On n-thin dense sets in powers of topological spaces

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Abstract. A subset of a product of topological spaces is called n-thin if every its two distinct points differ in at least n coordinates. We generalize a construction of Gruenhage, Natkaniec, and Piotrowski, and obtain, under CH, a countable T_3 space X without isolated points such that X^n contains an n-thin dense subset, but X^{n+1} does not contain any n-thin dense subset. We also observe that part of the construction can be carried out under MA.

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1. Introduction

We start by summarizing the definitions of thin-type subsets of products of topological spaces.

Definition 1.1. Let D be a subset of a product topological space $\prod_{\alpha\in A} X_\alpha$. We say that the set D is

- *thin* if $\forall x \neq y \in D\colon |\{\alpha \in A : x_\alpha \neq y_\alpha\}| \geq 2$, i.e. if every two distinct points of D differ in at least two coordinates (of course they differ in at least one coordinate);
- *very thin* if $(\forall x \neq y \in D)(\forall \alpha \in A)\colon x_\alpha \neq y_\alpha$, i.e. if every two distinct points of D differ in all coordinates;
- *κ-thin* if $\forall x \neq y \in D\colon |\{\alpha \in A : x_\alpha \neq y_\alpha\}| \geq \kappa$, i.e. if every two distinct points of D differ in at least κ coordinates;
- *$<\kappa$-thin* if $\forall x \neq y \in D\colon |\{\alpha \in A : x_\alpha = y_\alpha\}| < \kappa$, i.e. if every two distinct points of D agree in less than κ coordinates;
- *almost very thin* if $\forall x \neq y \in D\colon |\{\alpha \in A : x_\alpha = y_\alpha\}| < \omega$, i.e. if every two distinct points of D differ in all but finitely many coordinates.

The notions of thin and very thin sets were introduced in [Pi]. However, there are intermediate conditions between these two extreme ones. We can either demand that every two distinct points differ in a large set of coordinates or that every two distinct points agree only in a small set of coordinates. These two kinds

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of conditions are different in infinite products and lead to the notions of κ-thin and $<\kappa$-thin sets. The notion of $<\kappa$-thin sets was defined in [HG, 4.1].

The (non)strictness of the defining inequalities is justified since the condition "every two distinct points differ in more than κ coordinates" is equivalent to being κ^+-thin, and the condition "every two distinct points agree in at most κ coordinates" is equivalent to being $<\kappa^+$-thin.

Remark 1.2. Note that the thin-type notions depend on a fixed product structure. Even though the spaces $(X \times Y) \times (X \times Y) \times (X \times Y)$ and $X \times Y \times X \times Y \times X \times Y$ are homeomorphic, the elements of the first one are triples of pairs, whereas the elements of the second one are sextuples, which is an essential difference when considering thin-type properties of a set.

Observation 1.3. *Let us observe the basic relations between the various thin-type notions.*

- *Clearly, every set is* 0*-thin, and every set in a nonempty product is* 1*-thin. The notion of thin set is the same as the notion of* 2*-thin set, which is the first nontrivial case.*
- *In a given product $\prod_{\alpha \in A} X_\alpha$ we consider κ-thin subsets only for $\kappa \leq |A|$. Every very thin set is κ-thin for every κ considered.*
- *For $\lambda \leq \kappa$, κ-thinness implies λ-thinness, but $<\lambda$-thinness implies $<\kappa$-thinness.*
- *Very thin sets are the same as <1-thin sets.*
- *Almost very thin sets are the same as $<\omega$-thin sets.*
- *Every subset of a finite product is almost very thin, whereas in an infinite product the notion is stronger than any κ-thinness considered.*
- *For the smallest nontrivial product, product of two spaces, the strongest notion of very thinness coincides with the weakest notion of thinness.*

Observation 1.4.

- *If D is a κ-thin subset of $\prod_{\alpha \in A} X_\alpha$ and $D' \subseteq D$, then D' is also κ-thin.*
- *If D is a $<\kappa$-thin subset of $\prod_{\alpha \in A} X_\alpha$ and $D' \subseteq D$, then D' is also $<\kappa$-thin.*

Hence, systems of all thin-type subsets of a product are closed under subsets.

As we can see, thin-type sets are small in a certain way. On the other hand, dense sets are large. We focus on subsets of a product which are both thin and dense. We include a basic example of a very thin dense set.

Example 1.5. Let $\{Q_k : k \in \omega\}$ be a collection of pairwise disjoint countable dense subsets of $\mathbb{R}$. Let $\{B_k : k \in \omega\}$ be a countable open base of $\mathbb{R}^n$. If we choose $x_k \in (Q_k)^n \cap B_k$ for every $k \in \omega$, then $D := \{x_k : k \in \omega\}$ is a very thin dense subset of $\mathbb{R}^n$.

Proposition 1.6. *Let X be an at least two point T_1 space with an isolated point. Then X^n does not contain a thin dense subset for any $n \in \omega$, $n \geq 1$, and X^κ does not contain a very thin dense subset for any $\kappa \geq \omega$.*

PROOF: Let $0 \in X$ denote an isolated point. Let D be a dense subset of X^κ and $\omega \ni n < \kappa$. Consider $U := \{x \in X^\kappa : \forall \alpha < n\ x(\alpha) = 0\}$. The set U is open and contains at least two points. If $x \in D \cap U$, then $U \setminus \{x\}$ is open and nonempty and there is some $y \in D \cap (U \setminus \{x\})$. Hence, D contains points $x \neq y$ such that $x(\alpha) = y(\alpha) = 0$ for every $\alpha < n$, and so D is not very thin and not even thin if $\kappa < \omega$. □

[GNP] contains several sufficient conditions and necessary conditions for existence of a very thin or thin dense subset. By [GNP, Theorem 2.4], if X is a topological space and $\kappa \geq \omega$, then X^κ contains a κ-thin dense subset. On the other hand, an isolated point is an obstacle for existence of a thin dense subset of a finite power by the previous proposition. In the next section we take a look at finite powers of spaces without isolated points.

2. The construction

[GNP, Example 2.6] provides under the continuum hypothesis a construction of a countable T_3 space X without isolated points such that X^2 contains a thin dense subset, but X^3 does not contain any such subset. In this section we generalize the construction in the following way: for every natural number n there is a countable T_3 space X without isolated points such that X^n contains an n-thin dense subset, but X^{n+1} does not contain any n-thin dense subset. In other words, X^n has a dense set in which every two points differ in every coordinate, but every dense set in X^{n+1} has a pair of points which agree in at least two coordinates.

We also assume the continuum hypothesis, but part of the construction can be carried out under Martin's axiom. In particular, for X^{n+1} not having any $(n+1)$-thin dense subset rather than n-thin dense subset, Martin's axiom is sufficient.

We start with a construction of topological spaces using independent families of subbasic clopen sets.

Definition 2.1. Let X be a set, $\{T_\alpha : \alpha < \kappa\}$ a family of subsets of X. For $\alpha \leq \kappa$ we define $\Sigma_\alpha := \{\sigma : \sigma \text{ a function to } \{0,1\}, \operatorname{dom}(\sigma) \text{ a finite subset of } \alpha\}$. For $\sigma \in \Sigma_\alpha$ we put $[\sigma] := \bigcap_{\alpha \in \operatorname{dom}(\sigma)} T_\alpha^{\sigma(\alpha)}$, where $T_\alpha^0 := T_\alpha$ and $T_\alpha^1 := X \setminus T_\alpha$. We also define $[\Sigma_\alpha] := \{[\sigma] : \sigma \in \Sigma_\alpha\}$.

For every $\alpha \leq \kappa$, the family $[\Sigma_\alpha]$ is closed under finite intersections (except for the case when the intersection is empty) and covers the set X. Hence, it forms a base of a topology. The *topology induced by the family of subbasic clopen sets* $\{T_\beta : \beta < \alpha\}$ is the topology on X with the base $[\Sigma_\alpha]$. The members of $[\Sigma_\alpha]$ are called *basic clopen sets*.

From now on, X_α denotes the set X endowed with the topology induced by the family of subbasic clopen sets $\{T_\beta : \beta < \alpha\}$.

We say that $\{T_\alpha : \alpha < \kappa\}$ is an *independent family* (*on* X) if the set $[\sigma]$ is infinite for every $\sigma \in \Sigma_\kappa$.

The following proposition summarizes some properties of the well-ordered system of topologies introduced by Definition 2.1.

Proposition 2.2. *Let $\{T_\alpha : \alpha < \kappa\}$ be an independent family on X.*

(i) *The topologies of the spaces X_α are getting finer as α increases. That is, if U is open in X_α, then it is open also in X_β for every β such that $\alpha \leq \beta \leq \kappa$.*

(ii) *All the spaces X_α have no isolated points.*

(iii) *All the spaces X_α are regular.*

(iv) *If any space X_α is T_0, then all the spaces X_β for $\beta \geq \alpha$ are T_3.*

(v) *If α is a limit ordinal and D is a dense subset of X_β for every $\beta \in B$ where B is a cofinal subset of α, then D is also dense in X_α.*

(vi) *If D is open dense in X_α, then it is open dense in X_β for every $\beta \geq \alpha$.*

(vii) *If N is nowhere dense in X_α, then it is nowhere dense in X_β for every $\beta \geq \alpha$.*

(viii) *All the previous propositions hold also for all powers $\{(X_\alpha)^\lambda : \alpha < \kappa\}$. That is, the propositions with $(X_\alpha)^\lambda$, $(X_\beta)^\lambda$ substituted for X_α, X_β, respectively, hold.*

Proof: (i) This follows clearly from the definition of the spaces X_α.

(ii) Since every family $\{T_\beta : \beta < \alpha\}$ is independent, every space X_α has a base consisting of infinite sets.

(iii) All the spaces X_α have a base consisting of clopen sets by the definition.

(iv) From the previous claims it follows that all the spaces $X_{\geq\alpha}$ are T_0 and regular, and hence they are Hausdorff and T_3.

(v) Since B is cofinal in α, it follows that $\Sigma_\alpha = \bigcup_{\beta\in B} \Sigma_\beta$. Hence, every basic clopen subset of X_α is a basic clopen subset of X_β for some $\beta \in B$ and so has nonempty intersection with D.

(vi) Let $\sigma \in \Sigma_{\alpha+1}$ and $\sigma' := \sigma \restriction \alpha$. Since D is open dense in X_α, there is $\sigma'' \in \Sigma_\alpha$, $\sigma'' \supseteq \sigma'$ such that $[\sigma''] \subseteq D \cap [\sigma']$. Hence, $\emptyset \neq [\sigma'' \cup \sigma] \subseteq D \cap [\sigma]$. That proves the induction step for successor ordinals. The limit steps are handled by the previous claim.

(vii) If N is nowhere dense in X_α and $\beta \geq \alpha$, then $X \setminus \mathrm{cl}_{X_\alpha}(N)$ is open dense in X_α and by the previous claim it is also open dense in X_β. Since the space X_β has a finer topology than X_α, it follows that $X \setminus \mathrm{cl}_{X_\beta}(N)$ is also open dense, which is equivalent to N being nowhere dense in X_β.

(viii) We proceed similarly as above. We use the standard product base, whose members are of form $\bigcap_{i\in F} \pi_i^{-1}[[\sigma_i]]$ for a finite set $F \subset \lambda$. When proving the power variant of the claim (iii) we also use the fact that a product of regular spaces is regular (see [En, 2.3.11, p. 80]). □

Definition 2.3. Until the end of the section we will use the following notation.

- We fix a natural number $n \geq 1$.
- The universe X of our topological space is ω.
- We define $D := \{\langle kn + i : i < n\rangle : k \in \omega\} \subseteq X^n$. This will be our n-thin dense subset of X^n.

Lemma 2.4. *There exists an independent family $\{T_\alpha : \alpha < \omega\}$ on X such that the space X_ω is T_3 and D is dense in $(X_\omega)^n$.*

Proof: We start with any independent family $\{T_\alpha : \alpha < \omega\}$ on X. It is a standard fact that there is even an independent family of size $\mathfrak{c}$ on ω (for example [Je, Lemma 7.7]). We may also assume that our family separates points, i.e. for each $x \neq y \in X$ there is $\alpha < \omega$ such that $|\{x, y\} \cap T_\alpha| = 1$. This is possible since any countable independent family can be extended to an independent family separating given pair, and there are only countably many pairs. So X_ω is T_3 by the previous proposition.

It is enough to show that there is a dense subset $\{x_k : k < \omega\} \subseteq (X_\omega)^n$ such that $f : \omega \times n \to \omega$ defined as $f(k, i) := x_k(i)$ is a bijection. Then we can enumerate ω in a way that our dense set becomes D. Let $\{\prod_{i<n} B_{k,i} : k < \omega\} \not\ni \emptyset$ be an open base of $(X_\omega)^n$. We define $x_k(i) := \min(B_{k,i} \setminus \{x_l(j) : \langle l, j\rangle <_{\text{lex}} \langle k, i\rangle\})$, that is the minimal element of $B_{k,i}$ not chosen so far. Clearly, the set $\{x_k : k < \omega\}$ is dense and f is injective. It is surjective as well since each number is in infinitely many sets $B_{k,i}$. □

Remark 2.5. The space X_ω from the previous lemma is T_3 and second countable, hence metrizable by Urysohn's metrization theorem. It is also countable without isolated points, so it is homeomorphic to $\mathbb{Q}$ by [En, 6.2.A (d), p. 370]. In the previous lemma we have just enumerated the rational numbers so that D becomes a dense set.

Definition 2.6. Let $\{T_\beta : \beta < \alpha\}$ be an independent family on X. We define

$$\mathcal{C} := \{C = \langle f_C, g_C\rangle : f : n \to \Sigma_\alpha,\ g : n \to \{0, 1\}\}.$$

The family $\mathcal{C}$ represents a collection of conditions, meaning of which is made clear in the following lemma.

Lemma 2.7. *Let $\{T_\beta : \beta < \alpha\}$ be an independent family on X, $\alpha \geq \omega$, $T_\alpha \subseteq X$. If*

$$\forall C \in \mathcal{C} : D \cap \textstyle\prod_{i<n}([f_C(i)] \cap T_\alpha^{g_C(i)}) \neq \emptyset,$$

where $T_\alpha^0 := T_\alpha$, $T_\alpha^1 := X \setminus T_\alpha$, then $\{T_\beta : \beta < \alpha + 1\}$ is an independent family and D is dense in $(X_{\alpha+1})^n$.

Proof: We can see that the sets $\prod_{i<n}[f_C(i)]$, $C \in \mathcal{C}$, form a base of $(X_\alpha)^n$, and the sets $\prod_{i<n}[f_C(i)] \cap T_\alpha^{g_C(i)}$, $C \in \mathcal{C}$, form a base of $(X_{\alpha+1})^n$. Hence, D is clearly dense in $(X_{\alpha+1})^n$. Also, the hypothesis implies that the sets $[f_C(i)] \cap T_\alpha$ and $[f_C(i)] \cap (X \setminus T_\alpha)$, $C \in \mathcal{C}$, are nonempty. Hence, for every $\sigma \in \Sigma_{\alpha+1}$ we have $[\sigma] \neq \emptyset$, which is enough for an infinite family to be independent. □

Definition 2.8. Let $x \in X^m$, $1 \leq m < \omega$, t is an injective mapping from a nonempty subset of n to m. We say that the *point x is of type t*, if its coordinates are distinct and there exists $k_x \in \omega$ such that

- $x_{t(i)} = k_x n + i$ for $i \in \text{dom}(t)$,

- $x_j < k_x n$ for $j \notin \operatorname{rng}(t)$.

We also say that a *set* $E \subseteq X^m$ *is of type* t if all its elements are of type t.

We can see that for every point $x \in X^m$ with distinct coordinates there exists unique t such that x is of type t. It is enough to choose k_x such that the coordinate of x with maximum value is of form $k_x n + i$ for some $i < n$, and then define t accordingly.

The following observation will be used later in proofs.

Observation 2.9. *Let* $k \in \omega$. *Let* $A, B \subseteq X$ *such that* $A \subseteq \{kn + i : i < n\}$ *and* $\max B < kn$. *Let* $e \in (A \cup B)^m \setminus B^m \subseteq X^m$ *be of type* t. *Then* $k_e = k$, *i.e.* $e_{t(i)} = kn + i$ *for* $i \in \operatorname{dom}(t)$ *and* $e_j < kn$ *for* $j \notin \operatorname{rng}(t)$. *And also*

$$I_{high} := \{i < m : e_i \in A\} = \operatorname{rng}(t),$$
$$I_{low} := \{i < m : e_i \in B\} = m \setminus \operatorname{rng}(t).$$

The following lemma allows us to extend an independent family on X while preserving the density of D, but preventing a given thin set E from being dense.

Lemma 2.10. *Let* $\{T_\beta : \beta < \alpha\}$ *be an independent family on* X *such that* D *is dense in* $(X_\alpha)^n$, $\omega \leq \alpha < \omega_1$. *Then there exists* $T_\alpha \subseteq X$ *such that* $\{T_\beta : \beta < \alpha{+}1\}$ *is an independent family and* D *is dense in* $(X_{\alpha+1})^n$. *Moreover, the following holds.*

(i) *If* $E \subseteq X^m$, $1 \leq m < \omega$, *is an* l*-thin set of type* t *for some* $l > |\operatorname{dom}(t)|$, *then we can arrange that* $E \cap (T_\alpha)^m = \emptyset$.

(ii) *If* $E \subseteq X^m$, $n < m < 2n$, *is an* n*-thin set of type* t *with* $\operatorname{dom}(t) = n$, *then we can arrange that* $E \cap (T_\alpha)^m$ *is nowhere dense in* $(X_{\alpha+1})^m$.

PROOF: We have $|\mathcal{C}| = |\Sigma_\alpha| = |\alpha| = \omega$. Hence, we can enumerate $\mathcal{C}$ as $\{\langle f_j, g_j\rangle : j < \omega\}$. We inductively define numbers $k_j \in \omega$ such that pairs of disjoint finite sets $A_j = \langle A_{j,0}, A_{j,1}\rangle$, and sets F_j and B_j satisfy

$$F_j := \textstyle\bigcup_{i<j} A_{i,0},$$
$$B_j := \{k \in \omega : k \leq \max(\textstyle\bigcup_{i<j} A_{i,0} \cup A_{i,1})\},$$
$$\langle k_j n + i : i < n\rangle \in D \cap \left(\textstyle\prod_{i<n} [f_j(i)]\right) \setminus (B_j)^n,$$
$$A_{j,0} := \{k_j n + i : g_j(i) = 0\}, \qquad A_{j,1} := \{k_j n + i : g_j(i) = 1\}.$$

Note that the intersections $D \cap \prod_{i<n} [f_j(i)]$ are infinite while the sets B_j are finite, so it is always possible to choose some k_j. If a set E is given, we choose such numbers k_j that $E \cap (F_{j+1})^m = E \cap (F_j)^m$ whenever it is possible. Finally, we define $T_\alpha := \bigcup_{j<\omega} A_{j,0} = \bigcup_{j<\omega} F_j$.

We have $D \cap \prod_{i<n}([f_j(i)] \cap A_{j,g_j(i)}) \neq \emptyset$ and $X \setminus T_\alpha \supseteq \bigcup_{j<\omega} A_{j,1}$. Hence, the family $\{T_\beta : \beta < \alpha + 1\}$ is independent and D is dense in $(X_{\alpha+1})^n$ by Lemma 2.7.

Case (i). Since $F_0 = \emptyset$, it is enough to show that we can always arrange $E \cap (F_{j+1})^m = E \cap (F_j)^m$. Consider a potential point $e \in E \cap F_{j+1}$. By Observation 2.9 applied to $(k_j,\, A_{j,0},\, F_j,\, e)$ we have $k_e = k_j$ and

$$\begin{aligned} I_{\text{high}} &:= \{i < m : e_i \in A_{j,0} = F_{j+1} \setminus F_j\} = \operatorname{rng}(t), \\ I_{\text{low}} &:= \{i < m : e_i \in F_j\} = m \setminus \operatorname{rng}(t). \end{aligned}$$

For any $e', e'' \in E$ such that $e'_i = e''_i$ for every $i \in I_{\text{low}}$, it holds that $e' = e''$, since E is l-thin and $|I_{\text{high}}| = |\operatorname{dom}(t)| < l$. Therefore, the point e is uniquely determined by its coordinates indexed by I_{low} whose values lie in the finite set F_j. Hence, there are only finitely many such points e and corresponding numbers $k_e \in \omega$. We can omit these when choosing k_j.

Case (ii). We will show by contradiction that $E \cap (T_\alpha)^m$ is nowhere dense in $(X_{\alpha+1})^m$. Otherwise, there exist sets C_i for $i < m$ that are basic clopen in $X_{\alpha+1}$ and such that $E \cap (T_\alpha)^m \cap \prod_{i<m} C_i$ is dense in $\prod_{i<m} C_i$. We may suppose that the sets C_i are of form $[\sigma_i] \cap T_\alpha$ where $\sigma_i \in \Sigma_\alpha$. So $C_i \subseteq T_\alpha$ and $E \cap \prod_{i<m} C_i$ is dense in $\prod_{i<m} C_i$.

We choose a point $e \in E \cap \prod_{i<m} C_i$. We have

$$e \in (T_\alpha)^m = (\textstyle\bigcup_{j<\omega} F_j)^m = \bigcup_{j<\omega} (F_j)^m = \bigcup_{j<\omega} ((F_{j+1})^m \setminus (F_j)^m),$$

hence there is some j such that $e \in (F_{j+1})^m \setminus (F_j)^m$. By Observation 2.9 applied to $(k_j,\, A_{j,0},\, F_j,\, e)$ it holds that $k_e = k_j$, i.e. $e_{t(i)} = k_j n + i \in A_{j,0} \cap [f_j(i)]$ for $i \in \operatorname{dom}(t) = n$. Thus, $g_j(i) = 0$ for every $i < n$. Define

$$\begin{aligned} U_{t(i)} &:= [f_j(i)], && i \in \operatorname{dom}(t), \\ U_i &:= X, && i \notin \operatorname{rng}(t), \end{aligned}$$

and put $V_i := C_i \cap U_i$ for $i < m$. Since $e \in \prod_{i<m} V_i$, the sets V_i are nonempty, and hence they are basic clopen in $X_{\alpha+1}$ and infinite. Therefore, the sets $V_i \setminus k_j n$ are nonempty and open in C_i, respectively, and the set $\prod_{i<m}(V_i \setminus k_j n)$ is nonempty and open in $\prod_{i<m} C_i$. It contains a point $e' \in E$, because of the density.

It holds that $e'_i \geq k_j n$ for $i < m$. Also, $k_{e'} n + i = e'_{t(i)} \in U_{t(i)} = [f_j(i)]$ for $i \in \operatorname{dom}(t) = n$. Hence, $k_{e'}$ is another candidate when choosing k_j in the j-th step of the induction. If F'_{j+1} denotes the corresponding alternative to F_{j+1}, then the equality $E \cap (F'_{j+1})^m = E \cap (F_j)^m$ cannot hold. Otherwise, we could not have chosen the original k_j, which does not satisfy the condition. Hence, there is a point $e'' \in E \cap (F'_{j+1})^m \setminus (F_j)^m$. By Observation 2.9 applied to $(k_{e'},\, F'_{j+1} \setminus F_j = \{k_{e'} n + i : i < n\},\, F_j,\, e'')$ we have $k_{e''} = k_{e'}$, i.e. $e''_{t(i)} = k_{e'} n + i = e'_{t(i)}$. Since $m > n$, there exists $i \in m \setminus \operatorname{rng}(t)$. For this i, we have $e'_i \geq k_j n$, but $e''_i \in F_j < k_j n$, hence $e' \neq e''$. Therefore, we have two distinct elements of E that agree on n coordinates. However, the total number of coordinates is $m < 2n$, and that is a contradiction with n-thinness of E. □

Finally, the proof of the main proposition follows.

Theorem 2.11 (CH). *For every natural number $n \geq 2$ there is a countable T_3 space X without isolated points such that X^n contains an n-thin dense subset, but X^{n+1} does not contain any n-thin dense subset.*

Proof: As we said before, the universe of our space will be $X = \omega$, and the set D will be dense in X^n.

Consider all n-thin sets $E \subseteq X^{n+1}$ of all types t. There are continuum many such sets, but because we assume the continuum hypothesis, we can enumerate them as $\{E_\alpha : \omega \leq \alpha < \omega_1\}$.

We will inductively construct topologies on X induced by independent families of subbasic clopen sets $\{T_\beta : \beta < \alpha\}$ for $\alpha < \omega_1$. We start with a T_3 space X_ω induced by the family $\{T_\beta : \beta < \omega\}$ from Lemma 2.4. Suppose we have an independent family $\{T_\beta : \beta < \alpha\}$ for some $\omega \leq \alpha < \omega_1$, i.e. we have the space X_α. We choose a set T_α such that the space $X_{\alpha+1}$ satisfies the following:

- $\{T_\beta : \beta < \alpha + 1\}$ is still an independent family,
- D is dense in $(X_{\alpha+1})^n$,
- $E_\alpha \cap (T_\alpha)^{n+1}$ is nowhere dense in $(X_{\alpha+1})^{n+1}$.

We use Lemma 2.10. If E_α is of a type t such that $|\operatorname{dom}(t)| < n$, we can use case (i) to obtain even such T_α that $E_\alpha \cap (T_\alpha)^{n+1} = \emptyset$. Otherwise, E_α is of a type t such that $|\operatorname{dom}(t)| \geq n$, and hence $\operatorname{dom}(t) = n$. In that case we use case (ii).

Now we show that the space X_{ω_1} has the desired properties. In particular, X_{ω_1} is a T_3 space without isolated points by Proposition 2.2, since even the space X_ω is Hausdorff. Next, D is dense in $(X_{\omega_1})^n$ because the density of D is preserved by our construction at the successor steps, and it is preserved automatically at the limit steps by Proposition 2.2. Finally, if E is any n-thin subset of X^{n+1}, we can decompose it as $E = \bigcup_{i<j<n+1} E_{ij} \cup \bigcup_{i<k} E_{\alpha_i}$, where $E_{ij} := \{e \in E : e_i = e_j\}$ for $i < j < n$ and each α_i satisfies $\omega \leq \alpha_i < \omega_1$ and $k \in \omega$. This is possible since any point $e \in E \setminus \bigcup_{i<j<n} E_{ij}$ is of some type t, there are only finitely many types, and each $E_t := \{e \in E : e \text{ is of type } t\}$ is n-thin, and hence $E_t = E_\alpha$ for some α. All the sets E_{ij} are nowhere dense. For any α_i we have that $E_{\alpha_i} \cap (T_{\alpha_i})^{n+1}$ is nowhere dense in $(X_{\alpha_i+1})^{n+1}$, and hence it is also nowhere dense in $(X_{\omega_1})^{n+1}$. Therefore, $E \cap U$ is nowhere dense in $(X_{\omega_1})^{n+1}$, where $U := (\bigcap_{i<k} T_{\alpha_i})^{n+1} \neq \emptyset$, and hence E is not dense. □

We have constructed a space that contains a very thin dense subset in any power up to X^n, but X^{n+1} does not contain any n-thin dense subset. In the case $n = 2$ we have a space X such that X^2 contains a thin dense subset, but X^3 does not contain such set. That is the original result [GNP, 2.6].

We can see that the previous proof works also for higher powers X^m. The additional assumption of Lemma 2.10(i), $|\operatorname{dom}(t)| <$ thinness of E, is satisfied for $m > n$ and E being $(n+1)$-thin. Case (ii) holds for $n < m < 2n$. Also, we can consider all the sets E_α in all powers X^m for $n < m < \omega$ together. There are still continuum many of them. In conclusion, the following strengthening of the theorem holds.

Theorem 2.12 (CH). *For every natural number $n \geq 1$ there is a countable T_3 space X without isolated points such that*

- *X^n contains an n-thin dense subset, and hence X^m, for any $m \leq n$, contains a very thin dense subset;*
- *X^m, for $n < m < 2n$, does not contain any n-thin dense subset;*
- *X^m, for $n < m < \omega$, does not contain any $(n+1)$-thin dense subset, and hence it does not contain any very thin dense subset.*

In the last part, we show that case (i) of Lemma 2.10 can be strengthened under Martin's axiom, and therefore the last theorem partially holds even under Martin's axiom.

Lemma 2.13 (MA). *Let $\{T_\beta : \beta < \alpha\}$ be an independent family on X such that D is dense in $(X_\alpha)^n$, $\omega \leq \alpha < \mathfrak{c}$. If $E \subseteq X^m$ for some m such that $1 \leq m < \omega$ is an l-thin set of type t for some $l > |\operatorname{dom}(t)|$, then there exists $T_\alpha \subseteq X$ such that $\{T_\beta : \beta < \alpha + 1\}$ is an independent family, D is dense in $(X_{\alpha+1})^n$, and $E \cap (T_\alpha)^m = \emptyset$.*

Proof: Consider the partially ordered set

$$\mathcal{A} := \{A = \langle A_0, A_1\rangle : A_0, A_1 \text{ finite disjoint subsets of } X,\ E \cap (A_0)^m = \emptyset\},$$
$$A' \leq A \ :\Longleftrightarrow\ (A'_0 \supseteq A_0) \wedge (A'_1 \supseteq A_1).$$

It holds that $|\mathcal{A}| = \omega$, and hence the ordered set $\mathcal{A}$ satisfies c. c. c. For $C \in \mathcal{C}$ we define

$$D_C := \{A \in \mathcal{A} : D \cap \textstyle\prod_{i<n}([f_C(i)] \cap A_{g_C(i)}) \neq \emptyset\}.$$

The sets D_C are dense with respect to the ordering of $\mathcal{A}$. For any $A \in \mathcal{A}$, we can choose $k \in \omega$ such that $\langle kn + i : i < n\rangle \in D \cap \prod_{i<n}[f_C(i)] \setminus (A_0 \cup A_1)^n$. This is possible since we are removing a finite set from the intersection $D \cap \prod_{i<n}[f_C(i)]$ which is infinite. If we put $A' := \langle A_0 \cup \{kn+i : g_C(i) = 0\}, A_1 \cup \{kn+i : g_C(i) = 1\}\rangle$, then $A' \leq A$. By an argument similar to that in Lemma 2.10(i), there are only finitely many numbers k such that $E \cap (A'_0)^m \neq \emptyset$ for corresponding sets A'_0. Hence, we can choose such k that $A' \in D_C$.

We have that $\mathcal{D} := \{D_C : C \in \mathcal{C}\}$ is a family of dense sets. Since $|\mathcal{D}| = |\mathcal{C}| = |\alpha| < \mathfrak{c}$, there exists a $\mathcal{D}$-generic filter F by Martin's axiom. The sets $\bigcup_{A\in F} A_0$ and $\bigcup_{A \in F} A_1$ are infinite and disjoint. If there were $A, A' \in F$ such that $x \in A_0 \cap A'_1$, there would be $A'' \in F$ such that $A'' \leq A, A'$ because F is a filter. Then, $x \in A''_0 \cap A''_1$, which is a contradiction. If we put $T_\alpha := \bigcup_{A\in F} A_0$, then $X \setminus T_\alpha \supseteq \bigcup_{A\in F} A_1$ and T_α satisfies the hypotheses of Lemma 2.7. Also, $E \cap (T_\alpha)^m = \emptyset$. Otherwise, there would be sets $A_i \in F$, $i < m$, and a point $x \in E \cap \prod_{i<m} A_{i,0}$. Hence $x \in E \cap (A'_0)^m$ for any $A' \in F$ such that $A' \leq A_i$ for any $i < m$, which is a contradiction. □

Corollary 2.14 (MA). *For every natural number $n \geq 1$ there is a countable T_3 space X without isolated points such that*

- *X^n contains an n-thin dense subset;*

- *X^m, for $n < m < \omega$, does not contain any $(n+1)$-thin dense subset.*

Proof: We proceed analogously to Theorem 2.11. For the sets E_α we take all $(n+1)$-thin subsets of all powers X^m of all possible types t. □

Example 2.15 (MA)**.** There exists a countable T_3 space X without isolated points such that X^n does not contain any thin dense subset for any natural number n.

Proof: It is a special case of the previous corollary for $n = 1$. □

Several questions arise naturally.

Question 2.16. Is it possible to prove Theorem 2.12 under Martin's axiom or even in ZFC?

Question 2.17. Does there exist a space X such that X^n contains a very thin dense subset, but X^{n+1} does not contain even a thin dense subset?

Question 2.18. Does there exist a space X such that X^n contains an l-thin dense subset for some $1 < l < n$, but X^{n+1} does not contain an l-thin dense subset?

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References

[En] Engelking R., *General Topology*, revised and completed edition, Sigma series in pure mathematics, 6, Heldermann, Berlin, 1989.

[GNP] Gruenhage G., Natkaniec T., Piotrowski Z., *On thin, very thin, and slim dense sets*, Topology Appl. **154** (2007), no. 4, 817–833.

[HG] Hutchison J., Gruenhage G., *Thin-type dense sets and related properties*, Topology Appl. **158** (2011), no. 16, 2174–2183.

[Je] Jech T., *Set Theory*, The Third Millennium Edition, revised and expanded, Springer, Berlin, 2002.

[Pi] Piotrowski Z., *Dense subsets of product spaces*, Questions Answers Gen. Topology **11** (1993), 313–320.

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