On matrix Lie rings over a commutative ring that contain the special linear Lie ring

Evgenii L. Bashkirov, Esra Pekönür

Abstract. Let K be an associative and commutative ring with 1, k a subring of K such that $1 \in k$, $n \geq 2$ an integer. The paper describes subrings of the general linear Lie ring $gl_n(K)$ that contain the Lie ring of all traceless matrices over k.

Keywords: Lie rings; commutative associative rings

Classification: 17B05

1. Introduction and the formulation of main results

This paper addresses a problem which has its origin in group theory, where the description of linear (matrix) groups lying between two given groups of matrices led to the creation of a deep and rather branched part of the theory of groups (see, for instance, [5]). During the recent decade, a number of papers have appeared in which the similar problems for Lie rings of matrices over commutative associative rings have been considered ([3], [1], [2], [4]). However, the results in all of these papers have been obtained under rather rigorous restrictions for the associative rings, namely, either matrices over the rings could be realized over no smaller ring ([4]), or the associative rings in question are fields, the larger of which being an algebraic extension of the smaller ([3], [1], [2]). The main result of the present note establishes that, at least for the Lie rings of all traceless matrices over commutative associative rings with 1, these restrictions are irrelevant. Roughly speaking, we show that each matrix Lie ring containing the Lie ring of traceless matrices over some commutative ring as a subring always contains the Lie ring of traceless matrices over some, possibly another, commutative ring as an ideal. First let us agree on notation which will be used throughout the paper and make some definitions.

All associative rings considered in the paper are assumed to be unitary and all their subrings have the same identity element 1. Let k be an associative ring. The multiplicative group of all invertible elements of k is denoted by $k^{\times}$. If $a, b \in k$, we write $[a, b]$ to denote the Lie product $ab - ba$. If $n \geq 2$ is an integer, then $M_n(k)$ is as usual the associative ring of all n by n matrices whose entries lie in k. By the standard matrix unit e_{ij} of the ring $M_n(k)$, we understand the matrix with

DOI 10.14712/1213-7243.2015.144

the (ij)-th entry equal to 1 and all other entries equal to 0. In the present paper, the (ij)-th entry of the matrix $x \in M_n(k)$ will be designated by x_{ij} so that

$$x = \sum_{1 \le i,j \le n} e_{ij} x_{ij}.$$

If $x \in M_n(k)$ and $L \subseteq k$, then xL denotes the set of all matrices xl with $l \in L$. Next assume that k is not only associative but also commutative, and let $sl_n(k)$ be the set of all matrices in $M_n(k)$ whose trace is 0. Since $sl_n(k)$ is an additive subgroup of $M_n(k)$ and is closed under Lie multiplication, it is a Lie ring, the special linear Lie ring of n by n matrices over k. The set $M_n(k)$ itself forms a Lie ring under the usual Lie multiplication. This ring is called the general linear Lie ring and will be denoted by $gl_n(k)$ in the sequel. In the present paper, we are interested in the special linear and the general linear Lie rings over various commutative associative rings. More precisely, we will prove the following.

Theorem 1. *Let K be an associative and commutative ring, k a subring of K, $n \ge 2$ an integer and let $\mathfrak{g}$ be a subring of the Lie ring $gl_n(K)$ such that $\mathfrak{g} \supseteq sl_n(k)$.*

Assume that either $n \ge 3$, or $n = 2$ and 2 is invertible in k. Then $\mathfrak{g}$ can be written as a sum

$$\mathfrak{g} = sl_n(L) + D,$$

where L is a subring of K containing k, and D is an abelian Lie ring consisting of certain diagonal matrices

$$e_{11}d_1 + e_{22}d_2 + \cdots + e_{nn}d_n$$

in $gl_n(K)$ such that $d_i - d_j \in L$ for all $i, j = 1, 2, \ldots, n$. In particular, $\mathfrak{g}$ contains $sl_n(L)$ as an ideal.

Applying Theorem 1 to the situation of Lie rings that are intermediate between two special linear Lie rings over various associative commutative rings allows us to deduce the following, to some extent, more precise assertion.

Theorem 2. *Let K be an associative and commutative ring, k a subring of K, $n \ge 2$ an integer and let $\mathfrak{g}$ be a subring of the special linear Lie ring $sl_n(K)$ such that $\mathfrak{g} \supseteq sl_n(k)$. If n is invertible in k, then $\mathfrak{g} = sl_n(L)$, where L is a subring of K containing k.*

Note that if $n \ge 3$ is not invertible in k, then — as we show in Section 3 — there exist situations when $\mathfrak{g}$ normalizes $sl_n(L)$ not coinciding with it.

2. Proofs of Theorems 1 and 2

For the sake of brevity, let us agree that in the proofs below all integers are assumed to be members of the collection $\{1, 2, \ldots, n\}$. By $|X|$ we denote the cardinality of a set X.

Proof of Theorem 1: Denote by L the set of nondiagonal elements of all matrices that belong to $\mathfrak{g}$. We shall distinguish between two cases which will be considered separately.

(i) Assume first $n \geq 3$. Let $\alpha \in L$. It is claimed that $e_{ij}\alpha \in \mathfrak{g}$ for all $i \neq j$. Indeed, the relation $\alpha \in L$ means that there exists a matrix $a \in \mathfrak{g}$ with $a_{i_0j_0} = \alpha$ for some $i_0 \neq j_0$. Due to the condition $n \geq 3$, one can find an integer s_0 which is distinct from both i_0 and j_0. Then $e_{s_0i_0}, e_{j_0s_0}$ are in $sl_n(k)$, hence in $\mathfrak{g}$, and the equation

$$[e_{s_0i_0}, [e_{j_0s_0}, [e_{j_0i_0}, a]]] = e_{j_0i_0}a_{i_0j_0} \tag{1}$$

shows that $e_{j_0i_0}\alpha \in \mathfrak{g}$.

If we take this $e_{j_0i_0}\alpha$ as a and interchange i_0 and j_0, the argument of the preceding passage allows us to conclude $e_{i_0j_0}\alpha \in \mathfrak{g}$ thus establishing the relation $e_{ij}\alpha \in \mathfrak{g}$ whenever

$$\text{(i)} \qquad |\{i,j\} \cap \{i_0,j_0\}| = 2.$$

It remains to show that $e_{ij}\alpha \in \mathfrak{g}$ when

$$\text{(ii)} \qquad |\{i,j\} \cap \{i_0,j_0\}| = 1,$$

or when

$$\text{(iii)} \qquad |\{i,j\} \cap \{i_0,j_0\}| = 0,$$

and it is easily seen that it suffices to consider case (ii) only. While considering this case, one of the following two possibilities may occur:

$$\text{(a)} \qquad \{i,j\} \cap \{i_0,j_0\} = \{i_0\},$$

$$\text{(b)} \qquad \{i,j\} \cap \{i_0,j_0\} = \{j_0\}.$$

Consider (a). If $i = i_0$, then it must be shown that $e_{i_0j}\alpha \in \mathfrak{g}$. But we have already proved that $e_{i_0j_0}\alpha \in \mathfrak{g}$. Moreover, $j \neq j_0$ according to the condition defining (a), so $\mathfrak{g} \ni [e_{i_0j_0}\alpha, e_{j_0j}] = e_{i_0j}\alpha$. Suppose next that $j = i_0$, and hence $i \neq j_0$. Now $i \neq i_0$, and it must be demonstrated that $e_{ii_0}\alpha \in \mathfrak{g}$. But as we have already established, the relation $e_{i_0j_0}\alpha \in \mathfrak{g}$ implies $e_{i_0i}\alpha \in \mathfrak{g}$, and therefore $e_{ii_0}\alpha \in \mathfrak{g}$ by case (i). Suppose further that possibility (b) takes place. Then $e_{j_0i_0}\alpha \in \mathfrak{g}$ by case (i), so the relation $e_{ij}\alpha \in \mathfrak{g}$ follows from (a).

Next it is claimed that L is a subring of K. To this end in view, we take $\alpha, \beta \in L$ and seek to show that both $\alpha + \beta, \alpha\beta$ are in L too. At this place, we again employ the assumption $n \geq 3$ to obtain by the above reasoning that all the matrices $e_{12}\alpha, e_{12}\beta, e_{23}\beta$ are in $\mathfrak{g}$. Therefore the matrices

$$e_{12}\alpha + e_{12}\beta = e_{12}(\alpha + \beta), \quad [e_{12}\alpha, e_{23}\beta] = e_{13}\alpha\beta$$

also belong to $\mathfrak{g}$, showing that $\alpha+\beta, \alpha\beta \in L$ as required. Clearly $k \subseteq L$. Note then that $e_{1j}L \subseteq \mathfrak{g}$ for $j \neq 1$, so

$$[e_{1j}L, e_{j1}] = (e_{11} - e_{jj})L \subseteq \mathfrak{g}$$

which completes proving the inclusion $sl_n(L) \subseteq \mathfrak{g}$. Furthermore, according to our previous considerations, if a is an arbitrary matrix in $\mathfrak{g}$, then $a_{ij}e_{ij} \in \mathfrak{g}$ for all $i \neq j$ implying

$$d(a) = e_{11}a_{11} + e_{22}a_{22} + \cdots + e_{nn}a_{nn} \in \mathfrak{g}.$$

Therefore,

$$[d(a), e_{1j}] = e_{1j}(a_{11} - a_{jj}) \in \mathfrak{g},$$

hence

$$a_{11} - a_{jj} \in L$$

by the definition of L. This means that the matrix $d(a)$ normalizes $sl_n(L)$, and so every matrix a of $\mathfrak{g}$ can be written as a sum of a matrix that belongs to $sl_n(L)$ and the diagonal matrix $d(a) \in gl_n(K)$ normalizing $sl_n(L)$ and lying in $\mathfrak{g}$. Letting D be the set of all $d(a)$, when a runs over $\mathfrak{g}$, we see that $\mathfrak{g}$ can be expressed as a sum of the form required, and hence $sl_n(L)$ is an ideal of $\mathfrak{g}$.

(ii) Assume $n = 2$. Equation (1) is senseless in this case, and therefore, we use the condition that $2 \in k^\times$ instead. Recall that here L is the set of all elements of K that occupy either position (12) or position (21) in some matrix of $\mathfrak{g}$.

If $a \in \mathfrak{g}$, then $\mathfrak{g}$ contains the matrix

$$a' = [a, e_{12}] = e_{12}(a_{11} - a_{22}) - (e_{11} - e_{22})a_{21},$$

whence we conclude that

$$a_{11} - a_{22} \in L. \tag{2}$$

Taking a' as a yields $-2e_{12}a_{21} \in \mathfrak{g}$, and so $e_{12}a_{21} \in \mathfrak{g}$ since 2 is invertible in k. Also $\mathfrak{g}$ contains $[e_{12}a_{21}, e_{21}] = (e_{11} - e_{22})a_{21}$ as well as $[\frac{1}{2}e_{21}, (e_{11} - e_{22})a_{21}] = e_{21}a_{21}$. Similarly,

$$e_{21}a_{12},\ (e_{11} - e_{22})a_{12},\ e_{12}a_{12} \in \mathfrak{g}.$$

Thus if α, β are arbitrary elements in L, then $e_{12}\beta, e_{21}\alpha$ are in $\mathfrak{g}$, and so $\mathfrak{g}$ contains

$$[[e_{12}\beta, e_{21}\alpha], \frac{1}{2}e_{12}] = e_{12}\alpha\beta$$

which means actually that L is a subring of K. In addition, our above argument shows that $e_{12}L, e_{21}L, (e_{11} - e_{22})L \subseteq \mathfrak{g}$ and thus $sl_2(L) \subseteq \mathfrak{g}$. Moreover, every $a \in \mathfrak{g}$ can be written as

$$a = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} + \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix},$$

where the first summand is in $sl_2(L)$ while the second, being in $\mathfrak{g}$, normalizes $sl_2(L)$ according to (2). The theorem is proved completely. $\square$

PROOF OF THEOREM 2: By Theorem 1 one can find a subring L of K containing k such that $\mathfrak{g} = sl_n(L) + D$ where D is an abelian Lie ring that consists of some diagonal matrices normalizing $sl_n(L)$. It is sufficient to show that $D \subseteq sl_n(L)$.

Let d be any matrix in D. Write $d = e_{11}d_1 + e_{22}d_2 + \cdots + e_{nn}d_n (d_i \in K)$. Then since d normalizes $sl_n(L)$, we have $d_1 - d_j \in L$ for all $j = 2, 3, \ldots, n$. Adding yields

$$\sum_{j=2}^{n}(d_1 - d_j) \in L.$$

But since $\mathfrak{g}$ is contained in $sl_n(L)$, the trace of d is 0, so we obtain that the left hand side of the last expression is

$$(n-1)d_1 - \sum_{j=2}^{n} d_j = nd_1 - \sum_{j=1}^{n} d_j = nd_1,$$

whence $nd_1 \in L$. According to our assumption, n^{-1} is defined in k, hence in L. Therefore L contains $n^{-1}(nd_1) = d_1$ which yields $d_j \in L$ for all $j = 1, 2, \ldots, n$. Thus $d \in sl_n(L)$ as we intended to prove. The theorem is proved. $\square$

3. Examples

First we give an example which shows that if $n \geq 3$ is noninvertible, under assumption of Theorem 2 there exist Lie rings of traceless matrices that contain $sl_n(L)$ as an ideal but not coincide with $sl_n(R)$ for any commutative associative ring R. Indeed, let $\mathfrak{g}$ be the Lie ring $sl_n(\mathbb{Z}) + x\mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers and

$$x = e_{11}\frac{1}{n} + e_{22}\frac{1}{n} + \cdots + e_{n-1,n-1}\frac{1}{n} - e_{nn}\frac{n-1}{n}.$$

Then x normalizes $sl_n(\mathbb{Z})$, hence $sl_n(\mathbb{Z})$ is an ideal of $\mathfrak{g}$. On the other hand, if $\mathfrak{g} = sl_n(R)$ for some ring R, then $1/n \in R$. So $1/n^2$ must be also in R and thus the matrix $e_{12}\frac{1}{n^2}$ must be in $\mathfrak{g}$ which is obviously false.

Next we show that if 2 is noninvertible in k, then the part of Theorem 1 related to Lie rings of degree 2 is not true at all. For instance, consider the Lie ring $\mathfrak{g}$ of all matrices

$$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$$

with $\alpha, \beta \in \mathbb{Z}[i], \gamma \in \mathbb{Z}[2i]$ $(i^2 = -1)$. Clearly,

$$sl_2(\mathbb{Z}) \subsetneqq \mathfrak{g} \subsetneqq sl_2(\mathbb{Z}[i]).$$

However, for any associative and commutative ring L, this $\mathfrak{g}$ does not contain $sl_2(L)$ as an ideal, since otherwise $[e_{12}i, e_{21}] = (e_{11} - e_{22})i \in sl_2(L)$, and so $i \in L$ implying $L = \mathbb{Z}[i]$ which is impossible.

References

[1] Bashkirov E.L., *Matrix Lie rings that contain a one-dimensional Lie algebra of semi-simple matrices*, J. Prime Res. Math. **3** (2007), 111–119.

[2] Bashkirov E.L., *Matrix Lie rings that contain an abelian subring*, J. Prime Res. Math. **4** (2008), 113–117.

[3] Wang D.Y., *Extensions of Lie algebras according to the extension of fields*, J. Math. Res. Exposition **25** (2005), no. 3, 543–547.

[4] Zhao Y.X., Wang D.Y., Wang Ch.H., *Intermediate Lie algebras between the symplectic algebras and the general linear Lie algebras over commutative rings*, J. Math. (Wuhan) **29** (2009), no. 3, 247-252.

[5] Vavilov N.A., *Intermediate subgroups in Chevalley groups*, Groups of Lie Type and Their Geometries (Como 1993), London Math. Soc. Lecture Note Ser., 207, Cambridge Univ. Press, Cambridge, 1995, pp. 233–280.

Evgenii L. Bashkirov:
Ton Duc Thang University, Ho Chi Minh City, Vietnam

E-mail: zh.bash@mail.ru
ebashkirov@tdt.edu.vn
bashkirov.e@gmail.com

Esra Pekönür:
Department of Mathematics, Fatih University 34 500 Buyukcekmece, Istanbul, Turkey

(*Received* December 4, 2014, *revised* March 30, 2015)