On τ-extending modules

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Abstract. In this paper we introduce the concept of τ-extending modules by τ-rational submodules and study some properties of such modules. It is shown that the set of all τ-rational left ideals of ${}_RR$ is a Gabriel filter. An R-module M is called τ-extending if every submodule of M is τ-rational in a direct summand of M. It is proved that M is τ-extending if and only if $M = Rej_M E(R/\tau(R)) \oplus N$, such that N is a τ-extending submodule of M. An example is given to show that the direct sum of τ-extending modules need not be τ-extending.

Keywords: torsion theory; τ-rational submodules; τ-closed submodules; τ-extending modules

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1. Introduction

Throughout this paper, R is an associative ring with identity and M is a unital left R-module. A subfunctor ρ is called a preradical if it satisfies the following properties:

(1) $\rho(M)$ is a submodule of an R-module M;
(2) if $f : M \longrightarrow N$ is an R-homomorphism, then $f(\rho(M)) \subseteq \rho(N)$ and $\rho(f) : f(M) \longrightarrow f(N)$ is the restriction of f to $\rho(M)$.

A preradical ρ is *idempotent* if $\rho(\rho(M)) = \rho(M)$, and *radical* when $\rho(M/\rho(M)) = 0$ for all $M \in R$-Mod.

For a preradical ρ, let $\mathbb{T}_\rho = \{N | \rho(N) = N\}$ and $\mathbb{F}_\rho = \{N | \rho(N) = 0\}$, $\mathbb{T}_\rho$ is called the *torsion class* of ρ and $\mathbb{F}_\rho$ the *torsion free class* of ρ. ρ is called left exact if $\rho(N) = N \cap \rho(M)$ for every module M and every submodule N of M. A preradical ρ is left exact if and only if ρ is idempotent and $\mathbb{T}_\rho$ is closed under submodules. A preradical ρ is called *coheriditary* if $\rho(M/N) = (\rho(M) + N)/N$ for every module M and every submodule N of M. ρ is cohereditary if and only if ρ is radical and $\mathbb{F}_\rho$ is closed under homomorphic images.

A pair $(\mathcal{T}, \mathcal{F})$ of classes of modules is called a *torsion theory* if the following conditions hold:

(i) $\mathrm{Hom}_R(A, B) = 0$ for every $A \in \mathcal{T}$ and every $B \in \mathcal{F}$;
(ii) $\mathcal{T}$ and $\mathcal{F}$ are maximal classes having property (i).

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The modules in $\mathcal{T}$ are called torsion modules of τ and the modules in $\mathcal{F}$ are torsion-free of τ. There is a 1-1 correspondence between torsion theories and idempotent radicals. In particular preradicals are connected to torsion theory as follows. If ρ is an idempotent radical in R-Mod, then $(\mathbb{T}_\rho, \mathbb{F}_\rho)$ is a torsion theory, where $\mathbb{T}_\rho = \{M \in \text{R}-\text{Mod}|\ \rho(M) = M\}$ and $\mathbb{F}_\rho = \{M \in \text{R}-\text{Mod}|\ \rho(M) = 0\}$. Now for any torsion theory $(\mathcal{T}, \mathcal{F})$, there is an associated idempotent radical τ_t (simply denoted by τ), called the torsion radical associated to torsion theory $(\mathcal{T}, \mathcal{F})$. Here for every module N, $\tau(N)$ will be the unique maximal submodule of N such that $\tau(N) \in \mathcal{T}$. Then τ is uniquely determined and $\mathcal{T}$ is exactly the set $\{M|\ \tau(M) = M\}$ and $\mathcal{F} = \{M|\ \tau(M) = 0\}$. Therefore we can denote this torsion theory by $\tau = (\mathcal{T}, \mathcal{F})$, where τ is an idempotent radical associative to $(\mathcal{T}, \mathcal{F})$. A torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ is called *hereditary* if $\mathcal{T}$ is closed under submodules. A torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ is hereditary if and only if $\mathcal{F}$ is closed under injective hulls if and only if t is a left exact radical. Thus there is a 1-1 correspondence between hereditary torsion theories and left exact radicals.

A module M is extending if every submodule of M is essential in a direct summand of M. In recent years, torsion-theoretic analogues of extending modules have been studied by many authors (see [4], [15], [5], [7], [16], [10], [8]).

In 2007, Charalambides and Clark [5] generalized extending modules to torsion theories. They defined that a module M is τ-extending if every τ-dense, closed submodule of M is a direct summand of M. In 2008, [15] the authors also studied τ-CS (extending) modules under the name of type 2-τ-extending modules. In [8] s-t-CS modules and CS modules were studied under the name of type 1 τ-extending modules and type 2 τ-extending modules respectively.

Following J. L. Gomez Pardo [10], a submodule N of an R-module M is called τ-large in M if, for $W \leq M$, $N \cap W \subseteq \tau(M)$ implies $W \subseteq \tau(M)$.

In [4] the authors say that M is τ-extending module if every submodule is τ-large in a direct summand of M. They showed that every τ-torsion module is τ-extending and they also proved that a τ-torsion free module is τ-extending if and only if it is extending. In this paper, we generalize extending modules by using hereditary torsion theories. We say that a submodule N of M is τ-rational in M if $Hom(M/N, E(R/\tau(R))) = 0$, where $E(R/\tau(R))$ is the injective hull of $R/\tau(R)$. We say that an R-module M is τ-extending if for every submodule X, there exists a direct summand D of M such that X is τ-rational in D, i.e, $Hom(D/X, E(R/\tau(R))) = 0$. We prove that a module M is τ-extending if and only if every τ-closed submodule of M is a direct summand of M. We show that M is τ-extending if and only if $M = Rej_M E(R/\tau(R)) \oplus N$, and N is a τ-extending submodule of M. We also prove that the class of τ-extending modules is closed under direct summands. It is proved that if M is a τ-extending module and $Rej_T(E(R/\tau(R))) = T$ for a module T, then $M \oplus T$ is τ-extending. Moreover, we prove that $M = M_1 \oplus M_2$ is τ-extending if and only if M_i are τ-extending and every τ-closed submodule K of $N_1 \oplus N_2$ with $K \cap N_1 K \cap N_2 = 0$ is a direct summand of M, where $M_i = Rej_{M_i}(E(R/\tau(R))) \oplus N_i$.

2. τ-rational modules

Throughout this paper τ is a hereditary preradical associative to a hereditary torsion theory.

Definition 2.1. We say that a submodule N of an R-module M is τ-rational in M, denoted by $N \leq_{\tau-r} M$, if $Hom(M/N, E(R/\tau(R))) = 0$. If $M \in \mathbb{T}_\tau$, then every submodule of M is τ-rational.

Let $\mathcal{U}$ be a class of modules. A module M is (finitely) cogenerated by $\mathcal{U}$ (or $\mathcal{U}$ (finitely) cogenerates M), in case there is an (a finite) indexed set $(U_\alpha)_{\alpha \in A}$ in $\mathcal{U}$ and a monomorphism

$$0 \longrightarrow M \longrightarrow \prod_A U_\alpha.$$

An R-module ${}_RC$ is said to be a cogenerator if ${}_RC$ cogenerates every R-module.

It is recalled that a submodule N of M is dense in M if $Hom_R(M/N, E(M)) = 0$, where $E(M)$ denotes the injective envelope of M.

Lemma 2.2. *Let $R/\tau(R)$ be an injective cogenerator in ${}_R\mathcal{M}$ and $N \leq_{\tau-r} M$. Then N is dense in M.*

Proof: As $N \leq_{\tau-r} M$, then $Hom_R(M/N, E(R/\tau(R))) = 0$. Since $R/\tau(R)$ is an injective cogenerator, there is a set A for which $E(M)$ can be embedded in $\prod_A R/\tau(R)$.

Thus $\prod_A Hom_R(M/N, R/\tau(R)) \simeq Hom_R(M/N, \prod_A R/\tau(R)) = 0$. It follows that $Hom(M/N, E(M)) = 0$ and so N is dense in M. $\square$

A ring R is called a left Kasch ring (or simply left Kasch) if every simple left module K embeds in ${}_RR$, equivalently if ${}_RR$ cogenerates K. Every semisimple artinian ring is right and left Kasch, and a local ring R is left Kasch if and only if $Soc_l(R) \neq 0$, because R has only one simple left module up to isomorphism.

Corollary 2.3. *Consider the trivial torsion theory $\tau = 0$. If R is a left Kasch ring and $N \leq_{\tau-r} M$, for R-modules M and N, then N is a dense submodule of M.*

Proof: This follows from the fact that ${}_RR$ is a left Kasch ring if and only if $E({}_RR)$ is a cogenerator in ${}_R\mathcal{M}$ and applying Lemma 2.2. $\square$

Definition 2.4. A non-empty set $\mathfrak{D}(R)$ of left ideals of R is called a filter radical if the following hold:

(i) for every $I \in \mathfrak{D}(R)$ and every $a \in R$, we have $(I : a) \in \mathfrak{D}(R)$, where $(I : a)$ is the ideal $\{r \in R \mid ra \in I\}$;

(ii) for every $J \in \mathfrak{D}(R)$ and every left ideal I of R with $(I : a) \in \mathfrak{D}(R)$ for each $a \in J$, we have $I \in \mathfrak{D}(R)$.

Proposition 2.5. *Let $\mathfrak{F}(R)$ be the set of all left ideals I such that ${}_RI$ is τ-rational in ${}_RR$. Then $\mathfrak{F}(R)$ is a filter radical.*

Proof: (i) Let $I \in \mathfrak{F}(R)$ and $a \in R$. Then $Hom_R(R/I, E(R/\tau(R))) = 0$ and by the injectivity of $E(R/\tau(R))$, $Hom_R((Ra+I)/I, E(R/\tau(R))) = 0$. As $R/(I : a) \simeq (Ra+I)/I$ then we get $Hom_R(R/(I : a), E(R/\tau(R))) = 0$. Hence for every $I \in \mathfrak{F}(R)$ and $a \in R$, we get $(I : a) \in \mathfrak{F}(R)$.

(ii) Assume that $J \in \mathfrak{F}(R)$ and there exists a left ideal I of R, such that $(I : a) \in \mathfrak{F}(R)$ for every $a \in J$, so that, $Hom_R(R/(I : a), E(R/\tau(R))) = 0$. If $f \in Hom_R(R/I, E(R/\tau(R)))$ then $(Ra+I)/I \simeq R/(I : a) \subseteq ker(f)$ for every $a \in J$. Hence $Hom_R((Ra+I)/I, E(R/\tau(R))) = 0$ for every $a \in J$ and so $Hom_R((J+I)/I, E(R/\tau(R))) = 0$. Thus f factors through $\overline{f} \in Hom_R(R/(I+J), E(R/\tau(R)))$. However $J \in \mathfrak{F}(R)$ implies $I+J \in \mathfrak{F}(R)$. Hence $\overline{f} = 0$ and so $f = 0$. This shows that $I \in \mathfrak{F}(R)$. □

Corollary 2.6. *Let I, J be left ideals of R. Then*

(i) *if $J \in \mathfrak{F}(R)$ and $J \subseteq I$, then $I \in \mathfrak{F}(R)$;*
(ii) *if $I, J \in \mathfrak{F}(R)$, then $I \cap J \in \mathfrak{F}(R)$;*
(iii) *if $I, J \in \mathfrak{F}(R)$, then $IJ \in \mathfrak{F}(R)$.*

Proof: This follows by [3]. □

Lemma 2.7. *If ${}_RI$ is τ-rational in ${}_RR$, then $(I+\tau(R))/\tau(R) \leq_{es} R/\tau(R)$.*

Proof: Suppose that there exists a nonzero left ideal $L/\tau(R)$ of $R/\tau(R)$ such that $L/\tau(R) \cap (I+\tau(R))/\tau(R) = 0$. As $I \subseteq I+L$, by Corollary 2.6(i), we have $I+L \in \mathfrak{F}(R)$. Hence $Hom_R(R/(L+I), E(R/\tau(R))) = 0$.

Since $Hom_R(R/I, E(R/\tau(R))) = 0$, then $Hom_R((L+I)/I, E(R/\tau(R))) = 0$. Thus $Hom(L/(L \cap I), E(R/\tau(R))) = 0$, and since $I \cap L \subseteq \tau(R)$,

$$Hom(L/\tau(R), E(R/\tau(R))) = 0,$$

a contradiction. □

The following examples show that Lemma 2.7 need not be true, for R-modules.

Example 2.8. Consider the torsion theory $(0, {}_R\mathcal{M})$ with associative radical $\tau = 0$, where $R = \mathbb{Z}$. Let $M = \mathbb{Z}_6$ and $N = 3\mathbb{Z}_6$, then $3\mathbb{Z}_6 \not\leq_e \mathbb{Z}_6$, however $N \leq_{\tau-r} M$ because $Hom_{\mathbb{Z}}(\mathbb{Z}_6/3\mathbb{Z}_6, \mathbb{Q}) = 0$. □

Example 2.9. Consider the torsion theory $({}_R\mathcal{M}, 0)$ with associative radical $\tau = id$, where $R = \mathbb{Z}$. Then for every R-module M, we have $Hom_{\mathbb{Z}}(M/N, 0) = 0$, for every $\mathbb{Z}$-submodule N of M, which implies that $N \leq_{\tau-r} M$. □

For each $M \in R$-Mod we define

$$\delta_\tau(M) = \{x \in M \mid (0 : x) \text{ is a } \tau\text{-rational left ideal in } R\}.$$

Proposition 2.10. *For an arbitrary ring R and a left R-module M the following assertions hold:*

(1) *$\delta_\tau(M)$ is a submodule of M;*

(2) $\delta_\tau(M/\delta_\tau(M)) = 0$;
(3) *for every R-homomorphism* $f : M \longrightarrow N$, $f(\delta_\tau(M)) \subseteq \delta_\tau(N)$;
(4) *for every* $K \leq M$ *we have* $\delta_\tau(K) = \delta_\tau(M) \cap K$.

Proof: (1) This is clear.

(2) Let $\overline{m} = m + \delta_\tau(M) \in M/\delta_\tau(M)$. Then $\overline{m} \in \delta_\tau(M/\delta_\tau(M))$ iff $(0 : \overline{m})$ is τ-rational in R. As $(0 : \overline{m}) = \{r \in R|\ rm \in \delta_\tau(M)\}$ and $(0 : rm) = ((0 : m) : r)$, then $(0 : \overline{m}) = \{r \in R|\ ((0 : m) : r) \leq_{\tau_r} R\}$ is τ-rational in R. Since the set of all τ-rational left ideals of R is a Gabriel filter, we get $(0 : m) \leq_{\tau_r} R$ and this shows that $m \in \delta_\tau(M)$, i.e; $\overline{m} = 0$.

(3) Let $m \in \delta_\tau(M)$. Then $(0 : m) \leq_{\tau_r} R$. As $(0 : m) \subseteq (0 : f(m))$ we get $(0 : f(m)) \leq_{\tau_r} R$.

(4) This is clear. □

Corollary 2.11. *Let* $(\mathbb{T}, \mathbb{F})$, *where* $\mathbb{T} = \{M|\ \delta_\tau(M) = M\}$, $\mathbb{F} = \{M|\ \delta_\tau(M) = 0\}$. *Then* $(\mathbb{T}, \mathbb{F})$ *is a hereditary torsion theory.*

Proposition 2.12. *If* $\delta_\tau(M/N) = M/N$ *then* N *is τ-rational in* M.

Proof: To the contrary assume that $\delta_\tau(M/N) = M/N$ but

$$Hom(M/N, E(R/\tau(R))) \neq 0.$$

Then we have $Hom((Rm + N)/N, E(R/\tau(R))) \neq 0$, for some $m \in M$. As $R/(N : m) \simeq (Rm + N)/N$ for any $m \in M$, this gives

$$Hom(R/(N : m), E(R/\tau(R))) \neq 0,$$

a contradiction to the fact that $(N : m) \leq_{\tau - r} R$. □

Corollary 2.13. *Let* $N \leq_{\tau-r} M$ *and* K *a submodule of* M. *Then* $N \cap K \leq_{\tau-r} K$.

Corollary 2.14. $\delta_\tau(M) = M$ *iff* $Hom_R(M, E(R/\tau(R))) = 0$.

Proposition 2.15. *Let* M *be an R-module and* $N, L \leq M$. *If* $N \subseteq L \subseteq M$, *then* $N \leq_{\tau-r} L \leq_{\tau-r} M$ *iff* $N \leq_{\tau-r} M$.

Proof: If $N \leq_{\tau-r} M$, then obviously $N \leq_{\tau-r} L \leq_{\tau-r} M$.

Conversely, let $N \leq_{\tau-r} L \leq_{\tau-r} M$. Consider the exact sequence

$$0 \longrightarrow L/N \longrightarrow M/N \longrightarrow M/L \longrightarrow 0.$$

Then, since $Hom(-, E(R/\tau(R)))$ is an exact functor we get the exact sequence

$$0 \longrightarrow Hom(M/N, E(R/\tau(R))) \longrightarrow 0.$$

Thus $Hom(M/N, E(R/\tau(R))) = 0$ and so $N \leq_{\tau-r} M$. □

Corollary 2.16. *N is τ-rational in M if and only if C is τ-rational in M, where* $C/N = \tau(M/N)$.

Lemma 2.17. *Let N be a submodule of M and suppose that every homomorphic image of M has a non-zero τ-torsion submodule. Then $N \leq_{\tau-r} M$.*

Proof: On the contrary, assume $0 \neq f \in Hom(M/N, E(R/\tau(R)))$. Then f factors through a monomorphism $0 \neq \overline{f} : M/kerf \longrightarrow E(R/\tau(R))$. As $\tau(M/ker(f)) \neq 0$ we get $0 \neq \tau(M/kerf) \subseteq ker\overline{f}$, a contradiction. □

Corollary 2.18. *If M/N is a τ-torsion module, then N is τ-rational in M.*

The following example shows that the converse of Corollary 2.18 need not be true.

Example 2.19. Consider the trivial torsion theory $(0, {}_R\mathcal{M})$ on ${}_{\mathbb{Z}}\mathcal{M}$. Then $Hom_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Q}) = 0$ while $\tau(\mathbb{Z}/4\mathbb{Z}) \neq \mathbb{Z}/4\mathbb{Z}$.

Definition 2.20. Let M be a module and $K \leq M$. We say that K is a τ-closed submodule of M, denoted by $K \leq_{\tau c} M$, if whenever for any submodule L of M, $Hom(L/K, E(R/\tau(R))) = 0$ implies $K = L$. If N is a submodule of M such that $K \leq_{\tau-r} N$ and N is τ-closed in M then N is called a τ-closure of K in M. Note that $N \leq_{\tau c} M$ if and only if for all $N < K \leq M$, $Rej_{K/N}(E(R/\tau(R))) = 0$.

Proposition 2.21. *Let $N' \leq N \leq M$. Then the following are true:*

1. *if N' is τ-closed in M, then N' is τ-closed in N;*
2. *$Rej_M(E(R/\tau(R))) \leq_{\tau c} M$ and $N \leq_{\tau c} M$, moreover $Rej_M(E(R/\tau(R))) \subseteq N$. Besides, $Rej_M(E(R/\tau(R)))$ is the intersection of all τ-closed submodules of M;*
3. *if $K \leq_{\tau c} M$, then M/K is a τ-torsion free module. Clearly the converse is not true;*
4. *if N' is τ-closed in N and N is τ-closed in M, then N' is τ-closed in M;*
5. *the class of τ-closed submodules of M is closed under intersections.*

Proof: (1) This is clear.

(2) Clearly $Rej_M(E(R/\tau(R))) \leq_{\tau c} M$. Now, on the contrary, assume that $N \leq_{\tau c} M$ and $N \not\supseteq Rej_M(E(R/\tau(R)))$. Then there is an

$$x \in Rej_M(E(R/\tau(R)))\backslash N$$

and so $Hom((Rx + N)/N, E(R/\tau(R))) \simeq Hom(Rx/(Rx \cap N), E(R/\tau(R))) = 0$, a contradiction.

(3) Assume that $K \leq_{\tau c} M$ and $\tau(M/K) = C/K \neq 0$. Then since $C/K \in \mathbb{T}_\tau$ and $E(R/\tau(R)) \in \mathbb{F}_\tau$, we get $Hom(C/K, E(R/\tau(R))) = 0$, a contradiction. Thus $C/K = 0$.

(4) If $Hom(L/N', E(R/\tau(R))) = 0$ for some $N' \lneq L \leq M$, then we have $L \nsubseteq N$ and $N' \subseteq L \cap N$. Therefore, $Hom_R((L \cap N)/N', E(R/\tau(R))) = 0$. Since N' is τ-closed in N, we get $(L \cap N) = N'$. Hence $Hom((L + N)/N, E(R/\tau(R))) = 0$ and so $N = N + L$ because $N \leq_{\tau c} M$. From $N = N + L$ we have $L \subseteq N$, a contradiction.

(5) Let $N_i \leq_{\tau c} M$ for every $i \in I$. Then $Rej_{M/N_i}(E(R/\tau(R))) = 0$, for every $i \in I$. Thus we have $\bigoplus_{i\in I} Rej_{M/N_i}(E(R/\tau(R))) = Rej_{\bigoplus_{i\in I} M/N_i}(E(R/\tau(R))) = 0$. Since there is a monomorphism $f : M/\cap_{i\in I} N_i \longrightarrow \bigoplus_{i\in I} M/N_i$, the injectivity of $E(R/\tau(R))$ implies $Rej_{M/\cap_{i\in I} N_i}(E(R/\tau(R))) = 0$ and so $\cap_{i\in I} N_i \leq_{\tau c} M$. □

Proposition 2.22. *For a module M, every submodule N of M has a τ-closure.*

Proof: If $Hom(M/N, E(R/\tau(R))) = 0$, then there is nothing to prove. Hence suppose that $Hom(M/N, E(R/\tau(R))) \neq 0$, $D/N = Rej_{M/N}(E(R/\tau(R)))$, for some submodule N of M. Then $Hom(D/N, E(R/\tau(R))) = 0$ and

$$Hom(D'/D, E(R/\tau(R))) \neq 0$$

for every $D < D' \leq M$. In this case D is a τ-closure of N in M. □

Example 2.23. Consider the Goldie torsion theory, where $\mathbb{T} = \{M|\ \mathcal{Z}_2(M) = M\}$, $\mathbb{F} = \{M|\ \mathcal{Z}_2(M) = 0\}$. It is not hard to see that the idempotent radical associated to Goldie torsion theory is $\mathcal{Z}_2$. If we take $\mathbb{Z}$ as a $\mathbb{Z}$-module, then we can easily check that $\mathcal{Z}_2(\mathbb{Z}) = 0$ and $E(\mathbb{Z}) = \mathbb{Q}$. Since $Hom_{\mathbb{Z}}(n\mathbb{Z}, \mathbb{Q}) \neq 0$, for every nonzero integer n, the $\mathcal{Z}_2$-closure of zero submodule is itself. Since for every nonzero integer m we have $Hom_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q}) = 0$, the $\mathcal{Z}_2$-closure of every nonzero $\mathbb{Z}$-submodule is $\mathbb{Z}$.

3. τ-extending modules

In this section we introduce the concept of τ-extending modules and give an example to show that the direct sum of τ-extending modules may not be τ-extending.

Definition 3.1. A module M is called τ-extending if every submodule of M is τ-rational in a direct summand of M.

From Example 2.23, it follows that $\mathbb{Z}$ is $\mathcal{Z}_2$-extending module.

Proposition 3.2. *A module M is τ-extending if and only if every τ-closed submodule of M is a direct summand of M.*

Proof: Suppose that M is τ-extending and N a τ-closed submodule of M. By hypothesis, N is τ-rational in a direct summand D of M, so $D = N$.
Conversely, assume that every τ-closed submodule of M is a direct summand of M. Let N be a submodule of M. Also, let $Rej_{M/N}(E(R/\tau(R))) = C/N$. Since C is τ-closed in M, then by assumption C is a direct summand of M. As N is τ-rational in C, M is τ-extending. □

Lemma 3.3. *The class of τ-extending modules is closed under direct summands.*

Proof: Let $M = M_1 \oplus M_2$ and $N_1 \leq_{\tau c} M_1$. We show that $N_1 \oplus M_2 \leq_{\tau c} M_1 \oplus M_2$. Let there be a submodule K such that $N_1 \oplus M_2 \leq K \leq M$ and $Hom(K/(N_1 \oplus M_2), E(R/\tau(R))) = 0$. By modularity $K = M_2 \oplus (K \cap M_1)$ and so $Hom((K \cap M_1)/N_1, E(R/\tau(R))) = 0$. This gives $K \cap M_1 = N_1$, because

$N_1 \leq_{\tau c} M_1$. Thus $K = N_1 \oplus M_2$ and so $N_1 \oplus M_2 \leq_{\tau c} M$. As M is τ-extending, then $N_1 \oplus M_2 \oplus L = M$. Therefore $M_1 = N_1 \oplus (M_1 \cap (M_2 \oplus L))$, so M_1 is τ-extending. □

Lemma 3.4. *Let M be τ-extending and K a module for which $Rej_K(E(R/\tau(R))) = K$. Then $M \oplus K$ is τ-extending.*

Proof: We may assume that $Rej_M(E(R/\tau(R))) = 0$. Let D be a τ-closed submodule of $M \oplus K$. Then $Rej_{M\oplus K}(E(R/\tau(R))) = Rej_K(E(R/\tau(R))) = K$ and by Proposition 2.21, we get $K \subseteq D$. This shows that $D = (M \cap D) \oplus K$. If $Hom(L/(M \cap D), E(R/\tau(R))) = 0$, for some submodule L of M that contains $M \cap D$, then $Hom((L + K)/D, E(R/\tau(R))) = 0$. Since D is τ-closed in $M \oplus K$, we get $L + K = D$ and so $M \cap D = L$. This shows that $M \cap D$ is τ-closed in M and since M is τ-extending $M = (M \cap D) \oplus N$, for some $N \leq M$. Thus $M \oplus K = (M \cap D) \oplus N \oplus K = D \oplus N$, which implies that D is a direct summand of $M \oplus K$. □

Corollary 3.5. *Let M be a τ-extending module and K a τ-torsion module. Then $M \oplus K$ is τ-extending.*

Lemma 3.6. *The following statements are equivalent for a module M:*

(i) *M is τ-extending;*
(ii) *$M = Rej_M E(R/\tau(R)) \oplus N$, and N is a τ-extending submodule of M.*

Proof: (i)$\Longrightarrow$(ii). As $Rej_M E(R/\tau(R)) \leq_{\tau c} M$ and M is τ-extending, we get $M = Rej_M E(R/\tau(R)) \oplus N$, where N is a τ-extending submodule of M, by Lemma 3.3.

(ii)$\Longrightarrow$(i). This follows by Lemma 3.4. □

The following example shows that a direct sum of τ-extending modules need not be τ-extending.

Example 3.7. Let $R = \mathbb{Z}$ and $\tau = 0$. Then $M_1 = M_2 = \mathbb{Z}$ are τ-extending, because $Hom_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q}) = 0$, for every nonzero ideal $m\mathbb{Z}$ of $\mathbb{Z}$. Next we show that $M_1 \oplus M_2$ is not τ-extending. For, let K be the $\mathbb{Z}$-submodule of $\mathbb{Z}\oplus\mathbb{Z}$ generated by $(2, 3)$, i.e, $K = \{(2n, 3n)|\ n \in \mathbb{Z}\}$. Then $f : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Q}$ defined by $f(1, 0) = 1/2$, $f(0, 1) = -1/3$, is a $\mathbb{Z}$-homomorphism with $ker(f) = \{(m, n)|\ f(m, n) = f(m, 0) + f(0, n) = mf(1, 0) + nf(0, 1) = m/2 - n/3 = 0\}$. Therefore $K = ker(f)$ and so $Hom_{\mathbb{Z}}((\mathbb{Z} \oplus \mathbb{Z})/K, \mathbb{Q}) \neq 0$. This shows that $\mathbb{Z} \oplus \mathbb{Z}$ is not τ-extending.

Theorem 3.8. *Let M_i $(i = 1, 2)$ be τ-extending modules and N_i a submodule of M_i such that $M_i = Rej_{M_i} E(R/\tau(R)) \oplus N_i$ for each $i = 1, 2$. Then $M = M_1 \oplus M_2$ is τ-extending if and only if every τ-closed submodule K of $N_1 \oplus N_2$ with $K \cap N_1 = K \cap N_2 = 0$ is a direct summand of M.*

Proof: Assume that $M = M_1 \oplus M_2$ is τ-extending. Then

$$Rej_{M_1\oplus M_2}(E(R/\tau(R))) = Rej_{M_1}(E(R/\tau(R))) \oplus Rej_{M_2}(E(R/\tau(R)))$$

is a direct summand of M, by Lemma 3.6. Thus there exists $N \leq M_1 \oplus M_2$ such that

$$M_1 \oplus M_2 = Rej_{M_1}(E(R/\tau(R))) \oplus Rej_{M_2}(E(R/\tau(R))) \oplus N.$$

By modularity we get

$$M_i = Rej_{M_i}(E(R/\tau(R))) \oplus ((N \oplus Rej_{M_j}(E(R/\tau(R)))) \cap M_i)$$

for $i, j = 1, 2$ with $i \neq j$. Suppose that $N_i = (N \oplus Rej_{M_j}(E(R/\tau(R)))) \cap M_i$. Then $N = N_1 \oplus N_2$ is τ-extending, and hence every τ-closed submodule of N is a direct summand of N and so a direct summand of M.

Conversely, assume that for each $i = 1, 2$, the module $M_i = Rej_{M_i}E(R/\tau(R)) \oplus N_i$ is τ-extending, such that every τ-closed submodule K of $N_1 \oplus N_2$ with $K \cap N_1 = K \cap N_2 = 0$ is a direct summand of M. We will show that every τ-closed submodule of $M_1 \oplus M_2$ is a direct summand of $M_1 \oplus M_2$. Let K be a τ-closed submodule of $M_1 \oplus M_2$ and $K \cap M_i = K_i$, for $i = 1, 2$.

Note that $Hom((M_i + K)/K, E(R/\tau(R))) = 0$ iff $M_i \subseteq K$, for $i = 1, 2$. Thus $Hom(M_i/K_i, E(R/\tau(R))) = 0$ iff $M_i = K_i$, for $i = 1, 2$. It follows that K_i are τ-closed submodule of M_i, for $i = 1, 2$ and by Proposition 2.21(2), $Rej_{M_i}E(R/\tau(R)) \subseteq K_i$. Hence $K_i = Rej_{M_i}E(R/\tau(R)) \oplus (K_i \cap N_i)$. It is not hard to see that $K_i \cap N_i$ is a τ-closed submodule of N_i, for $i = 1, 2$.

Since N_1 and N_2 are τ-extending, $N_i = (K_i \cap N_i) \oplus L_i$, for some $L_i \subseteq N_i$. This shows that $K = K_1 \oplus K_2 \oplus (K \cap (L_1 \oplus L_2))$. We can easily check that $K \cap (L_1 \oplus L_2)$ is a τ-closed submodule of $N_1 \oplus N_2$ with $K \cap (L_1 \oplus L_2) \cap N_i = 0$, for $i = 1, 2$. By assumption $K \cap (L_1 \oplus L_2)$ is a direct summand of M. Assume that $(K \cap (L_1 \oplus L_2)) \oplus S = M$. Then $L_1 \oplus L_2 = (K \cap (L_1 \oplus L_2)) \oplus ((L_1 \oplus L_2) \cap S)$. It follows that $M = K_1 \oplus K_2 \oplus (K \cap (L_1 \oplus L_2)) \oplus ((L_1 \oplus L_2) \cap S) = K \oplus ((L_1 \oplus L_2) \cap S)$, which implies that every τ-closed submodule of M is a direct summand of M. Thus M is a τ-extending module, by Proposition 3.2. □

Lemma 3.9. *Let M_1 be an M_2-injective module. Then $M = M_1 \oplus M_2$ is τ-extending if and only if M_1, M_2 are τ-extending.*

Proof: Let M_i be τ-extending, for $i = 1, 2$. Then by Lemma 3.6, there exist $L_i \leq M_i$ such that $M_i = Rej_{M_i}(E(R/\tau(R))) \oplus L_i$. Applying Theorem 3.8, it suffices to show that every τ-closed submodule K of $L_1 \oplus L_2$, with $K \cap L_1 = K \cap L_2 = 0$, is a direct summand of $L_1 \oplus L_2$. As M_1 is an M_2-injective, then by [9, Lemma 7.5], there exists a submodule L' of $L_1 \oplus L_2$ for which $K \subseteq L'$ and $L_1 \oplus L' = L_1 \oplus L_2$. As L_2 is τ-extending and $L' \simeq L_2$, then by Proposition 3.2, L' is τ-extending. Hence K is a direct summand of L' and so a direct summand of $L_1 \oplus L_2$. By Proposition 3.2, $M_1 \oplus M_2$ is τ-extending. □

Corollary 3.10. *Let $M = M_1 \oplus M_2$ be an injective module. Then M is τ-extending if and only if M_1, M_2 are τ-extending modules.*

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