Notes on strongly Whyburn spaces

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Abstract. We introduce the notion of a strongly Whyburn space, and show that a space X is strongly Whyburn if and only if $X \times (\omega+1)$ is Whyburn. We also show that if $X \times Y$ is Whyburn for any Whyburn space Y, then X is discrete.

Keywords: Whyburn; strongly Whyburn; Fréchet-Urysohn

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1. Introduction

Throughout this paper, all spaces are assumed to be T_2, unless a specific separation axiom is indicated.

A space X is said to be *Fréchet-Urysohn* if $A \subset X$ and $p \in \overline{A}$ imply that there is a sequence $\{p_n : n \in \omega\} \subset A$ converging to p. A space X is said to be *strongly Fréchet-Urysohn* [14] (or, *countably bi-sequential* [9]) if for a decreasing sequence $\{A_n : n \in \omega\}$ of subsets of X, $p \in \bigcap\{\overline{A_n} : n \in \omega\}$ implies that there are points $p_n \in A_n$ converging to p. Every strongly Fréchet-Urysohn space is Fréchet-Urysohn. Michael [9, Proposition 4.D.5] showed that a space X is strongly Fréchet-Urysohn if and only if $X \times \mathbb{I}$ is Fréchet-Urysohn, where $\mathbb{I}$ is the closed unit interval. In this result, $\mathbb{I}$ can be replaced by the convergent sequence $\omega+1$: see the proof of [9, Proposition 4.D.5].

According to recent literature (e.g., [4], [12]), a space X is said to be *Whyburn* if $A \subset X$ and $p \in \overline{A} \setminus A$ imply that there is a subset $B \subset A$ such that $\overline{B} = \{p\} \cup B$. Every Fréchet-Urysohn space is Whyburn, because the convergent sequence is closed in a T_2-space. This notion was considered in Whyburn [16], and was called property H. Whyburn showed in [16, Corollary 1] that every quotient map onto a T_1-space Y having property H is pseudo-open (=hereditarily quotient). Later, introducing the notion of an accessibility space [17] which is weaker than property H, he sharpened this result. He showed that for a T_1-space Y, every quotient map onto Y is pseudo-open if and only if Y is an accessibility space. A space having property H is always an accessibility space, and conversely a regular accessibility space has property H. A Whyburn space is sometimes called an *AP-space* according to [13].

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Even if a space X is Whyburn, $X \times (\omega+1)$ need not be Whyburn. Such examples are given in Bella and Yaschenko [5]. Aull [3, Theorem 11] showed that a T_2-space X is a k-space and an accessibility space if and only if it is Fréchet-Urysohn.[1] Hence we have:

Proposition 1.1. *For a k-space X, $X \times (\omega+1)$ is Whyburn if and only if X is strongly Fréchet-Urysohn.*

Proof: Assume that $X \times (\omega+1)$ is Whyburn. Since $X \times (\omega+1)$ is a k-space [6, Theorem 3.3.27], by Aull's result, $X \times (\omega+1)$ is Fréchet-Urysohn. Thus X is strongly Fréchet-Urysohn by Michael's result. The converse immediately follows from Michael's result mentioned above. □

Let S_ω be the space obtained by identifying the limits of countably many convergent sequences. This space is Fréchet-Urysohn (hence, a k-space), but not strongly Fréchet-Urysohn. Therefore, $S_\omega \times (\omega+1)$ is not Whyburn by the preceding proposition. One purpose of this paper is to make clear when $X \times (\omega+1)$ is Whyburn. Another topic is when $X \times Y$ is Whyburn for any Whyburn space Y.

2. Strongly Whyburn spaces

Definition 2.1. A space X is *strongly Whyburn* if for any sequence $\{A_n : n \in \omega\}$ of subsets in X and a point $p \in X \setminus \bigcup\{A_n : n \in \omega\}$, $p \in \bigcap\{\overline{\bigcup_{m \geq n} A_m} : n \in \omega\}$ implies that there is a sequence $\{B_n : n \in \omega\}$ of closed subsets in X such that $B_n \subset A_n$ and $\{p\} = \bigcap\{\overline{\bigcup_{m \geq n} B_m} : n \in \omega\}$.

In the definition above, some B_n may be empty, and note that the condition $\{p\} = \bigcap\{\overline{\bigcup_{m \geq n} B_m} : n \in \omega\}$ holds if and only if (a) the closed family $\{B_n : n \in \omega\}$ in X is locally finite at any point in $X \setminus \{p\}$, and (b) $p \in \overline{\bigcup\{B_n : n \in \omega\}}$ holds. If all A_n's are identical with a set A, there is an F_σ-subset $F \subset A$ in X such that $\overline{F} = \{p\} \cup F$. Therefore, every strongly Whyburn space is Whyburn. Moreover, we can easily observe that every strongly Fréchet-Urysohn space is strongly Whyburn. Thus we have the implications below.

$$\begin{array}{ccc} \text{strongly Fréchet-Urysohn} & \rightarrow & \text{Fréchet-Urysohn} \\ \downarrow & & \downarrow \\ \text{strongly Whyburn} & \rightarrow & \text{Whyburn} \end{array}$$

Theorem 2.2. *For a space X, the following are equivalent:*

(1) *X is strongly Whyburn,*
(2) *$X \times (\omega+1)$ is Whyburn.*

Proof: (1)→(2) We have only to check the Whyburn property at a point $(p,\omega) \in X \times (\omega+1)$. Let $A \subset X \times (\omega+1)$ and assume $(p,\omega) \in \overline{A} \setminus A$. If $(p,\omega) \in \overline{A \cap (X \times \{\omega\})}$, using the Whyburn property of X, we can take a subset $B \subset$

[1] In particular, every compact T_2 Whyburn space is Fréchet-Urysohn. This fact was given in [1, Proposition 1 and Theorem 1] and [8, Theorem 1].

$A \cap (X \times \{\omega\})$ such that $\overline{B} = \{(p,\omega)\} \cup B$. Therefore, we may put $A = \bigcup\{A_n \times \{n\} : n \in \omega\}$ for some $A_n \subset X$. If $p \in A_n$ for infinitely many $n \in \omega$, then we can take a sequence in A converging to (p,ω). Therefore, we may assume $p \notin \bigcup\{A_n : n \in \omega\}$. The condition $(p,\omega) \in \overline{A}$ implies $p \in \bigcap\{\overline{\bigcup_{m \geq n} A_m} : n \in \omega\}$, so there are closed subsets B_n in X such that $B_n \subset A_n$ and $\{p\} = \bigcap\{\overline{\bigcup_{m \geq n} B_m} : n \in \omega\}$. Let $B = \bigcup\{B_n \times \{n\} : n \in \omega\}$. The condition $p \in \bigcap\{\overline{\bigcup_{m \geq n} B_m} : n \in \omega\}$ obviously implies $(p,\omega) \in \overline{B}$. We observe that $\{(p,\omega)\} \cup B$ is closed. Let $q \in X \setminus \{p\}$. By $\{p\} = \bigcap\{\overline{\bigcup_{m \geq n} B_m} : n \in \omega\}$, there are a neighborhood U of q and some $n \in \omega$ such that $U \cap (\bigcup\{B_m : m \geq n\}) = \emptyset$. Then we have $(U \times [n,\omega]) \cap B = \emptyset$. Thus $(q,\omega) \notin \overline{B}$.

(2)→(1) Assume that $A_n \subset X$, $p \in X \setminus \bigcup\{A_n : n \in \omega\}$ and $p \in \bigcap\{\overline{\bigcup_{m \geq n} A_m} : n \in \omega\}$. Let $A = \bigcup\{A_n \times \{n\} : n \in \omega\}$. Then obviously $(p,\omega) \in \overline{A}$. Since $X \times (\omega+1)$ is Whyburn, there is a subset $B \subset A$ such that $\overline{B} = \{(p,\omega)\} \cup B$. We can put $B = \bigcup\{B_n \times \{n\} : n \in \omega\}$ for some $B_n \subset A_n$. Then each B_n is closed in X, and the condition $(p,\omega) \in \overline{B}$ implies $p \in \bigcap\{\overline{\bigcup_{m \geq n} B_m} : n \in \omega\}$. Let $q \in X \setminus \{p\}$. By the condition $(q,\omega) \notin \overline{B}$, there are a neighborhood U of q and some $n \in \omega$ such that $(U \times [n,\omega]) \cap B = \emptyset$. Hence we have $q \notin \overline{\bigcup\{B_m : m \geq n\}}$. Consequently we have $\{p\} = \bigcap\{\overline{\bigcup_{m \geq n} B_m} : n \in \omega\}$. □

Corollary 2.3. *For a k-space X, X is strongly Whyburn if and only if it is strongly Fréchet-Urysohn.*

Unfortunately, the author does not know if for a strongly Whyburn space X, $X \times \mathbb{I}$ is Whyburn. A space X is said to have *countable fan-tightness* [2] if whenever $A_n \subset X$ and $p \in \bigcap\{\overline{A_n} : n \in \omega\}$, there are finite subsets $F_n \subset A_n$ such that $p \in \overline{\bigcup\{F_n : n \in \omega\}}$. It is known [5, Corollary 3.4] that if a regular space X has countable fan-tightness and every point of X is a G_δ-set, then X is Whyburn. Note that if a space X has countable fan-tightness, so does $X \times Y$ for any first-countable space Y. Therefore we can say that if a regular space X has countable fan-tightness and every point of X is a G_δ-set, then $X \times Y$ is Whyburn for any first-countable space Y (in particular, X is strongly Whyburn).

A space is said to be *submaximal* if every dense subset is open (equivalently, every subset with the empty interior is closed and discrete). Every regular submaximal space is Whyburn [5, Proposition 1.3], but if X is a countable dense-in-itself submaximal space, $X \times (\omega+1)$ is not Whyburn [5, Theorem 2.3]. Hence, a countable submaximal dense-in-itself space cannot be strongly Whyburn. It looks interesting to give a direct proof of this fact, using the definition of the strong Whyburn property. Our idea owes to Bella and Yaschenko [5].

Proposition 2.4. *If a space X is countable, dense-in-itself and submaximal, then it is not strongly Whyburn.*

Proof: Fix a point $p \in X$, and let $X \setminus \{p\} = \{x_n : n \in \omega\}$. Let $A_n = \{x_n\}$ for each $n \in \omega$. Then obviously $p \in \bigcap\{\overline{\bigcup_{m \geq n} A_m} : n \in \omega\}$. Assume that there is a sequence $\{B_n : n \in \omega\}$ of closed subsets in X such that $B_n \subset A_n$ and $\{p\} = \bigcap\{\overline{\bigcup_{m\geq n} B_m} : n \in \omega\}$. Then $B_n = \emptyset$, or $B_n = \{x_n\}$. Let $I = \{n \in \omega : B_n \neq \emptyset\}$. Since the family $\{B_n : n \in I\}$ is locally finite at each point in $X \setminus \{p\}$, the set $C = \{x_n : n \in I\}$ is a discrete subspace of X, so C has the empty interior. Hence C is closed in X. This is a contradiction, because of $p \in \overline{C}$. □

We give one application of Theorem 2.2. For a Tychonoff space X, we denote by $C_p(X)$ the space of all real-valued continuous functions with the topology of pointwise convergence.

Lemma 2.5 ([11, Theorem 2.10])**.** *If $X \times Y$ contains a homeomorphic copy of S_ω and X is first-countable, then Y contains a homeomorphic copy of S_ω.*

Proposition 2.6. *If $C_p(X)$ is Whyburn, then S_ω cannot be embedded into $C_p(X)$.*

Proof: Fix a point $x \in X$. Note that $C_p(X)$ is homeomorphic to $C_p(X, x) \times \mathbb{R}$, where $C_p(X, x) = \{f \in C_p(X) : f(x) = 0\}$ and $\mathbb{R}$ is the real line. Since $C_p(X)$ is Whyburn, $C_p(X, x) \times (\omega + 1)$ is also Whyburn, so $C_p(X, x)$ is strongly Whyburn. If $C_p(X)$ has a homeomorphic copy of S_ω, by the preceding lemma, $C_p(X, x)$ has a homeomorphic copy of S_ω. This is a contradiction. □

The Whyburn property for $C_p(X)$ were investigated in [5], [10] and [15]. So far the author knows, there is no precise characterization (in terms of X) for $C_p(X)$ to be Whyburn.

Let $\mathcal{F}$ be a filter on a set. Then $\mathcal{F}$ is said to be *free* if $\bigcap \mathcal{F} = \emptyset$ holds, and have the *countable intersection property* if for each countable subfamily $\mathcal{G} \subset \mathcal{F}$, $\bigcap \mathcal{G} \neq \emptyset$ holds. If $\mathcal{F}$ is an ultrafilter, then $\bigcap \mathcal{G} \neq \emptyset$ is equivalent to $\bigcap \mathcal{G} \in \mathcal{F}$. For the discrete space $D(\kappa)$ of cardinality $\kappa \geq \omega$, let $p \in \beta D(\kappa) \setminus D(\kappa)$, where $\beta D(\kappa)$ is the Stone-Čech compactification of $D(\kappa)$ (i.e., p is a free ultrafilter on $D(\kappa)$). Let $X(p) = \{p\} \cup D(\kappa)$ be the subspace of $\beta D(\kappa)$. We examine whether $X(p)$ is strongly Whyburn.

A space is said to be a *P-space* if every G_δ-subset is open. There are many non-discrete Whyburn P-spaces, for example, consider the one-point Lindelöfication of the discrete space of cardinality ω_1. In contrast with this fact, we have the following.

Lemma 2.7. *Every strongly Whyburn P-space is discrete.*

Proof: Let X be a strongly Whyburn space and assume that there is a non-isolated point $p \in X$. Then $p \in \overline{X \setminus \{p\}}$, so there is an F_σ-subset $F \subset X \setminus \{p\}$ in X such that $p \in \overline{F}$. This implies that X is not a P-space. □

Theorem 2.8. *Let $p \in \beta D(\kappa) \setminus D(\kappa)$. Then the following assertions are equivalent:*

(1) *$X(p)$ is strongly Whyburn,*
(2) *p does not have the countable intersection property,*
(3) *$X(p) \times Y$ is Whyburn for any first-countable space Y.*

PROOF: (1)→(2) If p has the countable intersection property, then $X(p)$ is obviously a P-space. By Lemma 2.7, $X(p)$ is not strongly Whyburn.

(3)→(1) is trivial.

(2)→(3) We have only to check the Whyburn property at $(p,y) \in X(p) \times Y$. Suppose $(p,y) \in \overline{A} \setminus A$ for some subset $A \subset X(p) \times Y$. Without loss of generality, we may assume $A \subset D(\kappa) \times Y$. We put $A = \bigcup\{\{\alpha\} \times A_\alpha : \alpha < \kappa\}$, where $A_\alpha \subset Y$ and some A_α may be empty. Let $\{U_n : n \in \omega\}$ be an open neighborhood base at y such that $U_n \supset U_{n+1}$. For each $n \in \omega$, we put $P_n = \{\alpha < \kappa : A_\alpha \cap U_n \neq \emptyset\}$. Then $P_n \supset P_{n+1}$, and $P_n \in p$ by the condition $(p,y) \in \overline{A}$. Using (2), we can take subsets $Q_n \subset P_n$ such that $Q_n \in p$, $Q_n \supset Q_{n+1}$ and $\bigcap\{Q_n : n \in \omega\} = \emptyset$. For each $n \in \omega$ and $\alpha \in Q_n \setminus Q_{n+1}$, take a point $y_{n,\alpha} \in U_n \cap A_\alpha$. We define a subset $B \subset A$ as follows:

$$B = \{(\alpha, y_{n,\alpha}) : n \in \omega, \alpha \in Q_n \setminus Q_{n+1}\}.$$

First we observe $(p,y) \in \overline{B}$. Let N be a neighborhood of (p,y) in $X(p) \times Y$. Take $R \in p$ and $n \in \omega$ satisfying $(\{p\} \cup R) \times U_n \subset N$. Since $R \cap Q_n \neq \emptyset$ and $\bigcap\{Q_k : k \in \omega\} = \emptyset$, there is some $k \geq n$ such that $R \cap (Q_k \setminus Q_{k+1}) \neq \emptyset$. If $\alpha \in R \cap (Q_k \setminus Q_{k+1})$, then $(\alpha, y_{k,\alpha}) \in B \cap ((\{p\} \cup R) \times U_n) \subset B \cap N$. Thus we have $(p,y) \in \overline{B}$. Next we observe $\overline{B} = B \cup \{(p,y)\}$. For a point $y' \in Y \setminus \{y\}$, we see $(p,y') \notin \overline{B}$. Since Y is T_2, there are an open neighborhood V of y' and $n \in \omega$ such that $V \cap U_n = \emptyset$. We consider the open neighborhood $(\{p\} \cup Q_n) \times V$ of (p,y'). Suppose $((\{p\} \cup Q_n) \times V) \cap B \neq \emptyset$. Then there are some $k \in \omega$ and $\alpha \in Q_k \setminus Q_{k+1}$ such that $(\alpha, y_{k,\alpha}) \in (\{p\} \cup Q_n) \times V$. The conditions $\alpha \notin Q_{k+1}$ and $\alpha \in Q_n$ imply $n \leq k$. On the other hand, $y_{k,\alpha} \in U_k$ and $y_{k,\alpha} \notin U_n$ (because, $y_{k,\alpha} \in V$) imply $k < n$. This is a contradiction. Thus we have $(p,y') \notin \overline{B}$. Therefore $X(p) \times Y$ is Whyburn. □

We refer to [7, Chapter 12] on measurable and non-measurable cardinals. What we have to recall is that for a set X, every ultrafilter p on X with the countable intersection property satisfies $\bigcap p \neq \emptyset$ if and only if the cardinality of X is non-measurable [7, 12.2]. By Theorem 2.8, we have the following.

Corollary 2.9. *The following assertions hold.*

(1) *If $\mathfrak{m}$ is a measurable cardinal and p is a free ultrafilter on $D(\mathfrak{m})$ with the countable intersection property, then $X(p)$ is not strongly Whyburn.*
(2) *If $\mathfrak{n}$ is a non-measurable cardinal and p is a free ultrafilter on $D(\mathfrak{n})$, then $X(p)$ is strongly Whyburn.*

3. κ-Whyburn spaces

Finally, in this section, we investigate when $X \times Y$ is Whyburn for any Whyburn space Y. If $X \times Y$ is Fréchet-Urysohn for any Fréchet-Urysohn space Y, then Y is discrete. Because, if X is not discrete, then X contains the convergent sequence $\omega + 1$, so the product $X \times S_\omega$ is not Fréchet-Urysohn.

Temporarily, for an infinite cardinal κ, a space X is said to be κ-*Whyburn* if $A \subset X$, $|A| \leq \kappa$ and $p \in \overline{A} \setminus A$ imply that there is a subset $B \subset A$ such that $\overline{B} = \{p\} \cup B$. Obviously a space is Whyburn if and only if it is κ-Whyburn for each infinite cardinal κ.

Theorem 3.1. *For an infinite cardinal κ and a space X, the following assertions are equivalent:*

(1) *every subset $A \subset X$ with $|A| \leq \kappa$ is closed (equivalently, closed and discrete) in X,*
(2) *$X \times Y$ is κ-Whyburn for any κ-Whyburn space Y,*
(3) *$X \times Y$ is κ-Whyburn for any Whyburn space Y.*

Proof: (1)→(2) Let Y be a κ-Whyburn space, and assume that $A \subset X \times Y$, $|A| \leq \kappa$ and $(p,q) \in \overline{A} \setminus A$. Let $\pi_X : X \times Y \to X$ be the projection. Since the set $\pi_X(A \setminus (\{p\} \times Y))$ is closed in X, we have $(p,q) \in \overline{A \cap (\{p\} \times Y)}$. Applying the κ-Whyburn property of Y, we can take a subset $B \subset A$ such that $\overline{B} = \{(p,q)\} \cup B$.

(2)→(3) is trivial.

We show (3) →(1). Note that X is, at least, κ-Whyburn. Assume the contrary of (1). Then there is a subset $A \subset X$ such that A is not closed in X and $|A| \leq \kappa$. Let $|A| = \lambda \leq \kappa$, and let $p \in \overline{A} \setminus A$. The subspace $S = \{p\} \cup A$ of X is Whyburn, because of $|S| \leq \kappa$. For each $\alpha < \lambda$, let $Y_\alpha = \{p_\alpha\} \cup A_\alpha$ be a homeomorphic copy of S, where $p_\alpha = p$ and $A_\alpha = A$. Let $Y = \{\tilde{p}\} \cup (\bigcup_{\alpha<\lambda} A_\alpha)$ be the quotient space of the topological sum of Y_α's obtained by collapsing the set $\{p_\alpha : \alpha < \lambda\}$ to one point $\tilde{p}$. It is not difficult to check that Y is Whyburn. Since $|S \times Y| \leq \kappa$, we have only to see that $S \times Y$ is not Whyburn. Let $f : A \to \lambda$ be a bijection. We put $E = \bigcup\{\{x\} \times A_{f(x)} : x \in A\}$, then obviously $(p, \tilde{p}) \in \overline{E} \setminus E$. If $S \times Y$ is Whyburn, there is a subset $F \subset E$ such that $\overline{F} = \{(p, \tilde{p})\} \cup F$. The set F is of the form $F = \bigcup\{\{x\} \times B_{f(x)} : x \in A\}$, where $B_{f(x)} \subset A_{f(x)}$. Since $\{(p, \tilde{p})\} \cup F$ is closed, $\bigcup\{B_{f(x)} : x \in A\}$ is closed in Y. This is a contradiction, because of $(p, \tilde{p}) \in \overline{F}$. Thus $S \times Y$ is not Whyburn. □

Applying the preceding theorem, we immediately have:

Corollary 3.2. *For a space X, $X \times Y$ is Whyburn for any Whyburn space Y if and only if X is discrete.*

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