The regularity of the positive part of functions in $L^2(I;H^1(\Omega)) \cap H^1(I;H^1(\Omega)^*)$ with applications to parabolic equations

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Abstract. Let $u \in L^2(I;H^1(\Omega))$ with $\partial_t u \in L^2(I;H^1(\Omega)^*)$ be given. Then we show by means of a counter-example that the positive part u^+ of u has less regularity, in particular it holds $\partial_t u^+ \notin L^1(I;H^1(\Omega)^*)$ in general. Nevertheless, u^+ satisfies an integration-by-parts formula, which can be used to prove non-negativity of weak solutions of parabolic equations.

Keywords: Bochner integrable function; projection onto non-negative functions; parabolic equation

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1. Introduction

In this note, we are concerned with the regularity of the positive part of functions from the function space

$$W := \{u \in L^2(I;H^1(\Omega)) : \partial_t u \in L^2(I;H^1(\Omega)^*)\}$$

of Bochner integrable functions. Here, $I = (0,T)$, $T > 0$, is an open interval, and $H^1(\Omega)$ denotes the usual Sobolev space on the domain $\Omega \subset \mathbb{R}^n$; $\partial_t u$ denotes the weak derivative of u with respect to the time variable $t \in I$. The underlying spaces form a so-called evolution triple (or Gelfand triple) $H^1(\Omega) \subset L^2(\Omega) = L^2(\Omega)^* \subset H^1(\Omega)^*$ with continuous and dense embeddings. In the sequel, we will use the commonly applied abbreviations

$$V := H^1(\Omega), \quad H := L^2(\Omega).$$

For an introduction to this kind of function spaces and their various properties, we refer to e.g. [1, Section IV.1], [3, Section 7.2], [4, Chapter 25].

Let $u \in W$ be given. Let us denote its positive part by u^+,

$$u^+(t,x) = \max(u(t,x),\ 0),\ t \in I,\ x \in \Omega.$$

Due to the embedding $W \hookrightarrow L^2(I\times\Omega)$, the positive part is well-defined. Moreover, since the mapping $u \mapsto u^+$ is bounded from $H^1(\Omega)$ to $H^1(\Omega)$, it follows that for

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$u \in W$ also $u^+ \in L^2(I;V)$ holds. Here, the question arises whether $u \in W$ also implies $u^+ \in W$. The aim of the short note is to provide a counter-example of this claim, see Theorem 2.7. Nevertheless, the following integration-by-parts formula holds true for all $u \in W$

$$\int_I \langle u_t(s), u^+(s)\rangle_{V^*,V}\,\mathrm{d}s = \frac{1}{2}\|u^+(T)\|_H^2 - \frac{1}{2}\|u^+(0)\|_H^2, \tag{1}$$

which enables us to show positivity of weak solutions of linear parabolic equations, see Section 3.

2. The regularity of the positive part

In this section, we study the mapping properties of $u \mapsto u^+$. First, let us state the following well-known result:

Proposition 2.1. *The mapping $u \mapsto u^+$ is Lipschitz continuous as mapping from H to H. Furthermore it is bounded from V to V, and for $u \in V$ it holds*

$$\nabla u^+(x) = \begin{cases} \nabla u(x) & \text{if } u(x) > 0 \\ 0 & \text{if } u(x) \le 0 \end{cases}, \quad x \in \Omega,$$

which implies $\|u^+\|_V \le \|u\|_V$.

The following result is an obvious consequence.

Corollary 2.2. *Let $u \in W$ be given. Then $u^+ \in L^2(I;V) \cap C(\bar{I};H)$, and it holds*

$$\|u^+\|_{L^2(I;V)},\ \|u^+\|_{C(\bar{I};H)} \ \le\ \|u\|_W.$$

With the same arguments that are classically used to prove Proposition 2.1, one can prove

Corollary 2.3. *Let $u \in W$ be given with $u_t \in L^2(I;H)$. Then $u^+ \in W$ with $u_t^+ \in L^2(I;H)$.*

Moreover, in this case, we have $\partial_t u^+ \in L^2(I\times\Omega)$, and we can write for almost all $(t,x) \in I \times \Omega$

$$\partial_t u^+(t,x) = \begin{cases} \partial_t u(t,x) & \text{if } u(t,x) > 0 \\ 0 & \text{if } u(t,x) \le 0. \end{cases} \tag{2}$$

Now, if $\partial_t u$ is in $L^2(I;V^*)$ only, the representation (2) makes no sense, as $\partial_t u(t,\cdot)$ is only in $H^1(\Omega)^*$ for almost all t.

In the following, we will construct a function $u \in W$ with $\partial_t u \notin L^2(I;H)$ such that $\partial_t u^+ \notin L^2(I;V^*)$. The key idea is the observation that the mapping $u \mapsto u^+$ for $u \in L^2(\Omega)$ is *not* bounded as mapping from $H^1(\Omega)^*$ to $H^1(\Omega)^*$.

To see this, set $\Omega=(0,1)$. Let us define $\psi_n(x)=\sin(2\pi nx)$. Then it is well-known that ψ_n converges weakly to zero in $L^2(\Omega)$, thus strongly to zero in $H^1(\Omega)^*$. However, a short computation shows that

$$\int_0^1 \psi_n^+(x)\,\mathrm{d}x=\int_0^1 \psi_1^+(x)\,\mathrm{d}x=\int_0^{1/2}\sin(2\pi x)dx=\frac{1}{\pi}\neq 0,$$

which implies that ψ_n^+ converges weakly to the constant function $\hat{\psi}(x)=1/\pi$ in $L^2(\Omega)$. Hence, ψ_n^+ cannot converge to zero in $H^1(\Omega)^*$.

In the sequel, we will equip V with the scalar product $(u,v)_V:=\int_\Omega \nabla u\cdot\nabla v+u\cdot v\,\mathrm{d}x$ and the associated norm. The space H is equipped with the standard $L^2(\Omega)$ inner product and norm. We consider the family of functions

$$\psi_n(x):=\cos(n\pi x),\ \ x\in\Omega \tag{3}$$

for $n\in\mathbb{N}$. Now, we will derive quantitative estimates of the norm of ψ_n in V, H, and V^* for $n\to\infty$.

Lemma 2.4. *Let $n\in\mathbb{N}$ be given. Then it holds*

$$\|\psi_n\|_V=\left(\frac{n^2\pi^2+1}{2}\right)^{1/2}\le n\pi,\quad \|\psi_n\|_H=\frac{1}{\sqrt{2}},\quad \|\psi_n\|_{V^*}\le\frac{1}{\sqrt{2}\,n\pi}\,.$$

Proof: The first two identities can be verified with elementary calculations. To prove the third, consider the solution $z\in V$ of $(z,v)_V=(\psi_n,v)_H$ for all $v\in V$. Then it follows $\|\psi_n\|_{V*}=\|z\|_V$. The function z is given by $z=\frac{1}{n^2\pi^2+1}\psi_n$, and hence the third estimate follows from the first. ☐

Let us show that the V^*-norm of ψ_n^+ is bounded away from zero.

Lemma 2.5. *There is $C>0$ such that*

$$\|\psi_n^+\|_{V^*}\ge C\quad\forall n.$$

Proof: Let $e\in H$ be defined by $e(x)=1$. Then we have

$$\begin{aligned}(\psi_n^+,e)_H&=\int_0^1\psi_n^+(x)\,\mathrm{d}x=\int_0^1(\cos(n\pi x))^+\,\mathrm{d}x\\&=n\int_0^{1/2n}\cos(n\pi x)\,\mathrm{d}x=\frac{1}{\pi}\,.\end{aligned}$$

Let now $v_e\in V$ be defined by $v_e(x)=\min(4x,\,1,\,4(1-x))$. Then it holds $\|v_e-e\|_H^2=2\int_0^{1/4}(4x)^2\,\mathrm{d}x=\frac{1}{6}$. Thus, we can estimate

$$\langle\psi_n^+,v_e\rangle_{V^*,V}\ge(\psi_n^+,e)_H-\|\psi_n^+\|_H\|v-e_e\|_H\ge\frac{1}{\pi}-\frac{1}{\sqrt{12}}=0.0296\dots\ge\frac{1}{5}\,.$$

Here, we used $\|\psi_n^+\|_H \le \|\psi_n\|_H = 1/\sqrt{2}$. The lower bound implies that $\|\psi_n^+\|_{V^*} \ge \frac{1}{5}\|v_e\|_V^{-1}$, and the claim is proven. □

Let us now introduce a family of functions on small time intervals, which will be used to define the counterexample by means of an infinite series.

Lemma 2.6. *Let $I := (0,1)$. Let $\phi \in H_0^1(I)$ be given. Define*

$$\phi_n(t) := n(n+1) \cdot \phi(n(n+1)t - n). \tag{4}$$

Then it holds $\operatorname{supp} \phi_n \subset (\frac{1}{n+1}, \frac{1}{n})$ *and*

$$\begin{aligned} \|\phi_n\|_{L^1(I)} &= \|\phi\|_{L^1(I)}, & \|\partial_t\phi_n\|_{L^1(I)} &\ge n^2\|\partial_t\phi\|_{L^1(I)}, \\ \|\phi_n\|_{L^2(I)} &\le \sqrt{2}n\|\phi\|_{L^2(I)}, & \|\partial_t\phi_n\|_{L^2(I)} &\le \sqrt{2}n^3\|\partial_t\phi\|_{L^2(I)}. \end{aligned}$$

Proof: This follows by elementary calculations. □

Let us now define the function

$$u(x,t) = \sum_{n=1}^{\infty} n^{-3}\phi_n(t)\psi_n(x). \tag{5}$$

Theorem 2.7. *Let $\phi \in H_0^1(I) \setminus \{0\}$ be given with $\phi \ge 0$. Then the function u defined in* (5) *with ψ_n and ϕ_n from* (3) *and* (4)*, respectively, belongs to W. However, the time derivative of its positive part $\partial_t u^+$ does not belong to $L^1(I;V^*)$.*

Proof: Let us define the partial sum $u_N := \sum_{n=1}^N \phi_n(t)\psi_n(x)$. We will exploit the fact that the supports of the functions ϕ_n are distinct. From the Lemmas 2.4, 2.5, and 2.6, we have

$$\begin{aligned} \|u_N\|^2_{L^2(I;V)} &= \sum_{n=1}^N n^{-6}\|\phi_n\|^2_{L^2(I)}\|\psi_n\|^2_V \le c\sum_{n=1}^N n^{-6}\cdot n^2\cdot n^2 = c\sum_{n=1}^N n^{-2}, \\ \|\partial_t u_N\|^2_{L^2(I;V^*)} &= \sum_{n=1}^N n^{-6}\|\partial_t\phi_n\|^2_{L^2(I)}\|\psi_n\|^2_{V^*} \le c\sum_{n=1}^N n^{-6}\cdot n^6\cdot n^{-2} = c\sum_{n=1}^N n^{-2}, \\ \|\partial_t u_N^+\|_{L^1(I;V^*)} &= \sum_{n=1}^N n^{-3}\|\partial_t\phi_n\|_{L^1(I)}\|\psi_n^+\|_{V^*} \ge c\sum_{n=1}^N n^{-3}\cdot n^2\cdot 1 = c\sum_{n=1}^N n^{-1}. \end{aligned}$$

This proves that (u_N) strongly converges in W to u. Since $u = u_N$ on $(\frac{1}{n+1}, 1)$, the weak derivative $\partial_t u^+$ exists almost everywhere on I, and belongs to the space $L^1_{\mathrm{loc}}(I;V^*)$. Suppose that $\partial_t u^+ \in L^1(I;V^*)$ holds. Then by the continuity of the integral it follows

$$\|\partial_t u^+\|_{L^1(I;V^*)} = \lim_{N\to\infty}\int_{1/(N+1)}^1 \|\partial_t u^+(t)\|_{V^*}\,\mathrm{d}t = \lim_{N\to\infty}\|\partial_t u_N\|_{L^1(I;V^*)} \to \infty,$$

which is a contradiction, hence $\partial_t u^+ \notin L^1(I;V^*)$. □

3. Positivity of weak solutions to parabolic equations

Let $\Omega \subset \mathbb{R}^n$ be a domain. Again, we make use of the evolution triple $V = H^1(\Omega)$, $H = L^2(\Omega)$, $V^* = (H^1(\Omega)^*)$. Due to the counter-example in the previous section, we cannot apply the well-known integration-by-parts results for functions in W to u^+. In order to prove formula (1), we recall the following density result

Proposition 3.1 ([3, Lemma 7.2]). *The space $C^\infty([0,T],V)$ is dense in W.*

First, let us prove the integration-by-parts formula for smooth u.

Lemma 3.2. *Let $u \in W$ with $\partial_t u \in L^2(I;L^2(\Omega))$ be given. Then it holds*

$$\begin{aligned}\int_0^T \langle \partial_t u(t),\ u^+(t)\rangle_{V^*,V}\,\mathrm{d}t &= \frac{1}{2}\int_0^T \partial_t\|u^+(t)\|_H^2 \\ &= \frac{1}{2}\left(\|u^+(t)\|_H^2 - \|u^+(0)\|_H^2\right).\end{aligned} \tag{6}$$

Proof: Since $\partial_t u \in L^2(I;L^2(\Omega))$, it holds $\partial_t u^+ \in L^2(I;L^2(\Omega))$. With the representation (2) it follows

$$\int_I\int_\Omega \partial_t u(x,t)u^+(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_I\int_\Omega \partial_t u^+(x,t)u^+(x,t)\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\int_0^T \partial_t\|u^+(t)\|_H^2\,\mathrm{d}t,$$

which proves the claim. □

Lemma 3.3. *Let $u \in W$ be given. Then it holds*

$$\int_0^T \langle \partial_t u(t),\ u^+(t)\rangle_{V^*,V}\,\mathrm{d}t = \frac{1}{2}\int_0^T \partial_t\|u^+(t)\|_H^2 = \frac{1}{2}\left(\|u^+(t)\|_H^2 - \|u^+(0)\|_H^2\right).$$

Proof: Let $u \in W$ be given. By density, there is (u_k) in $C^\infty([0,T],V)$ with $u_k \to u$ in W. By continuity of the projection, it follows $u_k^+ \to u^+$ in $C([0,T],H)$.

Moreover, the sequence u_k^+ is bounded in $L^2(V)$. Hence, there is a weakly converging subsequence with weak limit $\tilde{u}$ in $L^2(V)$. Due to $u_k^+ \to u^+$ in $C([0,T],H)$, it follows $\tilde{u} = u^+$, and the whole sequence converges weakly, $u_k^+ \rightharpoonup u^+$ in $L^2(V)$.

Since u_k is smooth enough, u_k satisfies (6). Moreover, the left-hand side and the right-hand side in (6) converge for $k \to \infty$, proving the claim. □

Let us remark that this result can be proven using difference quotients, see e.g. [2, Lemma 2.5].

The integration-by-parts formula (1) can be applied to prove non-negativity of weak solutions of parabolic equations with non-negative data. Let $f \in L^1(I;L^2) + L^2(I;V')$ and $u_0 \in H$ be given. Then $u \in W$ is a weak solution of the parabolic equation with homogeneous Neumann boundary conditions

$$\partial_t u - \Delta u = f \text{ on } I \times \Omega, \quad \partial_n u = 0 \text{ on } I \times \partial\Omega, \quad u(0) = u_0(x), \tag{7}$$

if the following equation is satisfied for all $v \in V$ and almost all $t \in I$

$$\langle \partial u(t),\, v\rangle_{V^*,V} + \int_\Omega \nabla u(x,t)\nabla v(x)\,\mathrm{d}x = \langle f(t),\, v\rangle_{V^*,V}.$$

Theorem 3.4. *Let $f \in L^1(I; L^2(\Omega)) + L^2(I; V^*)$ be given, with $f \geq 0$, which is $\langle f, v\rangle \geq 0$ for all $v \in L^2(V) \cap C(I; H)$ with $v \geq 0$. Let $u_0 \in H$ be given with $u_0 \geq 0$. Let u be a weak solution of the parabolic equation* (7)*. Then it holds $u \geq 0$.*

Proof: Let us denote $u^- = -(-u)^+ \in L^2(V) \cap C(I; H)$. Testing the weak formulation with u^-, integrating from 0 to t, and using Proposition 2.1 and Lemma 3.3 yields

$$\begin{aligned}
0 &\geq \int_0^t \langle f(s), u^-(s)\rangle_{V^*,V}\,\mathrm{d}s \\
&= \int_0^t \langle \partial_t u(s),\, u^-(s)\rangle_{V^*,V}\,\mathrm{d}s + \int_0^t \int_\Omega \nabla u(x,s)\nabla u^-(x,s)\,\mathrm{d}x\,\mathrm{d}s \\
&= \frac{1}{2}\left(\|u^-(t)\|_H^2 - \|u^-(0)\|_H^2\right) + \|\nabla u^-\|^2_{L^2(0,t;L^2(\Omega))} \\
&\geq \frac{1}{2}\|u^-(t)\|_H^2.
\end{aligned}$$

Hence, it follows $u^-(t) = 0$ for almost all $t \in I$, which implies $u^- = 0$ almost everywhere on $I \times \Omega$. $\square$

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