$, $\langle 5_2\,0_3\,6_5\rangle$, $\langle 6_5\,0_3\,3_0\rangle$, $\langle 4_0\,0_3\,3_2\rangle$, $\langle 3_2\,0_3\,2_5\rangle$, $\langle 2_5\,0_3\,4_0\rangle$, $\langle 4_2\,0_5\,1_4\rangle$, $\langle 1_4\,0_5\,2_3\rangle$, $\langle 2_3\,0_5\,4_2\rangle$.

Example A.14. ULSOQ(17).
$Q = \mathbb{Z}_4 \times \mathbb{Z}_4 \cup \{\infty\}$.
The quasigroup is obtained by defining $\infty * x = x$ and developing the following partial Cayley table under the action of the automorphism $i_j \mapsto (i+1)_j$ with ∞ as a fixed point:

$*$	∞	0_0	1_0	2_0	3_0	0_1	1_1	2_1	3_1	0_2	1_2	2_2	3_2	0_3	1_3	2_3	3_3
0_0	1_0	∞	3_0	0_1	0_0	3_3	3_1	2_0	0_3	2_1	2_2	1_1	1_2	0_2	2_3	3_2	1_3
0_1	1_1	3_0	0_1	0_2	3_1	∞	3_3	2_3	1_2	1_3	1_0	0_3	2_2	2_1	3_2	2_0	0_0
0_2	1_2	1_1	0_2	0_3	3_2	1_0	0_1	0_0	2_0	∞	1_3	3_1	2_3	3_3	3_0	2_1	2_2
0_3	1_3	0_3	2_2	3_3	2_3	1_2	0_2	3_1	3_2	0_1	3_0	1_0	0_0	∞	2_1	1_1	2_0

Example A.15. ULSOQ(26).
$Q = \mathbb{Z}_5 \times \mathbb{Z}_5 \cup \{\infty\}$.
The quasigroup is obtained by defining $\infty * x = x$ and developing the following partial Cayley table under the action of the automorphism $i_j \mapsto (i+1)_j$ with ∞ as a fixed point:

$*$	∞	0_0	1_0	2_0	3_0	4_0	0_1	1_1	2_1	3_1	4_1	0_2	1_2	2_2	3_2	4_2	0_3	1_3	2_3	3_3	4_3	0_4	1_4	2_4	3_4	4_4
0_0	1_0	∞	0_1	4_0	3_1	0_0	1_1	4_4	3_4	3_0	2_0	2_2	2_1	3_2	1_2	4_1	1_3	0_3	4_3	0_2	4_2	2_3	2_4	1_4	0_4	3_3
0_1	1_1	3_0	0_2	4_1	2_0	0_1	∞	4_3	1_4	0_4	4_4	2_4	1_0	4_0	3_4	0_3	3_1	2_1	0_0	4_2	2_2	1_3	1_2	3_2	3_3	2_3
0_2	1_2	3_1	2_2	0_2	0_3	4_2	1_0	0_1	4_0	1_1	3_0	∞	1_3	4_3	4_4	2_3	3_3	3_4	2_4	1_4	0_4	0_0	3_2	2_1	4_1	2_0
0_3	1_3	2_2	0_3	2_3	4_3	0_4	2_1	0_2	3_2	1_2	4_2	3_4	0_1	1_0	2_4	3_3	∞	2_0	1_4	3_0	1_1	4_0	4_4	4_1	0_0	3_1
0_4	1_4	3_3	0_4	4_4	3_4	2_4	2_2	0_3	2_3	4_3	1_3	2_1	4_1	3_0	1_0	3_2	4_2	1_1	4_0	2_0	3_1	∞	0_2	0_1	1_2	0_0

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Charles University, Faculty of Mathematics and Physics, Department of Algebra, Sokolovská 83, 186 75 Praha 8, Czech Republic

E-mail: andrew.kozlik@gmail.com

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