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A proof of the independence of the Axiom of Choice from the Boolean Prime Ideal Theorem

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Abstract. We present a proof of the Boolean Prime Ideal Theorem in a transitive model of ZF in which the Axiom of Choice does not hold. We omit the argument based on the full Halpern-Läuchli partition theorem and instead we reduce the proof to its elementary case.

Keywords: Boolean Prime Ideal Theorem; the Axiom of Choice

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Let us recall the following result.

Theorem 1 (Halpern and Lévy [2]). *There is a transitive model of ZF in which the Boolean Prime Ideal Theorem holds and the Axiom of Choice fails.*

In the paper, we assume $V \models$ ZFC and we consider the following transitive model M (see [3, pp. 184–187] or [4, pp. 221–223]). Let P be the set of finite functions p such that $\mathrm{dom}(p) \subseteq \omega\times\omega$ and $\mathrm{rng}(p) \subseteq \{0,1\}$. Let $G \subseteq P$ be a generic set of conditions. For $i \in \omega$ let

$$a_i(n) = \begin{cases} 1, & \text{if } (\exists p \in G)\ p(i,n) = 1, \\ 0, & \text{otherwise,} \end{cases}$$
$$A = \{a_i : i \in \omega\},$$
$$M = \mathrm{HOD}^{V[G]}(A).$$

Then M is a transitive model of ZF and $A \in M$. The Axiom of Choice does not hold in M because the set A is infinite and has no countable subset in M (see [3]).

We prove the Boolean Prime Ideal Theorem in $M = \mathrm{HOD}^{V[G]}(A)$. The present proof uses the same ideas as the proof in [2] but its exposition relies on [3]. We also omit the argument from [2] based on the full Halpern-Läuchli partition theorem [1] and instead we reduce the proof to its elementary case substantiated in [2].

Recall that $[u] = \{x \in {}^\omega 2 : u \subseteq x\}$ for any finite function u such that $\mathrm{dom}(u) \subseteq \omega$ and $\mathrm{rng}(u) \in \{0,1\}$. For $t \in {}^m({}^\omega 2)$ and $k \in \omega$, $[t{\restriction}{\restriction}k] = \prod_{i<m}[t(i){\restriction}k]$ denotes a basic clopen set in ${}^m({}^\omega 2)$.

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Lemma 2 (Schema of continuity)**.** *Let $\varphi(x_1,\dots,x_n,s,A)$ be a formula of ZF with no free variables other than $x_1, \dots, x_n, s, A$. If $x_1,\dots,x_n \in V$, $m\in\omega$, $s\in {}^mA$ is a sequence of distinct members of A, and $\varphi(x_1,\dots,x_n,s,A)$ holds in $V[G]$, then there is a basic clopen set $U\subseteq {}^m({}^\omega 2)$ with pairwise disjoint projections in ${}^\omega 2$ such that $s\in U$ and $\varphi(x_1,\dots,x_n,t,A)$ holds in $V[G]$ for every $t\in U\cap {}^mA$.*

PROOF: Let W be the set of all one-to-one functions in ${}^m\omega$. For $h\in W$ let $h^*\in {}^mA$ be defined by $h^*(i)=a_{h(i)}$. For $h\in W$ let

$$\begin{aligned} b(h) &= \|\varphi(x_1,\dots,x_n,\dot h^*,\dot A)\|,\\ c(h) &= \textstyle\bigvee_{k\in\omega}\bigwedge_{z\in W} - \|\dot z^*\in[\dot h^*{\restriction\restriction}k]\|\vee\|\varphi(x_1,\dots,x_n,\dot z^*,\dot A)\|\\ &= \|(\exists k\in\omega)(\forall z\in W)\ \dot z^*\in[\dot h^*{\restriction\restriction}k]\to\varphi(x_1,\dots,x_n,\dot z^*,\dot A)\| \end{aligned}$$

where $\dot h^*$, $\dot z^*$, and $\dot A$ denote the canonical names for h^*, z^*, and A constructed by means of the canonical names $\dot a_i$ for $i\in\omega$. The inequality $b(h)\le c(h)$ means that if $\varphi(x_1,\dots,x_n,s,A)$ holds in $V[G]$ for $s=h^*$, then there is $k\in\omega$ such that the conclusion of the lemma holds for the clopen set $U=[s{\restriction\restriction}k]$. Then, since s is one-to-one, the projections of U are pairwise disjoint if k is sufficiently large. We prove $b(h)\le c(h)$ for all $h\in W$.

Let $p'\in P$ satisfy $p'\le b(h)$ and we find $p\le p'$ such that $p\le c(h)$. Extend p' to a condition $p\supseteq p'$ so that $\operatorname{dom}(p)=k\times k$ for some $k\in\omega$, $\operatorname{rng}(h)\subseteq k$, and for all $i<j<k$ there is $l<k$ such that $p(i,l)\ne p(j,l)$. For every $q\in P$ let q_i be defined by $q_i(j)=q(i,j)$. Then $p_i\in {}^k2$ for $i<k$ are pairwise incompatible and $p\Vdash[\dot h^*{\restriction\restriction}k]=\prod_{i<m}[p_{h(i)}]$. We prove that $p\le c(h)$.

To get a contradiction assume that for some $z\in W$ there is $r\le p$ such that $r\Vdash\dot z^*\in[\dot h^*{\restriction\restriction}k]$ and $r\Vdash\neg\varphi(x_1,\dots,x_n,\dot z^*,\dot A)$; the former assumption is equivalent to saying that $r_{z(i)}{\restriction}k=p_{h(i)}$ for all $i<m$. If $z(i)\ne h(i)$, then $z(i)>h(i)$ because p_j for $j<k$ are pairwise incompatible. Let π be the permutation of ω that interchanges $h(i)$ and $z(i)$ for all $i<m$ and $\pi(j)=j$ otherwise. The permutation π induces an automorphism of P and an automorphism of V^P, i.e., for $p,q\in P$, $q=\pi(p)$ if $q(\pi(i),j)=p(i,j)$. By the symmetry lemma $\pi(r)\Vdash\neg\varphi(x_1,\dots,x_n,\pi(\dot z^*),\pi(\dot A))$ which is impossible because $\pi(r)$ and p are compatible, $\pi(\dot z^*)=\dot h^*$, $\pi(\dot A)=\dot A$, and $p\Vdash\varphi(x_1,\dots,x_n,\dot h^*,\dot A)$. This contradiction proves that there is no such r and hence $p\le c(h)$. □

Let $F\in[A]^m$. We say that a sequence $\langle U_i : i<m\rangle$ of pairwise disjoint basic open sets in ${}^\omega 2$ distinguishes F, if $|F\cap U_i|=1$ for all $i<m$.

Corollary 3. *Let $\varphi(x_1,\dots,x_n,F)$ be a formula of ZF with no free variables other than $x_1, \dots, x_n, F$. If $s\in {}^{<\omega}A$, $x_1,\dots,x_n\in \mathrm{OD}^{V[G]}[A,s]$, $F'\subseteq A\setminus\operatorname{rng}(s)$ is a finite set, $m=|F'|$, and $\varphi(x_1,\dots,x_n,F')$ holds in $V[G]$, then there is a sequence of basic open sets $\langle U_i : i<m\rangle$ in ${}^\omega 2$ disjoint from $\operatorname{rng}(s)$ and distinguishing members of F' such that $\varphi(x_1,\dots,x_n,F)$ holds in $V[G]$ for every $F\in[A]^m$ such that $|F\cap U_i|=1$ for all $i<m$.*

PROOF: Assume $|s| = k$ and let $t' : m \to F'$ be any one-to-one enumeration. There is a formula ψ such that for some ordinals $\alpha_1, \ldots, \alpha_r$,

$$V[G] \vDash (\forall t)\ \psi(\alpha_1, \ldots, \alpha_r, s^\frown t, A) \to \varphi(x_1, \ldots, x_n, \mathrm{rng}(t)),\ \text{and}$$
$$V[G] \vDash \psi(\alpha_1, \ldots, \alpha_r, s^\frown t', A).$$

By Lemma 2 there is a disjoint sequence of basic open sets $\langle V_i : i < k+m\rangle$ in ${}^\omega 2$ such that $s^\frown t' \in \prod_{i<k+m} V_i$ and $\psi(\alpha_1, \ldots, \alpha_r, t, A)$ holds in $V[G]$ for every $t \in \prod_{i<k+m} V_i$. Take $U_i = V_{k+i}$ for $i < m$. □

Now we prove the Boolean Prime Ideal Theorem in $M = \mathrm{HOD}^{V[G]}(A)$.

Let $(B, \vee, \wedge, -, 0, 1)$ be a Boolean algebra in M. Then there is $f \in {}^{<\omega}A$ such that $B \in \mathrm{OD}^{V[G]}[A, f]$. The class $\mathrm{OD}^{V[G]}[A, f]$ has a well-ordering ordinal-definable from A and f. Using this well-ordering by transfinite recursion we can define a proper ideal $I \subseteq B$ maximal ordinal-definable from A and f. Hence, for every $x \in B$ which is ordinal-definable from A and f, either $x \in I$ or $-x \in I$. Clearly $I \in M$ because $I \subseteq B \subseteq M$. We prove that I is a prime ideal of B in M.

Suppose that I is not prime and let $k \in \omega$ be the least natural number such that for some $h' \in {}^{k+1}A$ there is an $x \in \mathrm{OD}^{V[G]}[A, f^\frown h']$ such that $x \in B \setminus I$ and $-x \in B \setminus I$. Let $a' = h'(k)$ and $h = h' \restriction k$. Then $B \in \mathrm{OD}^{V[G]}[A, f^\frown h]$ and by minimality of k it is obvious that $a' \notin \mathrm{rng}(f) \cup \mathrm{rng}(h)$ and I is a maximal ideal of B in $\mathrm{OD}^{V[G]}[A, f^\frown h]$ because I is a prime ideal there. There is a formula φ such that

$$x = \{u \in V[G] : V[G] \vDash \varphi(u, \alpha_1, \ldots, \alpha_n, f^\frown h, a', A)\}$$

for some ordinals $\alpha_1, \ldots, \alpha_n$. Since $f^\frown h$, $\alpha_1, \ldots, \alpha_n$ are fixed throughout the proof we shall denote

$$d(a) = \{u \in V[G] : V[G] \vDash \varphi(u, \alpha_1, \ldots, \alpha_n, f^\frown h, a, A)\}.$$

Hence $d(a') \in B \setminus I$ and $-d(a') \in B \setminus I$. By Corollary 3 there is a basic open set $U \subseteq {}^\omega 2$ such that $a' \in U$, $U \cap \mathrm{rng}(f^\frown h) = \emptyset$, and

$$(\forall a \in U \cap A)\ -d(a) \in B \setminus I \ \text{ and }\ d(a) \in B \setminus I. \tag{1}$$

The ideal of B generated by $I \cup \{d(a) : a \in U \cap A\}$ is in $\mathrm{OD}^{V[G]}[A, f^\frown h]$ and it coincides with B by maximality of I. Therefore for some finite set $F_1' \subseteq U \cap A$ we have $\bigwedge_{a \in F_1'} -d(a) \in I$. Similarly, if we consider the ideal generated by $I \cup \{-d(a) : a \in U \cap A\}$ we obtain a finite set $F_2' \subseteq U \cap A$ such that $\bigwedge_{a \in F_2'} d(a) \in I$. Denote $F' = F_1' \cup F_2'$ and $m = |F'|$. Then

$$\bigwedge_{a \in F'} -d(a) \in I \quad \text{and} \quad \bigwedge_{a \in F'} d(a) \in I.$$

By Corollary 3, there is a sequence of basic open sets $\langle U_i : i < m\rangle$ distinguishing F', such that each set U_i is a subset of U (this is possible because $F' \subseteq U$), hence disjoint from $\operatorname{rng}(f^\frown h)$, and for every $F \in [A]^m$ such that $(\forall i < m)$ $F \cap U_i \neq \emptyset$,

$$\bigwedge_{a\in F} -d(a) \in I \quad \text{and} \quad \bigwedge_{a\in F} d(a) \in I. \tag{2}$$

For every $i < m$, (1) holds with U replaced with U_i because $U_i \subseteq U$. Replacing U with U_i in the argument that leads to (2) we obtain a sequence of pairwise disjoint basic open sets $\langle U_{i,j} : j < m_i\rangle$ which are subsets of U_i such that for every $i < m$, and for every $F \subseteq A \cap U$ with $(\forall j < m_i)$ $F \cap U_{i,j} \neq \emptyset$, we have

$$\bigwedge_{a\in F} -d(a) \in I \quad \text{and} \quad \bigwedge_{a\in F} d(a) \in I. \tag{3}$$

The system $S = \{U_{i,j} : i < m \text{ and } j < m_i\}$ is a pairwise disjoint system of basic clopen sets in ${}^\omega 2$ and A is a dense subset of ${}^\omega 2$. Let $y \subseteq A \cap U$ be a finite set of the size $|S|$ such that $(\forall V \in S)$ $|y \cap V| = 1$. Then for every $z \subseteq y$,

$$\bigwedge_{a\in z} d(a) \wedge \bigwedge_{a\in y\setminus z} -d(a) \in I. \tag{4}$$

To prove this let us consider these two possibilities.

(i) For every $i < m$, $z \cap U_i \neq \emptyset$. Then by (2), $\bigwedge_{a\in z} d(a) \in I$ and hence (4) holds.

(ii) There is $i < m$ such that $z \cap U_i = \emptyset$. Then $(\forall j < m_i)$ $(y \setminus z) \cap U_{i,j} \neq \emptyset$, and by (3), $\bigwedge_{a\in y\setminus z} -d(a) \in I$, and hence (4) holds.

Using (4) we obtain a contradiction as follows: $1 = \bigwedge_{a\in y}(d(a) \vee -d(a)) = \bigvee_{z\subseteq y}[\bigwedge_{a\in z} d(a) \wedge \bigwedge_{a\in y\setminus z} -d(a)] \in I$. This contradiction proves that I is prime in M.

References

[1] Halpern J.D., Läuchli H., *A partition theorem*, Trans. Amer. Math. Soc. **124** (1966), 360–367.

[2] Halpern J.D., Lévy A., *The Boolean Prime Ideal Theorem does not imply the Axiom of Choice*, In: Axiomatic Set Theory, Proceedings of Symposia in Pure Mathematics, vol. XIII, Part I, pp. 83–134, AMS, Providence, 1971.

[3] Jech T., *Set Theory*, Academic Press, New York-London, 1978.

[4] Jech T., *Set Theory*, the third millennium edition, revised and expanded, Springer Monographs in Mathematics, Springer, Berlin, 2003.

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