A weighted inequality for the Hardy operator involving suprema

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Abstract. Let u be a weight on $(0,\infty)$. Assume that u is continuous on $(0,\infty)$. Let the operator S_u be given at measurable non-negative function φ on $(0,\infty)$ by

$$S_u\varphi(t) = \sup_{0<\tau\le t} u(\tau)\varphi(\tau).$$

We characterize weights v, w on $(0,\infty)$ for which there exists a positive constant C such that the inequality

$$\left(\int_0^\infty [S_u\varphi(t)]^q w(t)\,dt\right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty [\varphi(t)]^p v(t)\,dt\right)^{\frac{1}{p}}$$

holds for every $0<p,q<\infty$. Such inequalities have been used in the study of optimal Sobolev embeddings and boundedness of certain operators on classical Lorenz spaces.

Keywords: Hardy operators involving suprema; weighted inequalities

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1. Introduction

In [1], it was characterized when the *Hardy–Littlewood maximal operator* M is bounded on the so-called classical Lorentz spaces. We recall that the operator M is defined at every $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ by

$$(Mf)(x) = \sup_{Q\ni x} |Q|^{-1} \int_Q |f(y)|\,dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes and $|E|$ denotes the n-dimensional Lebesgue measure of $E \subset \mathbb{R}^n$. To prove this result, two ingredients have been used. First of them was the well-known two-sided estimate for the non-increasing rearrangement of Mf in terms of the maximal non-increasing rearrangement. This result is due to Riesz, Wiener, Stein and Herz (cf. [2, Chapter 3, Theorem 3.8]). Second key ingredient was the characterization of the boundedness of the *Hardy averaging operator*

$$Af(t) := \frac{1}{t}\int_0^t f(s)\,ds$$

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on the cone of non-increasing functions in a weighted Lebesgue space. An analogous problem was later in [4] considered for the *fractional maximal operator*. This operator, denoted by M_γ, where $\gamma \in (0,n)$, is defined at $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$M_\gamma f(x) = \sup_{Q \ni x} |Q|^{\frac{\gamma}{n}-1} \int_Q |f(y)|\,dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes. It turned out that in order to handle the fractional maximal operator one needs to characterize a weighted inequality involving a substantially different operator than the Hardy's average integral operator. Namely, the operator R_γ was employed, which is defined at a measurable and positive on $(0,\infty)$ function g by

$$R_\gamma g(t) = \sup_{t \le s < \infty} s^{\frac{\gamma}{n}-1} g(s), \quad t \in (0,\infty).$$

The operator R_γ is a typical example of what we may call *a Hardy-type operator involving suprema*. In [10], a more general (weighted) version of such operator was studied. We recall that by a *weight* we shall throughout understand a positive measurable function on $(0,\infty)$. For a weight u, the operator R_u was defined in [10] at each non-negative measurable function g by

$$R_u g(t) = \sup_{t \le s < \infty} u(s) g(s), \quad t \in (0,\infty).$$

An analogous, in a certain sense, dual operator, denoted by S_u and defined by

$$S_u g(t) = \sup_{0 < s \le t} u(s) g(s), \quad t \in (0,\infty),$$

has been recently proved useful in various applications. These cover, for example, the search for optimal pairs of rearrangement-invariant norms for which a Sobolev-type inequality holds either in the Euclidean space (see e.g. [11], [12]) or in the product probability spaces of which the Gaussian space is a key example ([5], [6]). They further constitute a useful tool for characterization of the associate norm of an operator-induced norm, which naturally appears as an optimal domain norm in a Sobolev embedding ([13]). Supremum operators are also very useful in limiting interpolation theory as can be seen from their appearance for example in [8], [9], [7] or [14].

Although both the operators R_u and S_u are of interest, a comprehensive study was so far devoted only to the operator R_u. In this paper we characterize a weighted inequality for the operator S_u, restricted to the cone of non-increasing functions. The method of the proof is in some sense similar to that used in [10] but the characterizing conditions are different in nature and the technical steps of the proof had to be modified in a corresponding way.

Let $0 < p, q < \infty$ and let u be a continuous weight. Our principal goal is to give a characterization of weights v and w such that inequality

$$\left(\int_0^\infty [S_u\varphi(t)]^q w(t)\,dt\right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty [\varphi(t)]^p v(t)\,dt\right)^{\frac{1}{p}} \tag{1.1}$$

holds for all non-negative and non-increasing functions φ on $(0,\infty)$. It will be useful to observe that, for every non-negative function φ, the function $S_u\varphi$ is non-decreasing on $(0,\infty)$.

We treat the cases $p \le q$ and $p > q$ separately since the techniques we use in their proofs are quite different.

As usual, here and below, by $A \lesssim B$ we mean that $A \le CB$, where C is a positive constant independent of appropriate quantities involved in the expressions A and B.

2. Main results

Theorem 1. *Let $0 < p \le q < \infty$ and let u be a continuous weight. Let v and w be weights such that $0 < \int_0^x v(t)\,dt < \infty$ and $0 < \int_x^\infty w(t)\,dt < \infty$ for every $x \in (0,\infty)$. Then inequality (1.1) is satisfied for all non-negative and non-increasing functions φ on $(0,\infty)$ if and only if*

$$\sup_{a\in(0,\infty)} \frac{\left(\int_0^a (\bar{u}(t))^q w(t)\,dt\right)^{\frac{1}{q}} + \bar{u}(a)\left(\int_a^\infty w(t)\,dt\right)^{\frac{1}{q}}}{\left(\int_0^a v(t)\,dt\right)^{\frac{1}{p}}} < +\infty, \tag{2.1}$$

where $\bar{u}(t) = \sup_{0<\tau\le t} u(\tau)$.

Proof: *Sufficiency.* We distinguish several cases. First, let $\int_0^\infty w(t)\,dt = \infty$ and $\int_0^\infty v(t)\,dt = \infty$. We define sequences $\{x_k\}_{k\in\mathbb{Z}}$ and $\{y'_s\}_{s\in\mathbb{Z}}$ by

$$\int_{x_k}^\infty w(t)\,dt = 2^{-k} \quad \text{and} \quad \int_0^{y'_s} v(t)\,dt = 2^s. \tag{2.2}$$

Then we have

$$(0,\infty) = \bigcup_{k\in\mathbb{Z}} [x_k, x_{k+1}) = \bigcup_{s\in\mathbb{Z}} [y'_s, y'_{s+1}). \tag{2.3}$$

Consequently, using (2.3), the definition of the operator S_u, its monotonicity and (2.2), we get

$$\begin{aligned}\int_0^\infty [S_u\varphi(t)]^q w(t)\,dt &= \sum_{k\in\mathbb{Z}} \int_{x_k}^{x_{k+1}} [S_u\varphi(t)]^q w(t)\,dt \\ &= \sum_{k\in\mathbb{Z}} \int_{x_k}^{x_{k+1}} [\sup_{0<\tau\le t} u(\tau)\varphi(\tau)]^q w(t)\,dt\end{aligned}$$

$$\leq \sum_{k\in\mathbb{Z}} \sup_{0<\tau\leq x_{k+1}} [u(\tau)\varphi(\tau)]^q \int_{x_k}^{x_{k+1}} w(t)\,dt$$
$$\leq \sum_{k\in\mathbb{Z}} 2^{-k-1} \sup_{-\infty<i\leq k} \sup_{x_i<\tau\leq x_{i+1}} [u(\tau)\varphi(\tau)]^q.$$

Using a simple upper estimate of a supremum by a corresponding sum, (2.2) and (2.3) again, and interchanging the sums, we obtain

$$\begin{aligned}\int_0^\infty [S_u\varphi(t)]^q w(t)\,dt &\leq \sum_{k\in\mathbb{Z}} 2^{-k-1} \sum_{i=-\infty}^{k} \sup_{x_i<\tau\leq x_{i+1}} [u(\tau)\varphi(\tau)]^q \\ &= \sum_{i\in\mathbb{Z}} \sup_{x_i<\tau\leq x_{i+1}} [u(\tau)\varphi(\tau)]^q \sum_{k=i}^{\infty} 2^{-k-1} \\ &= \sum_{i\in\mathbb{Z}} 2^{-i} \sup_{x_i<\tau\leq x_{i+1}} [u(\tau)\varphi(\tau)]^q \\ &= \sum_{i\in\mathbb{Z}} \int_{x_i}^{\infty} w(t)\,dt \sup_{x_i<\tau\leq x_{i+1}} [u(\tau)\varphi(\tau)]^q \\ &\lesssim \sum_{i\in\mathbb{Z}} \int_{x_{i+1}}^{x_{i+2}} w(t)\,dt \sup_{x_i<\tau\leq x_{i+1}} [u(\tau)\varphi(\tau)]^q.\end{aligned}$$

Now, given $i \in \mathbb{Z}$, let us find points $z_i \in [x_i, x_{i+1}]$ such that

$$\sup_{x_i<\tau\leq x_{i+1}} [u(\tau)\varphi(\tau)]^q \leq 2[u(z_i)\varphi(z_i)]^q. \tag{2.4}$$

Thus, $[x_{i+1}, x_{i+2}] \subseteq [z_i, z_{i+2}]$ and

$$\int_0^\infty [S_u\varphi(t)]^q w(t)\,dt \lesssim \sum_{i\in\mathbb{Z}} \left(\int_{z_i}^{z_{i+2}} w(t)\,dt\right) [u(z_i)\varphi(z_i)]^q.$$

For a technical reason we divide the sum in two parts, write

$$\sum_{k\in\mathbb{Z}} \left(\int_{z_{2k}}^{z_{2k+2}} w(t)\,dt\right) [u(z_{2k})\varphi(z_{2k})]^q =: S_{even},$$
$$\sum_{k\in\mathbb{Z}} \left(\int_{z_{2k+1}}^{z_{2k+3}} w(t)\,dt\right) [u(z_{2k+1})\varphi(z_{2k+1})]^q =: S_{odd}.$$

We shall estimate S_{even}. First, we reduce the sequence $\{y'_s\}$. Fix $k \in \mathbb{Z}$. If the interval $[z_{2k}, z_{2k+2})$ contains more than one element of the sequence $\{y'_s\}$, we delete from this sequence all such elements except the one which lies nearest to z_{2k}. Thus, every interval $[z_{2k}, z_{2k+2})$, $k \in \mathbb{Z}$, now contains at most one element of the reduced sequence, which we denote by $\{y_n\}_{n\in\mathbb{Z}}$. More formally, we denote $Y_k = \{s \in \mathbb{Z}; y'_s \in [z_{2k}, z_{2k+2})\}, k \in \mathbb{Z}$, further $J = \{k \in \mathbb{Z}; Y_k \neq 0\}$, $\theta_k =$

$\min\{y'_s; s \in Y_k\}$, $k \in J$, and finally $Y = \{\theta_k\}_{k\in J}y$. Then Y is a subsequence of $\{y'_s\}$, which we enumerate as $\{y_n\}_{n\in\mathbb{Z}}$. Clearly, $y_n < y_{n+1}$ for all $n \in \mathbb{Z}$ and this sequence is a covering sequence having the following properties: Suppose that for some $n, k, s \in \mathbb{Z}$ we have

$$y_n < z_{2k} \le y_{n+1} = y'_s. \tag{2.5}$$

Then one can easily check that

$$y_{n-1} \le y'_{s-2}, \tag{2.6}$$

$$y'_{s-1} < z_{2k}, \tag{2.7}$$

$$y_{n-1} < z_{2k-2}. \tag{2.8}$$

By (2.6) and (2.7), for all $n, k, s \in \mathbb{Z}$ satisfying (2.5),

$$\int_0^{y_{n+1}} v(t)\,dt = 4\int_{y'_{s-2}}^{y'_{s-1}} v(t)\,dt \le 4\int_{y_{n-1}}^{z_{2k}} v(t)\,dt.$$

We need to estimate $\varphi^p(z_{2k})$ and to use this estimate in inequality for S_{even}. So, since the function φ is non-increasing, we have

$$\varphi^p(z_{2k}) = \frac{\int_{y_{n-1}}^{z_{2k}} v(t)\,dt}{\int_{y_{n-1}}^{z_{2k}} v(t)\,dt}\,\varphi^p(z_{2k}) \le \left(\int_{y_{n-1}}^{z_{2k}} v(t)\,dt\right)^{-1} \int_{y_{n-1}}^{z_{2k}} \varphi^p(t)v(t)\,dt. \tag{2.9}$$

Hence

$$\varphi^q(z_{2k}) \lesssim \left(\int_0^{y_{n+1}} v(t)\,dt\right)^{-\frac{q}{p}} \left(\int_{y_{n-1}}^{y_{n+1}} \varphi^p(t)v(t)\,dt\right)^{\frac{q}{p}}. \tag{2.10}$$

Let us still write

$$u^q(x) \le \left(\sup_{0<\tau\le t} u(\tau)\right)^q = [\bar{u}(t)]^q \quad \text{for all} \quad t \ge x.$$

Denote $A_n = \{k \in \mathbb{Z}; y_n < z_{2k} \le y_{n+1}\}$, $n \in \mathbb{Z}$. Then

$$S_{even} = \sum_{n\in\mathbb{Z}}\sum_{k\in A_n} \int_{z_{2k}}^{z_{2k+2}} w(t)\,dt\ [u(z_{2k})\varphi(z_{2k})]^q.$$

Fix $n \in \mathbb{Z}$ and define numbers $l_1^n = \min\{k; k \in A_n\}$ and $l_2^n = \max\{k; k \in A_n\}$. Thanks to (2.4), the definition of l_1^n and l_2^n and the fact that φ is non-increasing, we get

$$\sum_{k\in A_n} \int_{z_{2k}}^{z_{2k+2}} w(t)\,dt\ [u(z_{2k})\varphi(z_{2k})]^q$$

$$\leq \left(\int_{z_{2l_1^n}}^{y_{n+1}} (\bar{u}(t))^q w(t)\, dt + [\bar{u}(y_{n+1})]^q \int_{y_{n+1}}^{z_{2l_2^n+2}} w(t)\, dt \right) [\varphi(z_{2l_1^n})]^q.$$

Thus by (2.5) and (2.10),

$$\begin{aligned}
&\sum_{k\in A_n} \int_{z_{2k}}^{z_{2k+2}} w(t)\, dt\; [u(z_{2k})\varphi(z_{2k})]^q \\
&\quad\leq \left(\int_0^{y_{n+1}} (\bar{u}(t))^q w(t)\, dt + (\bar{u}(y_{n+1}))^q \int_{y_{n+1}}^{\infty} w(t)\, dt \right) [\varphi(z_{2l_1^n})]^q \\
&\quad\lesssim \sum_{n\in\mathbb{Z}} \left(\int_0^{y_{n+1}} (\bar{u}(t))^q w(t)\, dt + (\bar{u}(y_{n+1}))^q \int_{y_{n+1}}^{\infty} w(t)\, dt \right) \\
&\quad\quad \times \left(\int_0^{y_{n+1}} v(t)\, dt \right)^{-\frac{q}{p}} \left(\int_{y_{n-1}}^{y_{n+1}} \varphi^p(t) v(t)\, dt \right)^{\frac{q}{p}} \\
&\quad\lesssim \sum_{n\in\mathbb{Z}} \left(\int_{y_{n-1}}^{y_{n+1}} \varphi^p(t) v(t)\, dt \right)^{\frac{q}{p}},
\end{aligned}$$

where in the last inequality we use the condition (2.1). Since $p \leq q$, we can use the convexity of the function $x^{\frac{q}{p}}$ and we have

$$\begin{aligned}
S_{even} &\lesssim \sum_{n\in\mathbb{Z}} \left(\int_{y_{n-1}}^{y_{n+1}} \varphi^p(t) v(t)\, dt \right)^{\frac{q}{p}} \\
&\lesssim \left(\sum_{n\in\mathbb{Z}} \int_{y_{n-1}}^{y_{n+1}} \varphi^p(t) v(t)\, dt \right)^{\frac{q}{p}} \\
&\lesssim \left(\int_0^{\infty} \varphi^p(t) v(t)\, dt \right)^{\frac{q}{p}}.
\end{aligned}$$

In order to estimate S_{odd}, we define a possibly different sequence $\{y_n\}_{n\in\mathbb{Z}}$. Again, we reduce the sequence y'_n in the same way, but this time in intervals $[z_{2k+1}, z_{2k+3})$. Now, it is clear that we can estimate S_{odd} in the same way as S_{even} was estimated. The main reason for the division into sums S_{even} and S_{odd} is to guarantee that the sets A_n are non-empty.

If $\int_0^\infty w(t)\, dt < \infty$, then we can without loss of generality assume that $\int_0^\infty w(t) dt = 1$ and work instead of the sequence $\{x_k\}_{k\in\mathbb{Z}}$ only with the reduced sequence $\{x_k\}_{k=0}^{\infty}$. In the case when moreover $\int_0^\infty v(t)\, dt < \infty$, then we replace the sequence $\{y_n\}_{n\in\mathbb{Z}}$ by a reduced sequence $\{y_n\}_{n=-\infty}^{N}$ with an appropriate $N \in \mathbb{Z}$.

This completes the proof of the sufficiency part.

Necessity. We first observe that

$$S_u\chi_{(0,a]}(t) = \bar{u}(t)\chi_{(0,a]}(t) + \bar{u}(a)\chi_{(a,\infty)}(t).$$

Now, testing the inequality (1.1) with functions $\varphi(t) = \chi_{(0,a]}(t), a \in (0,\infty)$, we get exactly the inequality (2.1). □

Our next aim is to handle the case when $0 < q < p < \infty$. We shall need the following special case of [10, Theorem 4.4].

Theorem 2. *Let U be a continuous weight and let V and W be weights such that $0 < \int_0^x V(t)\,dt < \infty$ and $0 < \int_0^x W(t)\,dt < \infty$ for every $x \in (0,\infty)$. Let $0 < Q < 1$ and let R be defined by*

$$\frac{1}{R} = \frac{1}{Q} - 1.$$

Then the inequality

$$\left(\int_0^\infty \left(\sup_{t\le s<\infty} \frac{U(s)}{s}\int_0^s g(y)\,dy\right)^Q W(t)\,dt\right)^{\frac{1}{Q}} \lesssim \int_0^\infty g(t)V(t)\,dt$$

holds for every non-negative measurable function g if and only if

$$\left(\int_0^\infty \left(\int_t^\infty \left(\frac{\tilde{U}(s)}{s}\right)^Q W(s)\,ds\right)^R \left(\frac{\tilde{U}(t)}{t}\right)^Q \left[\operatorname*{ess\,sup}_{a<t<b} \frac{1}{V(t)}\right]^R W(t)\,dt\right)^{\frac{1}{R}} < \infty$$

and

$$\left(\int_0^\infty \left(\int_0^t W(s)\,ds\right)^R \left[\sup_{t\le\tau<\infty} \frac{\tilde{U}(\tau)}{\tau} \operatorname*{ess\,sup}_{a<t<b} \frac{1}{V(t)}\right]^R W(t)\,dt\right)^{\frac{1}{R}} < \infty,$$

where

$$\tilde{U}(t) = t \sup_{t\le\tau<\infty} \frac{U(\tau)}{\tau}, \quad t \in (0,\infty).$$

Theorem 3. *Let $0 < q < p < \infty$ and let u be a continuous weight. Let v and w be weights such that $0 < \int_0^x v(t)\,dt < \infty$ and $0 < \int_x^\infty w(t)\,dt < \infty$ for every $x \in (0,\infty)$. Then inequality (1.1) is satisfied for all non-negative and non-increasing functions φ on $(0,\infty)$ if and only if the following two conditions are* *satisfied:*

$$\begin{aligned}(2.11)\qquad &\int_0^\infty \left(\int_0^t \sup_{0<\tau\le s} u(\tau)^{\frac{q}{p}} w(s)\,ds\right)^{\frac{q}{q-p}} \sup_{0<y\le t} u(y)^{\frac{q}{p}} \\ &\quad\times w(t)\left(\int_0^t v(s)\,ds\right)^{-\frac{q}{p-q}} dt<\infty\end{aligned}$$

and

$$(2.12)\qquad \int_0^\infty \left(\int_t^\infty w(y)\,dy\right)^{\frac{q}{p-q}} \left(\sup_{0<\tau\le t} \frac{\sup_{0<z\le\tau} u(z)}{\int_0^\tau v(y)\,dy}\right)^{\frac{q}{p-q}} w(t)\,dt<\infty.$$

Proof: Changing variables ($y=\frac{1}{t}$) on both sides of the inequality (1.1), we get

$$\left(\int_0^\infty \left(\sup_{0<\tau\le\frac{1}{y}} u(\tau)\varphi(\tau)\right)^q w(\tfrac{1}{y})\frac{dy}{y^2}\right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty \varphi^p(\tfrac{1}{y})v(\tfrac{1}{y})\frac{dy}{y^2}\right)^{\frac{1}{p}}.$$

On denoting $z=\frac{1}{\tau}$, we arrive at the inequality

$$\left(\int_0^\infty \left(\sup_{0<\frac{1}{z}\le\frac{1}{y}} u(\tfrac{1}{z})\varphi(\tfrac{1}{z})\right)^q w(\tfrac{1}{y})\frac{dy}{y^2}\right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty \varphi^p(\tfrac{1}{y})v(\tfrac{1}{y})\frac{dy}{y^2}\right)^{\frac{1}{p}}$$

for every non-increasing positive function φ. Noting that $0<\frac{1}{z}\le\frac{1}{y}$ is equivalent to $y\le z<\infty$, we actually have

$$\left(\int_0^\infty \left(\sup_{y\le z<\infty} u(\tfrac{1}{z})\varphi(\tfrac{1}{z})\right)^q w(\tfrac{1}{y})\frac{dy}{y^2}\right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty \varphi^p(\tfrac{1}{y})v(\tfrac{1}{y})\frac{dy}{y^2}\right)^{\frac{1}{p}}.$$

By a simple re-scaling, this is equivalent to

$$\left(\int_0^\infty \left(\sup_{y\le z<\infty} u^p(\tfrac{1}{z})\varphi^p(\tfrac{1}{z})\right)^{\frac{q}{p}} w(\tfrac{1}{y})\frac{dy}{y^2}\right)^{\frac{p}{q}} \lesssim \int_0^\infty \varphi^p(\tfrac{1}{y})v(\tfrac{1}{y})\frac{dy}{y^2}.$$

Since φ is a non-increasing positive function, the function $z\mapsto\varphi^p(\frac{1}{z})$ is positive and non-decreasing on $(0,\infty)$ in the variable z. By a standard approximation argument based on the Monotone Convergence Theorem (see, e.g., [3]), one can equivalently reduce the last inequality to the same one but restricted only to functions of the form

$$\varphi^p(\tfrac{1}{z})=\int_0^z h(s)\,ds.$$

We thus get

$$\left(\int_0^\infty \left(\sup_{y\le z<\infty} u^p(\tfrac{1}{z})\int_0^z h(s)\,ds\right)^{\frac{q}{p}} w(\tfrac{1}{y})\frac{dy}{y^2}\right)^{\frac{p}{q}} \lesssim \int_0^\infty \int_0^t h(s)ds\, v(\tfrac{1}{t})\frac{dt}{t^2}$$

for every measurable non-negative function h on $(0,\infty)$. By the Fubini theorem, this is nothing else than

$$\left(\int_0^\infty \left(\sup_{y\le z<\infty} u^p(\tfrac{1}{z})\int_0^z h(s)\,ds\right)^{\frac{q}{p}} w(\tfrac{1}{y})\frac{dy}{y^2}\right)^{\frac{p}{q}} \lesssim \int_0^\infty h(s)\int_s^\infty v(\tfrac{1}{t})\frac{dt}{t^2}\,ds,$$

that is,

$$\left(\int_0^\infty \left(\sup_{y\le z<\infty} u^p(\tfrac{1}{z})\int_0^z h(s)\,ds\right)^{\frac{q}{p}} w(\tfrac{1}{y})\frac{dy}{y^2}\right)^{\frac{p}{q}} \lesssim \int_0^\infty h(s)\int_0^{\frac{1}{s}} v(y)\,dy\,ds.$$

Theorem 2 applied to parameters

$$Q=\tfrac{q}{p},\ U(z)=zu^p(\tfrac{1}{z}),\ W(y)=w(\tfrac{1}{y})y^{-2},\ V(s)=\int_0^{\frac{1}{s}} v(y)\,dy$$

now shows that the latter inequality holds if and only if the conditions (2.11) and (2.12) are satisfied. The proof is complete. □

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References

[1] Ariño M., Muckenhoupt B., *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions*, Trans. Amer. Math. Soc. **320** (1990), 727–735.

[2] Bennett C., Sharpley R., *Interpolation of Operators*, Pure and Applied Mathematics, 129, Academic Press, Boston, 1988.

[3] Carro M., Gogatishvili A., Martín J., Pick L., *Weighted inequalities involving two Hardy operators with applications to embeddings of function spaces*, J. Operator Theory **59** (2008), no. 2, 101–124.

[4] Cianchi A., Kerman R., Opic B., Pick L., *A sharp rearrangement inequality for fractional maximal operator*, Studia Math. **138** (2000), 277-284.

[5] Cianchi A., Pick L., *Optimal Gaussian Sobolev embeddings*, J. Funct. Anal. **256** (2009), no. 11, 3588–3642.

[6] Cianchi A., Pick L., Slavíková L., *Higher-order Sobolev embeddings and isoperimetric inequalities*, preprint, 2012.

[7] Cwikel M., Pustylnik E., *Weak type interpolation near "endpoint" spaces*, J. Funct. Anal. **171** (1999), 235–277.

[8] Doktorskii R.Ya., *Reiteration relations of the real interpolation method*, Soviet Math. Dokl. **44** (1992), 665–669.

[9] Evans W.D., Opic B., *Real interpolation with logarithmic functors and reiteration*, Canad. J. Math. **52** (2000), 920–960.

[10] Gogatishvili A., Opic B., Pick L., *Weighted inequalities for Hardy-type operators involving suprema*, Collect. Math. **57** (2006), no. 3, 227–255.

[11] Kerman R., Pick L., *Optimal Sobolev imbeddings*, Forum Math. **18** (2006), no. 4, 535–570.

[12] Kerman R., Pick L., *Optimal Sobolev imbedding spaces*, Studia Math. **192** (2009), no. 3, 195–217.

[13] Pick L., *Supremum operators and optimal Sobolev inequalities*, Function Spaces, Differential Operators and Nonlinear Analysis, Vol. 4, Proceedings of the Spring School held in Syöte, June 1999, V. Mustonen and J. Rákosník (Eds.), Mathematical Institute of the Academy of Sciences of the Czech Republic, Praha, 2000, pp. 207–219.

[14] Pustylnik E., *Optimal interpolation in spaces of Lorentz-Zygmund type*, J. Anal. Math. **79** (1999), 113–157.

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