Several quantitative characterizations of some specific groups

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Abstract. Let G be a finite group and let $\pi(G) = \{p_1, p_2, \ldots, p_k\}$ be the set of prime divisors of $|G|$ for which $p_1 < p_2 < \cdots < p_k$. The Gruenberg-Kegel graph of G, denoted $\mathrm{GK}(G)$, is defined as follows: its vertex set is $\pi(G)$ and two different vertices p_i and p_j are adjacent by an edge if and only if G contains an element of order p_ip_j. The degree of a vertex p_i in $\mathrm{GK}(G)$ is denoted by $d_G(p_i)$ and the k-tuple $D(G) = (d_G(p_1), d_G(p_2), \ldots, d_G(p_k))$ is said to be the degree pattern of G. Moreover, if $\omega \subseteq \pi(G)$ is the vertex set of a connected component of $\mathrm{GK}(G)$, then the largest ω-number which divides $|G|$, is said to be an order component of $\mathrm{GK}(G)$. We will say that the problem of OD-characterization is solved for a finite group if we find the number of pairwise non-isomorphic finite groups with the same order and degree pattern as the group under study. The purpose of this article is twofold. First, we completely solve the problem of OD-characterization for every finite non-abelian simple group with orders having prime divisors at most 29. In particular, we show that there are exactly two non-isomorphic finite groups with the same order and degree pattern as $U_4(2)$. Second, we prove that there are exactly two non-isomorphic finite groups with the same order components as $U_5(2)$.

Keywords: OD-characterization of finite group; prime graph; degree pattern; simple group; 2-Frobenius group

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1. Introduction

Throughout this article, all the groups under consideration are *finite*, and simple groups are *non-abelian*. Given a group G, the spectrum $\omega(G)$ of G is the set of orders of elements in G. Clearly, the spectrum $\omega(G)$ is closed and partially ordered by the divisibility relation, and hence is uniquely determined by the set $\mu(G)$ of its elements which are maximal under the divisibility relation. If n is a natural number, then $\pi(n)$ denotes the set of all prime divisors of n, in particular, we set $\pi(G) = \pi(|G|)$.

One of the most well-known graphs associated with G is the *Gruenberg-Kegel graph* (or *prime graph*) denoted by $\mathrm{GK}(G)$. The vertex set of this graph is $\pi(G)$ and two distinct vertices p and q are joined by an edge (abbreviated $p \sim q$) if and only if $pq \in \omega(G)$. The number of connected components of $\mathrm{GK}(G)$ is denoted

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by $s(G)$ and the sets of vertices of connected components of $\mathrm{GK}(G)$ are denoted as $\pi_i = \pi_i(G) (i = 1, 2, \ldots, s(G))$. If G is a group of even order, then we put $2 \in \pi_1(G)$. The vertex sets of connected components of all finite simple groups are obtained in [16] and [36].

Given a group G, suppose that $\pi(G) = \{p_1, p_2, \ldots, p_k\}$ in which $p_1 < p_2 < \cdots < p_k$. The *degree* $d_G(p_i)$ of a vertex p_i in the prime graph $\mathrm{GK}(G)$ is the number of edges incident on p_i. We define $D(G) = (d_G(p_1), d_G(p_2), \ldots, d_G(p_k))$, and we call this k-tuple the *degree pattern of* G. In addition, we denote by $\mathcal{OD}(G)$ the set of pairwise non-isomorphic finite groups with the same order and degree pattern as G, and we put $h_{\mathrm{OD}}(G) = |\mathcal{OD}(G)|$. Clearly, there are only finitely many isomorphism types of groups of order $|G|$, because there are just finitely many ways that an $|G| \times |G|$ multiplication table can be filled in. Finally, for each group G, it is clear that $1 \leq h_{\mathrm{OD}}(G) < \infty$.

Definition 1.1. A group G is called *k-fold OD-characterizable* if $h_{\mathrm{OD}}(G) = k$. Usually, a 1-fold OD-characterizable group is simply called *OD-characterizable*, and it is called *quasi OD-characterizable* if it is k-fold OD-characterizable for some $k > 1$.

We will say that the OD-characterization problem is solved for a group G if the value of $h_{\mathrm{OD}}(G)$ is known. Studies in recent years by several researchers have shown that many simple groups are OD-characterizable. Some of these results are summarized in Table 1.

In connection with the simple groups which are quasi OD-characterizable, it was shown in [4], [29] and [30] that:

$$\begin{aligned}
\mathcal{OD}(\mathbb{A}_{10}) &= \{\mathbb{A}_{10},\ \mathbb{Z}_3 \times J_2\}, \\
\mathcal{OD}(S_6(5)) &= \{S_6(5), O_7(5)\}, \\
\mathcal{OD}(S_{2m}(q)) &= \{S_{2m}(q), O_{2m+1}(q)\}, \quad m = 2^f \geq 2, \ \left|\pi\left(\frac{q^m+1}{2}\right)\right| = 1, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad q \text{ odd prime power}, \\
\mathcal{OD}(S_{2p}(3)) &= \{S_{2p}(3), O_{2p+1}(3)\}, \quad \left|\pi\left(\frac{3^p-1}{2}\right)\right| = 1, \ p \text{ odd prime.}
\end{aligned}$$

In addition to the above results, it has been shown that in [22] there exist many infinite families of alternating and symmetric groups, $\{\mathbb{A}_n\}$ and $\{\mathbb{S}_n\}$, which are quasi OD-characterizable, with $h_{\mathrm{OD}}(G) \geq 3$.

Here we consider the simple groups S such that $\pi(S) \subseteq \pi(29!)$, and we denote the set of all these simple groups by $\mathcal{S}_{\leq 29}$. Using the classification of finite simple groups it is not hard to obtain a full list of all groups in $\mathcal{S}_{\leq 29}$. Actually, there are 110 such groups (see [24, Table 4] or [40, Table 1]). For convenience, the groups S in $\mathcal{S}_{\leq 29}$ and their orders are listed in Table 2. The comparison between simple groups listed in Table 1 and the simple groups in $\mathcal{S}_{\leq 29}$, shows that there are only 5 groups in $\mathcal{S}_{\leq 29}$, that is $L_3(11)$, $U_4(2^3)$, ${}^2E_6(2)$, $S_4(17)$ and $U_4(17)$, for which

Table 1. Some OD-characterizable groups.

G	Conditions on G	$h_{\mathrm{OD}}(G)$	References
$\mathbb{A}_n$	$n = p, p+1, p+2$ (p a prime, $p \geq 5$)	1	[27], [30]
	$5 \leq n \leq 100, n \neq 10$	1	[9], [15], [24], [28], [31]
	$n = 106, 112, 116, 134$	1	[37], [38]
$L_2(q)$	$q \neq 2, 3$	1	[30], [43]
$L_3(q)$	$\left\lvert \pi\left(\frac{q^2+q+1}{d}\right) \right\rvert = 1$, $d = (3, q-1)$	1	[30]
$L_4(q)$	$q \leq 17$ and $q = 19, 23, 27, 29, 31, 32, 37$	1	[1], [3], [5]
$L_n(2)$	$n = p$ or $p+1$, $2^p - 1$ is Mersenne prime	1	[5]
$L_n(2)$	$n = 9, 10, 11$	1	[13], [26]
$L_3(9)$		1	[32]
$L_6(3)$		1	[2]
$U_3(q)$	$\left\lvert \pi\left(\frac{q^2-q+1}{d}\right) \right\rvert = 1$, $d = (3, q+1)$, $q > 5$	1	[30]
$U_4(q)$	$q = 5, 7$	1	[2], [5]
$U_6(2)$		1	[42]
$R(q)$	$\lvert\pi(q \pm \sqrt{3q} + 1)\rvert = 1, q = 3^{2m+1}, m \geq 1$	1	[30]
$\mathrm{Sz}(q)$	$q = 2^{2n+1} \geq 8$	1	[30]
$O_5(q) \cong S_4(q)$	$\lvert\pi((q^2+1)/2)\rvert = 1$, $q \neq 3$	1	[4]
$O_{2n+1}(q) \cong S_{2n}(q)$	$n = 2^m \geq 2$, $2 \mid q$, $\lvert\pi(q^n+1)\rvert = 1$, $(n, q) \neq (2, 2)$	1	[4]
$S_6(4)$		1	[21]
G	G is a sporadic group	1	[30]
G	$\lvert G\rvert \leq 10^8, G \neq \mathbb{A}_{10}, U_4(2)$	1	[33]
G	$\lvert\pi(G)\rvert = 4, G \neq \mathbb{A}_{10}$	1	[41]
G	G is a simple with $\pi_1(G) = \{2\}$	1	[27]
G	G is a simple with $\pi(G) \subseteq \pi(17!), G \neq \mathbb{A}_{10}, U_4(2)$	1	[25]

the OD-characterization problem has not been solved. Therefore, one of the aims of this article is to prove these groups are OD-characterizable.

Theorem 1.2. *The simple groups $L_3(11)$, $U_4(2^3)$, ${}^2E_6(2)$, $S_4(17)$ and $U_4(17)$ are OD-characterizable.*

We recall that a group G is called a 2-Frobenius group if $G = ABC$, where A and AB are normal subgroups of G, B is a normal subgroup of BC, and AB and BC are Frobenius groups. Zinov'eva and V.D. Mazurov observed that the prime graph of a 2-Frobenius group is always disconnected, more precisely, it is the union of two connected components each of which is a complete graph [45, Lemma 3(a)]. On the other hand, Mazurov constructed a 2-Frobenius group of the same order as the simple group $U_4(2)$ ([20], [44]). In particular, this shows that $h_{\mathrm{OD}}(U_4(2)) \geq 2$ (see also [33]). In this article we also prove the following result.

Theorem 1.3. *The simple group $U_4(2)$ is 2-fold OD-characterizable. In fact, there exists a unique 2-Frobenius group $F = (2^4 \times 3^4) : 5 : 4$ with the same order and degree pattern as $U_4(2)$, and so $\mathcal{OD}(U_4(2)) = \{U_4(2), F\}$.*

As an immediate consequence of Theorems 1.2, 1.3 and the results in [20], [29], [30], we have the following corollary.

Corollary 1.4. *All simple groups in the class $\mathcal{S}_{\leq 29}$, other than $\mathbb{A}_{10}$, $S_6(3)$, $O_7(3)$ and $U_4(2)$, are OD-characterizable. In addition, each of these groups is 2-fold OD-characterizable.*

Given a group G, the order of G can be expressed as a product of some coprime natural numbers $m_i = m_i(G)$, $i = 1, 2, \ldots, s(G)$, with $\pi(m_i) = \pi_i$. The numbers $m_1, m_2, \ldots, m_{s(G)}$ are called the *order components of* G. We set

$$\mathrm{OC}(G) = \left\{m_1, m_2, \ldots, m_{s(G)}\right\}.$$

In the similar manner, we denote by $\mathcal{OC}(G)$ the set of isomorphism classes of finite groups with the same set $\mathrm{OC}(G)$ of order components, and we put $h_{\mathrm{OC}}(G) = |\mathcal{OC}(G)|$. Again, in terms of function $h_{\mathrm{OC}}(\cdot)$, the groups G are classified as follows:

Definition 1.5. A group G is called *k-fold OC-characterizable*, if $h_{\mathrm{OC}}(G) = k$. Usually, a 1-fold OC-characterizable group is simply called *OC-characterizable*, and it is called *quasi OC-characterizable* if it is k-fold OC-characterizable for some $k > 1$.

Obviously, if p is a prime number, then $h_{\mathrm{OC}}(\mathbb{Z}_p) = 1$ while $h_{\mathrm{OC}}(\mathbb{Z}_{p^2}) = h_{\mathrm{OC}}(\mathbb{Z}_p \times \mathbb{Z}_p) = 2$. Examples of OC-characterizable groups are abundant (see for instance, [10], [11], [12] and [14]). Also, one family examples of simple groups S with $h_{\mathrm{OC}}(S) = 2$ is given in [12], namely

$$\mathcal{OC}(O_{2n+1}(q)) = \mathcal{OC}(S_{2n}(q)) = \{O_{2n+1}(q), S_{2n}(q)\} \quad (q \text{ odd}, \ n = 2^m \geq 4).$$

As the reader might have noticed, the values of the functions h_{OD} and h_{OC} may be different. For example, there are exactly two non-isomorphic groups of order $1814400 = 2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ and degree pattern $(2, 3, 2, 1)$, they are $\mathbb{A}_{10}$ and $\mathbb{Z}_3 \times J_2$, and hence $h_{\mathrm{OD}}(\mathbb{A}_{10}) = 2$. However, since the prime graph $\mathrm{GK}(\mathbb{A}_{10})$ is connected, $\mathrm{OC}(\mathbb{A}_{10}) = \{|\mathbb{A}_{10}|\}$, and so we obtain $h_{\mathrm{OC}}(\mathbb{A}_{10}) > \nu_{\mathrm{a}}(|\mathbb{A}_{10}|) = 150$, where $\nu_{\mathrm{a}}(m)$ denotes the number of types of abelian groups of order m. Therefore, we have $h_{\mathrm{OD}}(\mathbb{A}_{10}) \neq h_{\mathrm{OC}}(\mathbb{A}_{10})$. The simple group $U_5(2)$ is another example of this type. On the one hand, we have $h_{\mathrm{OD}}(U_5(2)) = 1$ by Theorem 3.3 in [41]. On the other hand, there exists a 2-Frobenius group F such that $|F| = |U_5(2)|$ (see [20]) which implies that $h_{\mathrm{OC}}(U_5(2)) \geq 2$. Hence, $h_{\mathrm{OD}}(U_5(2)) < h_{\mathrm{OC}}(U_5(2))$. Finally, we show the following.

Theorem 1.6. *The simple group $U_5(2)$ is 2-fold OC-characterizable. In fact, there exists a unique 2-Frobenius group $F = (2^{10} \times 3^5) : 11 : 5$ with the same order components as $U_5(2)$, and so we have $\mathcal{OC}(U_5(2)) = \{U_5(2), F\}$.*

It is worth noting that the pair $\{U_5(2), (2^{10} \times 3^5) : 11 : 5\}$ is the first pair of a finite simple group and a solvable group with the same order components. Note that these groups have different prime graphs: the first connected component of $\mathrm{GK}(U_5(2))$ is the path $2 \sim 3 \sim 5$, while the first connected component of $(2^{10} \times 3^5) : 11 : 5$ is the complete subgraph $2 \sim 3 \sim 5 \sim 2$.

Table 2. Simple groups with orders having prime divisors at most 29 except alternating ones.

S	$\lvert S\rvert$	S	$\lvert S\rvert$
$U_4(2)$	$2^6\cdot 3^4\cdot 5$	Fi_{22}	$2^{17}\cdot 3^9\cdot 5^2\cdot 7\cdot 11\cdot 13$
$L_2(7)$	$2^3\cdot 3\cdot 7$	$L_2(17)$	$2^4\cdot 3^2\cdot 17$
$L_2(2^3)$	$2^3\cdot 3^2\cdot 7$	$L_2(2^4)$	$2^4\cdot 3\cdot 5\cdot 17$
$U_3(3)$	$2^5\cdot 3^3\cdot 7$	$S_4(2^2)$	$2^8\cdot 3^2\cdot 5^2\cdot 17$
$L_2(7^2)$	$2^4\cdot 3\cdot 5^2\cdot 7^2$	He	$2^{10}\cdot 3^3\cdot 5^2\cdot 7^3\cdot 17$
$U_3(5)$	$2^4\cdot 3^2\cdot 5^3\cdot 7$	$O_8^-(2)$	$2^{12}\cdot 3^4\cdot 5\cdot 7\cdot 17$
$L_3(2^2)$	$2^6\cdot 3^2\cdot 5\cdot 7$	$L_4(2^2)$	$2^{12}\cdot 3^4\cdot 5^2\cdot 7\cdot 17$
J_2	$2^7\cdot 3^3\cdot 5^2\cdot 7$	$S_8(2)$	$2^{16}\cdot 3^5\cdot 5^2\cdot 7\cdot 17$
$U_4(3)$	$2^7\cdot 3^6\cdot 5\cdot 7$	$U_4(2^2)$	$2^{12}\cdot 3^2\cdot 5^3\cdot 13\cdot 17$
$S_4(7)$	$2^8\cdot 3^2\cdot 5^2\cdot 7^4$	$U_3(17)$	$2^6\cdot 3^4\cdot 7\cdot 13\cdot 17^3$
$S_6(2)$	$2^9\cdot 3^4\cdot 5\cdot 7$	$O_{10}^-(2)$	$2^{20}\cdot 3^6\cdot 5^2\cdot 7\cdot 11\cdot 17$
$O_8^+(2)$	$2^{12}\cdot 3^5\cdot 5^2\cdot 7$	$L_2(13^2)$	$2^3\cdot 3\cdot 5\cdot 7\cdot 13^2\cdot 17$
$L_2(11)$	$2^2\cdot 3\cdot 5\cdot 11$	$S_4(13)$	$2^6\cdot 3^2\cdot 5\cdot 7^2\cdot 13^4\cdot 17$
M_{11}	$2^4\cdot 3^2\cdot 5\cdot 11$	$L_3(2^4)$	$2^{12}\cdot 3^2\cdot 5^2\cdot 7\cdot 13\cdot 17$
M_{12}	$2^6\cdot 3^3\cdot 5\cdot 11$	$S_6(2^2)$	$2^{18}\cdot 3^4\cdot 5^3\cdot 7\cdot 13\cdot 17$
$U_5(2)$	$2^{10}\cdot 3^5\cdot 5\cdot 11$	$O_8^+(2^2)$	$2^{24}\cdot 3^5\cdot 5^4\cdot 7\cdot 13\cdot 17^2$
M_{22}	$2^7\cdot 3^2\cdot 5\cdot 7\cdot 11$	$F_4(2)$	$2^{24}\cdot 3^6\cdot 5^2\cdot 7^2\cdot 13\cdot 17$
M^cL	$2^7\cdot 3^6\cdot 5^3\cdot 7\cdot 11$	${}^2E_6(2)$	$2^{36}\cdot 3^9\cdot 5^2\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19$
HS	$2^9\cdot 3^2\cdot 5^3\cdot 7\cdot 11$	$U_3(2^3)$	$2^9\cdot 3^4\cdot 7\cdot 19$
$U_6(2)$	$2^{15}\cdot 3^6\cdot 5\cdot 7\cdot 11$	$U_4(2^3)$	$2^{18}\cdot 3^7\cdot 5\cdot 7^2\cdot 13\cdot 19$
$L_3(3)$	$2^4\cdot 3^3\cdot 13$	$L_3(7)$	$2^5\cdot 3^2\cdot 7^3\cdot 19$
$L_2(5^2)$	$2^3\cdot 3\cdot 5^2\cdot 13$	$L_4(7)$	$2^9\cdot 3^4\cdot 5^2\cdot 7^6\cdot 19$
$U_3(2^2)$	$2^6\cdot 3\cdot 5^2\cdot 13$	$L_3(11)$	$2^4\cdot 3\cdot 5^2\cdot 7\cdot 11^3\cdot 19$
$S_4(5)$	$2^6\cdot 3^2\cdot 5^4\cdot 13$	$L_2(19)$	$2^2\cdot 3^2\cdot 5\cdot 19$
$L_4(3)$	$2^7\cdot 3^6\cdot 5\cdot 13$	$U_3(19)$	$2^5\cdot 3^2\cdot 5^2\cdot 7^3\cdot 19^3$
${}^2F_4(2)'$	$2^{11}\cdot 3^3\cdot 5^2\cdot 13$	J_1	$2^3\cdot 3\cdot 5\cdot 7\cdot 11\cdot 19$
$L_2(13)$	$2^2\cdot 3\cdot 7\cdot 13$	J_3	$2^7\cdot 3^5\cdot 5\cdot 17\cdot 19$
$L_2(3^3)$	$2^2\cdot 3^3\cdot 7\cdot 13$	F_5	$2^{14}\cdot 3^6\cdot 5^6\cdot 7\cdot 11\cdot 19$
$G_2(3)$	$2^6\cdot 3^6\cdot 7\cdot 13$	$L_2(23)$	$2^3\cdot 3\cdot 11\cdot 23$
${}^3D_4(2)$	$2^{12}\cdot 3^4\cdot 7^2\cdot 13$	$U_3(23)$	$2^7\cdot 3^2\cdot 11\cdot 13^2\cdot 23^3$
$\mathrm{Sz}(2^3)$	$2^6\cdot 5\cdot 7\cdot 13$	M_{23}	$2^7\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 23$
$L_2(2^6)$	$2^6\cdot 3^2\cdot 5\cdot 7\cdot 13$	M_{24}	$2^{10}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 23$
$U_4(5)$	$2^7\cdot 3^4\cdot 5^6\cdot 7\cdot 13$	Co_1	$2^{21}\cdot 3^9\cdot 5^4\cdot 7^2\cdot 11\cdot 13\cdot 23$
$L_3(3^2)$	$2^7\cdot 3^6\cdot 5\cdot 7\cdot 13$	Co_2	$2^{18}\cdot 3^6\cdot 5^3\cdot 7\cdot 11\cdot 23$
$S_6(3)$	$2^9\cdot 3^9\cdot 5\cdot 7\cdot 13$	Co_3	$2^{10}\cdot 3^7\cdot 5^3\cdot 7\cdot 11\cdot 23$
$O_7(3)$	$2^9\cdot 3^9\cdot 5\cdot 7\cdot 13$	Fi_{23}	$2^{18}\cdot 3^{13}\cdot 5^2\cdot 7\cdot 11\cdot 13\cdot 17\cdot 23$
$G_2(2^2)$	$2^{12}\cdot 3^3\cdot 5^2\cdot 7\cdot 13$	$U_4(17)$	$2^{11}\cdot 3^7\cdot 5\cdot 7\cdot 13\cdot 17^6\cdot 29$
$S_4(2^3)$	$2^{12}\cdot 3^4\cdot 5\cdot 7^2\cdot 13$	$S_4(17)$	$2^{10}\cdot 3^4\cdot 5\cdot 17^4\cdot 29$
$O_8^+(3)$	$2^{12}\cdot 3^{12}\cdot 5^2\cdot 7\cdot 13$	$L_2(17^2)$	$2^5\cdot 3^2\cdot 5\cdot 17^2\cdot 29$
$L_5(3)$	$2^9\cdot 3^{10}\cdot 5\cdot 11^2\cdot 13$	$L_2(29)$	$2^2\cdot 3\cdot 5\cdot 7\cdot 29$
$L_6(3)$	$2^{11}\cdot 3^{15}\cdot 5\cdot 7\cdot 11^2\cdot 13^2$	Ru	$2^{14}\cdot 3^3\cdot 5^3\cdot 7\cdot 13\cdot 29$
Suz	$2^{13}\cdot 3^7\cdot 5^2\cdot 7\cdot 11\cdot 13$	Fi_{24}'	$2^{21}\cdot 3^{16}\cdot 5^2\cdot 7^3\cdot 11\cdot 13\cdot 17\cdot 23\cdot 29$

We introduce some more notation. Let Γ be a simple graph. An *independent set* of vertices in Γ is a set of vertices that are pairwise non-adjacent to each other in Γ. We denote by $\alpha(\Gamma)$ the maximal number of vertices in independent sets of Γ. Given a group G, we put $t(G) = \alpha(\mathrm{GK}(G))$. Moreover, for each prime $r \in \pi(G)$, $t(r, G)$ denotes the maximal number of vertices in independent sets of $\mathrm{GK}(G)$ containing r. Generally, our notation for simple groups follows [8]. Especially, the alternating and symmetric group on n letters are denoted by $\mathbb{A}_n$ and $\mathbb{S}_n$, respectively. We also denote by $\mathrm{Syl}_p(G)$ the set of all Sylow p-subgroups of G, where $p \in \pi(G)$.

The sequel of this article is organized as follows: In Section 2, we recall some basic results, especially, on the spectra of certain finite simple groups. Section 3 is devoted to the proofs of main results (Theorems 1.2, 1.3, 1.6). We conclude our article with some open problems in Section 4.

2. Preliminaries

In this section we consider some results which will be needed for our further investigations.

Lemma 2.1 ([35]). *Let G be a finite group with $t(G) \geq 3$ and $t(2, G) \geq 2$, and let K be the maximal normal solvable subgroup of G. Then the quotient group G/K is an almost simple group, i.e., there exists a non-abelian simple group P such that $P \leq G/K \leq \mathrm{Aut}(P)$.*

Lemma 2.2 ([17, Lemma 8]). *Let G be a finite group with $|\pi(G)| \geq 3$. If there exist prime numbers $r, s, t \in \pi(G)$ such that $\{tr, ts, rs\} \cap \omega(G) = \emptyset$, then G is non-solvable.*

According to Table 4 in [24], we have the following result:

Lemma 2.3. *If $S \in \mathcal{S}_{\leq 29}$, then either $\mathrm{Out}(S) = 1$ or $\pi(\mathrm{Out}(S)) \subseteq \{2, 3\}$.*

Lemma 2.4 ([34]). *Suppose that $q = p^n$, where p is an odd prime. Then we have*

$$\mu(L_2(q)) = \left\{p,\ \frac{q-1}{2},\ \frac{q+1}{2}\right\}.$$

Lemma 2.5 ([23]). *Suppose that $q = p^n$, where p is an odd prime. Then there holds:*

$$\mu(L_3(q)) = \begin{cases} \left\{q^2+q+1,\ q^2-1,\ p(q-1)\right\} & \text{if } q \not\equiv 1 \pmod 3, \\ \left\{\frac{q^2+q+1}{3},\ \frac{q^2-1}{3},\ \frac{p(q-1)}{3},\ q-1\right\} & \text{if } q \equiv 1 \pmod 3. \end{cases}$$

Lemma 2.6 ([19]). *Let q be a power of prime 2. Then there holds:*

$$\mu(U_4(q)) = \left\{(q^2+1)(q-1),\ q^3+1,\ 2(q^2-1),\ 4(q+1)\right\}.$$

Lemma 2.7 ([39]). *Let q be a power of an odd prime p. Denote $d = \gcd(4, q+1)$. Then $\mu(U_4(q))$ contains the following (and only the following) numbers:*

(i) $\frac{q^4-1}{d(q+1)}$, $\frac{q^3+1}{d}$, $\frac{p(q^2-1)}{d}$, q^2-1;
(ii) $p(q+1)$, *if and only if* $d = 4$;
(iii) 9, *if and only if* $p = 3$.

Lemma 2.8 ([18]). *Let $q = p^n$, where $p > 3$ is an odd prime. Then there holds:*

$$\mu(S_4(q)) = \left\{\frac{q^2+1}{2},\ \frac{q^2-1}{2},\ p(q+1),\ p(q-1)\right\}.$$

Using Corollaries 2.5, 2.6, 2.7, 2.8, [24, Table 4] and [8] some results are summarized in Table 3. In this table we assume that $s = |\mathrm{Out}(S)|$.

Table 3. Some simple groups in $\mathcal{S}_{\leq 29}$.

S	$\lvert S\rvert$	$\mu(S)$	$\mathrm{D}(S)$	s
$L_3(11)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11^3 \cdot 19$	110, 120, 133	(3, 2, 3, 1, 2, 1)	2
$U_4(2^3)$	$2^{18} \cdot 3^7 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$	36, 126, 455, 513	(2, 3, 2, 4, 2, 1)	6
${}^2E_6(2)$	$2^{36} \cdot 3^9 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	13, 16, ..., 22, 24, 28, 30, 33, 35	(4, 4, 3, 3, 2, 0, 0, 0)	6
$S_4(17)$	$2^{10} \cdot 3^4 \cdot 5 \cdot 17^4 \cdot 29$	144, 145, 272, 306	(2, 2, 1, 2, 1)	2
$U_4(17)$	$2^{11} \cdot 3^7 \cdot 5 \cdot 7 \cdot 13 \cdot 17^6 \cdot 29$	288, 2320, 2448, 2457	(4, 4, 2, 2, 2, 2, 2)	4

The following proposition is taken from [33].

Proposition 2.9 ([33]). *Let M be a simple group whose order is less than 10^8. If G is a finite group with the same order and degree pattern as M, then the following statements hold:*

(a) *If $M \neq A_{10}, U_4(2)$, then $G \cong M$;*
(b) *If $M = A_{10}$, then $\mathcal{OD}(M) = \{A_{10}, J_2 \times \mathbb{Z}_3\}$;*
(c) *If $M = U_4(2)$, then G is isomorphic to M or a 2-Frobenius group.*

In particular, item (c) of Proposition 2.9 shows that $h_{\mathrm{OD}}(U_4(2)) \geq 2$. As we mentioned in the Introduction, in fact, there is such a 2-Frobenius group (see [20]). Indeed, when we have a Frobenius group, say, $F = K : C$ with abelian kernel K, and a faithful irreducible $\mathbb{Z}_pF$-module V, then the semidirect product VF is a 2-Frobenius group. Now, we consider the general linear groups $\mathrm{GL}(4, 2)$ and $\mathrm{GL}(4, 3)$. In $\mathrm{GL}(4, 2)$ and also in $\mathrm{GL}(4, 3)$ there exists a Frobenius group $F = K : C$ of order 20 such that K acts fixed-point-freely on corresponding natural modules V_1 of dimension 4 over $\mathbb{F}_2$ and V_2 of dimension 4 over $\mathbb{F}_3$. Now, we take $(V_1 \times V_2) \cdot F$ with the natural action of F on direct factors. Then we obtain a 2-Frobenius group $(2^4 \times 3^4) : 5 : 4$ with the same order as $U_4(2)$. Note that the prime graphs of $U_4(2)$ and $(2^4 \times 3^4) : 5 : 4$ coincide.

3. Main results

In this section we will prove Theorems 1.2, 1.3 and 1.6. Before beginning the proof of Theorem 1.2, we draw the prime graphs of the groups $L_3(11)$, $U_4(2^3)$, ${}^2E_6(2)$, $S_4(17)$ and $U_4(17)$ in Figure 1.

Proof of Theorem 1.2: Let S be one of the following simple groups $L_3(11)$, $U_4(2^3)$, ${}^2E_6(2)$ or $U_4(17)$. Suppose that G is a finite group such that $|G| = |S|$ and $D(G) = D(S)$. We have to prove that $G \cong S$. In all cases we will prove that $t(G) \geq 3$ and $t(2, G) \geq 2$. Therefore, it follows from Lemma 2.1 that there exists a simple group P such that $P \leq G/K \leq \mathrm{Aut}(P)$, where K is the maximal normal

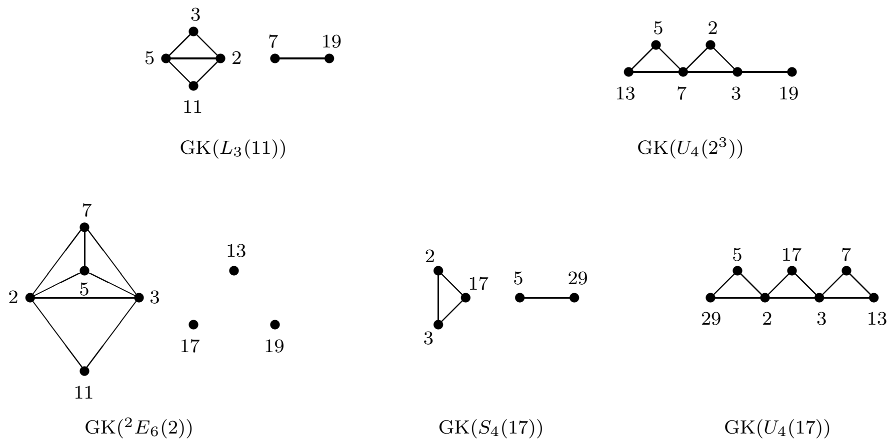

Figure 1. The prime graph of some simple groups.

solvable subgroup of G. In addition, we will prove that $P \cong S$, which implies that $K = 1$ and since $|G| = |S|$, G is isomorphic to S, as required. We handle every case singly.

(a) $S = L_3(11)$. Let G be a finite group such that

$$|G| = |S| = 2^4 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11^3 \cdot 19 \quad \text{and} \quad D(G) = D(S) = (3, 2, 3, 1, 2, 1).$$

According to our hypothesis there are five possibilities for the prime graph of G, as shown in Figure 2. Here, $p_1, p_2 \in \{2, 5\}$, $p_3, p_4 \in \{3, 11\}$, $p_5, p_6 \in \{7, 19\}$.

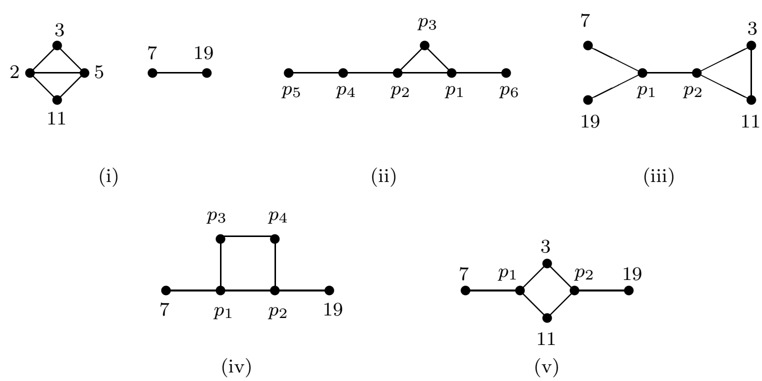

Figure 2. All possibilities for the prime graph of G.

We now consider two subcases separately.

(a.1) *Assume first that* GK(G) *is disconnected.* In this case we immediately imply that GK(G) = GK($L_3(11)$), and the hypothesis that $|G| = |L_3(11)|$

yields $\mathrm{OC}(G) = \mathrm{OC}(L_3(11))$. Now, by the Main Theorem in [10], G is isomorphic to $L_3(11)$, as required.

(a.2) *Assume next that* $\mathrm{GK}(G)$ *is connected.* In this case $7 \nsim 19$ in $\mathrm{GK}(G)$. Since $\{7, 19, p_3\}$ is an independent set, $t(G) \geq 3$, and so by Lemma 2.2, G is a non-solvable group. Moreover, since $d_G(2) = 3$ and $|\pi(G)| = 6$, $t(2, G) \geq 2$. Thus by Lemma 2.1 there exists a simple group P such that $P \leq G/K \leq \mathrm{Aut}(P)$, where K is the maximal normal solvable subgroup of G. We claim that K is a $\{7, 11, 19\}'$-group. We first show that K is a $\{7, 19\}'$-group. If $\{7, 19\} \subseteq \pi(K)$, then a Hall $\{7, 19\}$-subgroup of K is an abelian group. Hence $7 \sim 19$ in $\mathrm{GK}(K)$, and so in $\mathrm{GK}(G)$, which is a contradiction. Let $\{r, s\} = \{7, 19\}$. Now assume that $r \in \pi(K)$ and $s \notin \pi(K)$. Let $T \in \mathrm{Syl}_r(K)$. By Frattini argument $G = KN_G(T)$. Therefore, the normalizer $N_G(T)$ contains an element of order s, say x. Now, $T\langle x \rangle$ is an abelian subgroup of G, so it leads to a contradiction as before.

Finally, suppose that $11 \in \pi(K)$ and $T \in \mathrm{Syl}_{11}(K)$. Then $G = KN_G(T)$ by Frattini argument. Evidently, $N_G(T)$ contains some elements of order 7 and 19, that we respectively denote by u and v. Now, $T\langle u \rangle$ and $T\langle v \rangle$ are nilpotent subgroups of G, of orders $11^3 \cdot 7$ and $11^3 \cdot 19$, respectively, which implies that $7 \sim 11 \sim 19$, a contradiction. Since K and $\mathrm{Out}(P)$ are $\{7, 11, 19\}'$-groups, $|P|$ is divisible by $7 \cdot 11^3 \cdot 19$. Considering the orders of simple groups in $\mathcal{S}_{\leq 29}$, we conclude that P is isomorphic to $L_3(11)$, and so $K = 1$ and since $|G| = |L_3(11)|$, G is isomorphic to $L_3(11)$. But then $\mathrm{GK}(G) = \mathrm{GK}(L_3(11))$ is disconnected, which is impossible.

(b) $S = U_4(2^3)$. Assume that G is a finite group such that

$$|G| = |S| = 2^{18} \cdot 3^7 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19 \quad \text{and} \quad D(G) = D(S) = (2, 3, 2, 4, 2, 1).$$

So, the prime graph of G is one of the following graphs as shown in Figure 3. Here $p_1, p_2, p_3 \in \{2, 5, 13\}$.

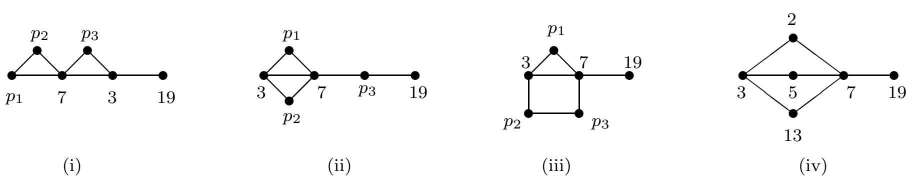

Figure 3. All possibilities for the prime graph of G.

In what follows, we will consider two subcases separately.

(b.1) *First, suppose that* $\mathrm{GK}(G)$ *is one of the graphs* (i), (iii) *or* (iv). Note that in each case $13 \nsim 19$ in $\mathrm{GK}(G)$ and $t(G) \geq 3$. Now, it follows from Lemma 2.2 that G is a non-solvable group. Moreover, since $d_G(2) = 2$ and $|\pi(G)| = 6$, $t(2, G) \geq 2$. Thus by Lemma 2.1 there exists a simple group P such that $P \leq G/K \leq \mathrm{Aut}(P)$, where K is the maximal normal solvable subgroup of G. As in the previous case, one can show that K is a $\{13, 19\}'$-group. Since K and $\mathrm{Out}(P)$ are $\{13, 19\}'$-groups, thus $|P|$ is divisible by $13 \cdot 19$. Considering the orders of simple groups in $\mathcal{S}_{\leq 29}$ yields P is isomorphic to $U_4(8)$, and so $K = 1$ and G is isomorphic to $U_4(8)$, because $|G| = |U_4(8)|$. Therefore, the prime graph of G and the graph (i) coincide, and in other cases we get a contradiction.

(b.2) *Next, suppose that* $\mathrm{GK}(G)$ *is the graph* (ii). In this case, 7 is not adjacent to 19 in $\mathrm{GK}(G)$. Since $\{p_1, p_2, 19\}$ is an independent set, $t(G) \geq 3$ and by Lemma 2.2, G is a non-solvable group. Moreover, since $d_G(2) = 2$ and $|\pi(G)| = 6$, $t(2, G) \geq 2$. Thus by Lemma 2.1 there exists a simple group P such that $P \leq G/K \leq \mathrm{Aut}(P)$, where K is the maximal normal solvable subgroup of G. Using similar arguments to those in the previous case, one can show that K is a $\{7, 19\}'$-group and G is isomorphic to $U_4(8)$. But then 3 is adjacent to 19 in $\mathrm{GK}(G)$, which is a contradiction.

(c) $S = {}^2E_6(2)$. Assume that G is a finite group such that

$$|G| = |S| = 2^{36} \cdot 3^9 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \quad \text{and} \quad D(G) = D(S) = (4, 4, 3, 3, 2, 0, 0, 0).$$

Then, the prime graphs of G and ${}^2E_6(2)$ coincide, and the hypothesis that $|G| = |{}^2E_6(2)|$ yields $\mathrm{OC}(G) = \mathrm{OC}({}^2E_6(2))$. Now, by [14], G is isomorphic to ${}^2E_6(2)$, as required.

(d) $S = U_4(17)$. Assume that G is a finite group such that

$$|G| = |S| = 2^{11} \cdot 3^7 \cdot 5 \cdot 7 \cdot 13 \cdot 17^6 \cdot 29 \quad \text{and} \quad D(G) = D(S) = (4, 4, 2, 2, 2, 2, 2).$$

According to our hypothesis there are four possibilities for the prime graph of G, as shown in Figure 4. Here $p_1, p_2, p_3, p_4, p_5 \in \{5, 7, 13, 17, 29\}$.

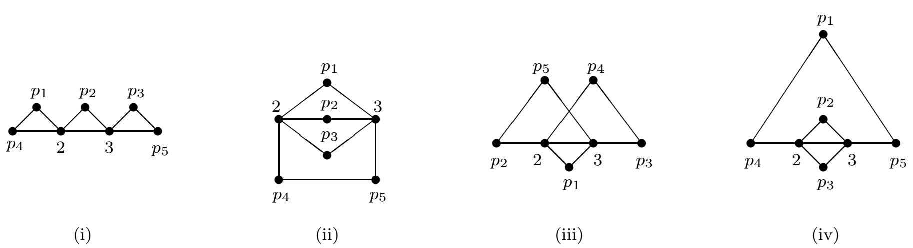

Figure 4. All possibilities for the prime graph of G.

In all cases $\{p_1, p_2, p_3\}$ is an independent set, and hence $t(G) \geq 3$. Moreover, since $d_G(2) = 4$ and $|\pi(G)| = 7$, $t(2, G) \geq 2$. Now, from Lemma 2.1 there exists a simple group P such that $P \leq G/K \leq \mathrm{Aut}(P)$. We claim now that K is a $\{2, 3\}$-group. In fact, if there exists $p_i \in \pi(K)$, for some i, then with similar arguments as before, we can verify that for each $j \neq i$, $p_i \sim p_j$ in $\mathrm{GK}(G)$, except $\{p_i, p_j\} = \{7, 29\}$, and this contradicts the fact that $d_G(p_i) = 2$. Hence K and

$\mathrm{Out}(P)$ are $\{2,3\}$-groups, thus $|P|$ is divisible by $5 \cdot 7 \cdot 13 \cdot 17^6 \cdot 29$. Again, considering the orders of simple groups in $\mathcal{S}_{\leq 29}$ yields P is isomorphic to $U_4(17)$, and so $K = 1$ and G is isomorphic to $U_4(17)$, because $|G| = |U_4(17)|$.

(e) $S = S_4(17)$. Assume that G is a finite group such that

$$|G| = |S| = 2^{10} \cdot 3^4 \cdot 5 \cdot 17^4 \cdot 29 \quad \text{and} \quad D(G) = D(S) = (2, 2, 1, 2, 1).$$

Under these conditions, there are two possibilities for the prime graph of G, as shown in Figure 5. Here $p_1, p_2, p_3 \in \{2, 3, 17\}$.

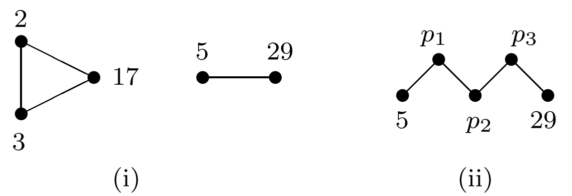

Figure 5. All possibilities for the prime graph of G.

We now consider two cases separately, depending on $\mathrm{GK}(G)$ is connected or disconnected.

(2.1) *Assume first that* $\mathrm{GK}(G)$ *is connected*. Since $\{5, p_2, 29\}$ is an independent set, $t(G) \geq 3$. Moreover, since $d_G(2) = 2$ and $|\pi(G)| = 5$, $t(2, G) \geq 2$. Thus by Lemma 2.1 there exists a simple group P such that $P \leq G/K \leq \mathrm{Aut}(P)$. We shall treat the cases $17 \nsim 29$ and $17 \sim 29$ in $\mathrm{GK}(G)$, separately.

(2.1.a) First we consider the case where $17 \nsim 29$ in $\mathrm{GK}(G)$. In this case as before, one can show that K is a $\{17, 29\}'$-group. Since K and $\mathrm{Out}(P)$ are $\{17, 29\}'$-groups, thus $|P|$ is divisible by $17^4 \cdot 29$. Considering the orders of simple groups in $\mathcal{S}_{\leq 29}$ yields P is isomorphic to $S_4(17)$, and so $K = 1$ and G is isomorphic to $S_4(17)$, because $|G| = |S_4(17)|$. But then $\mathrm{GK}(G) = \mathrm{GK}(S_4(17))$ is disconnected, which is impossible.

(2.1.b) Next we discuss the case where $17 \sim 29$ in $\mathrm{GK}(G)$. An argument similar to that in the above paragraphs shows that K is a $\{3, 29\}'$-group. Since K and $\mathrm{Out}(P)$ are $\{3, 29\}'$-groups, thus $|P|$ is divisible by $3^4 \cdot 29$. Considering the orders of simple groups in $\mathcal{S}_{\leq 29}$ yields P is isomorphic to $S_4(17)$, and so $K = 1$ and G is isomorphic to $S_4(17)$, because $|G| = |S_4(17)|$. But then $\mathrm{GK}(G) = \mathrm{GK}(S_4(17))$ is disconnected, which is impossible.

(2.2) *Assume next that* $\mathrm{GK}(G)$ *is disconnected*. In this case, it is easy to see that the prime graphs of G and $S_4(17)$ coincide. Now, by the main theorem in [11], G is isomorphic to $S_4(17)$.

This completes the proof of Theorem 1.2. $\square$

Proof of Theorem 1.3: Let G be a finite group satisfying

$$(1)\ |G| = |U_4(2)| = 2^6 \cdot 3^4 \cdot 5, \quad \text{and} \quad (2)\ D(G) = D(U_4(2)) = (1,1,0).$$

By Proposition 2.9, G is isomorphic to $U_4(2)$ or a 2-Frobenius group. First of all, it should be noted that the existence of a 2-Frobenius group satisfying conditions (1) and (2) is guaranteed by [20], [44]. To prove uniqueness, we note that any such group will be a subdirect product of 2-Frobenius groups of orders $2^4{\cdot}5{\cdot}4$ and $3^4{\cdot}5{\cdot}4$. As a matter of fact, since 4 is the order of 2 modulo 5, 4 is the smallest dimension of an irreducible module for $\mathbb{Z}_5$ over $\mathbb{F}_2$, so there is a unique Frobenius group of order $2^4 \cdot 5$ and its kernel is elementary abelian. Actually, this is a subgroup of the 1-dimensional affine group over $\mathbb{F}_{2^4}$ which is denoted by $\mathrm{AGL}(1, \mathbb{F}_{2^4})$. We can now extend this subgroup by an element of order 4 acting as a field automorphism of $\mathbb{F}_{2^4}$, giving a unique isomorphism class of 2-Frobenius groups of order $2^4 \cdot 5 \cdot 4$. Another way of looking at it is that the normalizer of a subgroup of order 5 in $\mathrm{GL}(4,2)$ is the semilinear group, which is metacyclic with structure $\mathbb{Z}_{15} : \mathbb{Z}_4$, and this has the Frobenius group $\mathbb{Z}_5 : \mathbb{Z}_4$ as a subgroup. Reasoning exactly as before, we can show that there is a unique 2-Frobenius group of order $3^4 \cdot 5 \cdot 4$, and it has elementary abelian normal subgroup of order 3^4. Now, taking the subdirect product of these gives a unique isomorphism class of 2-Frobenius groups of order $2^6 \cdot 3^4 \cdot 5$. This completes the proof. $\square$

Proof of Theorem 1.6: Let G be a finite group satisfying

$$\mathrm{OC}(G) = \mathrm{OC}(U_5(2)) = \{2^{10} \cdot 3^5 \cdot 5, 11\}.$$

Clearly $|G| = |U_5(2)| = 2^{10} \cdot 3^5 \cdot 5 \cdot 11$ and $s(G) = 2$, in fact, we have $\pi_1(G) = \{2,3,5\}$ and $\pi_2(G) = \{11\}$. Then, by Theorem A in [36], one of the following statements holds:

(1) G is a Frobenius group,
(2) G is a 2-Frobenius group, or
(3) G has a normal series $1 \trianglelefteq H \triangleleft K \trianglelefteq G$ such that H is a nilpotent π_1-group, K/H is a non-abelian simple group, G/K is a π_1-group, and any odd order component of G is equal to one of the odd order components of K/H.

If G is a Frobenius group with kernel K and complement C, then $\mathrm{OC}(G) = \{|K|, |C|\}$, and since $|C| < |K|$, the only possibility is $|K| = 2^{10} \cdot 3^5 \cdot 5$ and $|C| = 11$. However, this is a contradiction because $|C| \nmid |K| - 1$.

If G is a 2-Frobenius group of order $2^{10} \cdot 3^5 \cdot 5 \cdot 11$, then, by the definition, $G = ABC$, where A and AB are normal subgroups of G and AB and BC are Frobenius groups with kernels A and B, respectively. Reasoning as in the proof of Theorem 1.3, we observe that there are unique 2-Frobenius groups A_1BC and A_2BC of orders $2^{10} \cdot 11 \cdot 5$ and $3^5 \cdot 11 \cdot 5$, respectively. Note that A_1 and A_2 are elementary abelian normal subgroups of orders 2^{10} and 3^5, respectively. Therefore,

G is a subdirect product $(A_1 \times A_2)BC = (2^{10} \times 3^5) : 11 : 5$ of A_1BC and A_2BC. So there is a unique 2-Frobenius group $G = ABC$ of order $|U_5(2)| = 2^{10} \cdot 3^5 \cdot 5 \cdot 11$.

Finally, we suppose that G satisfies condition (3). Then, by Table 2, K/H is isomorphic to one of the simple groups $L_2(11)$, M_{11}, M_{12}, or $U_5(2)$. We see that, in general, $K/H \leq G/H \leq \mathrm{Aut}(K/H)$. Let $K/H \cong L_2(11)$. Since $|\mathrm{Aut}(K/H)| = 2^3 \cdot 3 \cdot 5 \cdot 11$ is not divisible by 3^2, it follows that 3^4 divides $|H|$. Let P be a Sylow 3-subgroup of H and let Q be a Sylow 11-subgroup of G . Then, P is a normal subgroup of G, because H is nilpotent. It now follows that PQ is a subgroup of G of order $3^4 \cdot 11$. Since all groups of order $3^4 \cdot 11$ are nilpotent, we conclude that 3 is adjacent to 11 in $\mathrm{GK}(G)$, which is a contradiction.

Reasoning exactly as above, we conclude that $K/H \ncong M_{11}, M_{12}$. Therefore, we deduce that $K/H \cong U_5(2)$, and since $|G| = |U_5(2)|$ it follows that $|H| = 1$ and $G = K \cong U_5(2)$. This completes the proof. $\square$

4. Some open problems

We conclude this article with some open problems. Actually, in this section, we restrict our attention to the relationship between degree patterns and prime graphs. A natural question is:

Question 4.1. Let G and M be two finite groups with $|G| = |M|$. Clearly $\mathrm{GK}(G) = \mathrm{GK}(M)$ implies $D(G) = D(M)$. Does the converse hold?

Assuming the converse is *true*, under these hypotheses we conclude that $\mathrm{OC}(G) = \mathrm{OC}(M)$, and so $h_{\mathrm{OD}}(M) \leq h_{\mathrm{OC}}(M)$. In particular, if M is OC-characterizable, then M is also OD-characterizable. In [15, Lemma 2.15] it was shown that if G is a finite group with $\pi(G) = \pi(M)$ and $D(G) = D(M)$, where M is an arbitrary alternating or symmetric group, then the prime graphs of G and M coincide. Therefore, we have the following consequence.

Corollary 4.2. *The symmetric and alternating groups which are OC-characterizable are also OD-characterizable.*

On the other hand, in view of the Main Theorem in [6], the symmetric groups S_p and S_{p+1}, and the alternating groups A_p, A_{p+1} and A_{p+2}, where $p \geq 3$ is a prime number, are OC-characterizable. Therefore, by Corollary 4.2, they are also OD-characterizable (see also [27, Theorem 1.5]). We notice that other alternating and symmetric groups are *not* OC-characterizable. In fact, for all alternating groups A_n ($n \geq 5$), except A_p, A_{p+1} and A_{p+2}, where p is a prime, the vertex 3 is adjacent to all other vertices in $\mathrm{GK}(A_n)$. Similarly, for all symmetric groups S_n ($n \geq 5$), except S_p and S_{p+1}, where p is a prime, the vertex 2 is adjacent to all other vertices in $\mathrm{GK}(S_n)$. Therefore, the prime graphs associated with these groups are connected. Assume now that G is the alternating group (resp. the symmetric group) on $n \geq 5$ letters, except A_p, A_{p+1}, A_{p+2} (resp. S_p, S_{p+1}) where p is a prime. Let H be a nilpotent group of order $|G|$ (for instance, consider a cyclic group of order $|G|$). Clearly, $\mathrm{GK}(H)$ is complete. Now, by the definition of order components, we have $\mathrm{OC}(H) = \mathrm{OC}(G) = \{|G|\}$, while H is not isomorphic to G. But the situation of OD-characterizability of alternating and symmetric groups looks a little differently. As pointed out in the Introduction, there are infinitely many alternating groups $\mathbb{A}_n$ (resp. symmetric groups $\mathbb{S}_n$) which satisfy $h_{\rm OD}(\mathbb{A}_n) \geq 3$ (resp. $h_{\rm OD}(\mathbb{S}_n) \geq 3$), in particular, neither $h_{\rm OD}(\mathbb{A}_n)$ nor $h_{\rm OD}(\mathbb{S}_n)$ is bounded above (see [22]).

We now focus our attention on the sporadic simple groups. By Table 1 in [30], it is easy to see that if G is a finite group with $\pi(G) = \pi(M)$ and $D(G) = D(M)$, where M is a sporadic simple group, then the prime graphs of G and M coincide. Moreover, it is proved in [7] that all sporadic simple groups are OC-characterizable, hence we conclude that they are also OD-characterizable (see [30, Proposition 3.1.]).

Finally, we consider the OD-characterizability of simple groups of Lie type. Studies show that between simple groups of Lie type there are many simple orthogonal and symplectic groups which are 2-fold OD-characterizable (see [4]). Moreover, by Theorem 1.3, we have $h_{\rm OD}(U_4(2)) = 2$. So far we have not found a simple group of Lie type S satisfying $h_{\rm OD}(S) > 2$. So it seems natural to ask the following question.

Question 4.3. Does there exist a finite simple group S of Lie type such that $h_{\rm OD}(S) \geq 3$?

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References

[1] Akbari B., Moghaddamfar A.R., *Recognizing by order and degree pattern of some projective special linear groups*, Internat. J. Algebra Comput. **22** (2012), no. 6, 1250051, 22 pages.

[2] Akbari B., Moghaddamfar A.R., *On recognition by order and degree pattern of finite simple groups*, Southeast Asian Bull. Math. **39** (2015), no. 2, 163–172.

[3] Akbari B., Moghaddamfar A.R., *OD-characterization of certain four dimensional linear groups with related results concerning degree patterns*, Front. Math. China **10** (2015), no. 1, 1–31.

[4] Akbari B., Moghaddamfar A.R., *Simple groups which are* 2*-fold OD-characterizable*, Bull. Malays. Math. Sci. Soc. **35** (2012), no. 1, 65–77.

[5] Akbari M., Moghaddamfar A.R., Rahbariyan S., *A characterization of some finite simple groups through their orders and degree patterns*, Algebra Colloq. **19** (2012), no. 3, 473–482.

[6] Alavi S.H, Daneshkhah A., *A new characterization of alternating and symmetric groups*, J. Appl. Math. Comput. **17** (2005), no. 1–2, 245–258.

[7] Chen G.Y., *A new characterization of sporadic simple groups*, Algebra Colloq. **3** (1996), no. 1, 49–58.

[8] Conway J.H., Curtis R.T., Norton S.P., Parker R.A., Wilson R.A., *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[9] Hoseini A.A., Moghaddamfar A.R., *Recognizing alternating groups A_{p+3} for certain primes p by their orders and degree patterns*, Front. Math. China **5** (2010), no. 3, 541–553.

[10] Iranmanesh A., Alavi S.H., Khosravi B., *A characterization of* $\mathrm{PSL}(3,q)$ *where q is an odd prime power*, J. Pure Appl. Algebra **170** (2002), no. 2–3, 243–254.

[11] Iranmanesh A., Khosravi B., *A characterization of $C_2(q)$ where $q > 5$*, Comment. Math. Univ. Carolin. **43** (2002), no. 1, 9–21.

[12] Khosravi A., Khosravi B., *r-recognizability of $B_n(q)$ and $C_n(q)$ where $n = 2^m \geq 4$*, J. Pure Appl. Algebra **199** (2005), no. 1–3, 149–165.

[13] Khosravi B., *Some characterizations of $L_9(2)$ related to its prime graph*, Publ. Math. Debrecen **75** (2009), no. 3–4, 375–385.

[14] Khosravi Beh., Khosravi Bah., *A characterization of ${}^2E_6(q)$*, Kumamoto J. Math. **16** (2003), 1–11.

[15] Kogani-Moghaddam R., Moghaddamfar A.R., *Groups with the same order and degree pattern*, Sci. China Math. **55** (2012), no. 4, 701–720.

[16] Kondrat'ev A.S., *On prime graph components of finite simple groups*, Math. Sb. **180** (1989), no. 6, 787–797.

[17] Lucido M.S., Moghaddamfar A.R., *Groups with complete prime graph connected components*, J. Group Theory **7** (2004), no. 3, 373–384.

[18] Mazurov V.D., *Recognition of the finite simple groups $S_4(q)$ by their element orders*, Algebra Logic **41** (2002), no. 2, 93–110.

[19] Mazurov V.D., Chen G.Y., *Recognizability of the finite simple groups $L_4(2^m)$ and $U_4(2^m)$ by the spectrum*, Algebra Logic **47** (2008), no. 1, 49–55.

[20] Moghaddamfar A.R., *A comparison of the order components in Frobenius and 2-Frobenius groups with finite simple groups*, Taiwanese J. Math. **13** (2009), no. 1, 67–89.

[21] Moghaddamfar A.R., *Recognizability of finite groups by order and degree pattern*, Proceedings of the International Conference on Algebra 2010, World Sci. Publ., Hackensack, NJ, 2012, pp. 422–433.

[22] Moghaddamfar A.R., *On alternating and symmetric groups which are quasi OD-characterizable*, J. Algebra Appl., **16** (2017), no. 2, 1750065, 14 pp.

[23] Moghaddamfar A.R., Darafsheh M.R., *A family of finite simple groups which are 2-recognizable by their elements order*, Comm. Algebra **32** (2004), no. 11, 4507–4513.

[24] Moghaddamfar A.R., Rahbarian S., *More on the OD-characterizability of a finite group*, Algebra Colloq. **18** (2011), 663–674.

[25] Moghaddamfar A.R., Rahbariyan S., *A quantitative characterization of some finite simple groups through order and degree pattern*, Note Mat. **34** (2014), no. 2, 91–105.

[26] Moghaddamfar A.R., Rahbarian S., *OD-characterization of some projective special linear groups over the binary field and their automorphism groups*, Comm. Algebra **43** (2015), no. 6, 2308–2334.

[27] Moghaddamfar A.R., Zokayi A.R., *Recognizing finite group through order and degree pattern*, Algebra Colloq. **15** (2008), no. 3, 449–456.

[28] Moghaddamfar A.R., Zokayi A.R., *OD-characterization of alternating and symmetric groups of degree* 16 *and* 22, Front. Math. China **4** (2009), 669–680.

[29] Moghaddamfar A.R., Zokayi A.R., *OD-characterization of certain finite groups having connected prime graphs*, Algebra Colloq. **17** (2010), no. 1, 121–130.

[30] Moghaddamfar A.R., Zokayi A.R., Darafsheh M.R., *A characterization of finite simple groups by the degrees of vertices of their prime graphs*, Algebra Colloq. **12** (2005), no. 3, 431–442.

[31] Shao C., Shi W., Wang L., Zhang L., *OD-characterization of $\mathbb{A}_{16}$*, Journal of Suzhou University (Natural Science Edition) **24** (2008), 7–10.

[32] Shao C., Shi W., Wang L., Zhang L., *OD-characterization of the simple group $L_3(9)$*, Journal of Guangxi University (Natural Science Edition) **34** (2009), 120–122.

[33] Shi W., Zhang L., *OD-characterization of all simple groups whose orders are less than* 10^8, Front. Math. China **3** (2008), 461–474.

[34] Suzuki M., *Group Theory I*, Springer, Berlin-New York, 1982.

[35] Vasil'ev A.V., Gorshkov I.B., *On the recognition of finite simple groups with a connected prime graph*, Sib. Math. J. **50** (2009), 233–238.

[36] Williams J.S., *Prime graph components of finite groups*, J. Algebra **69** (1981), no. 2, 487–513.

[37] Yan Y., Chen G.Y., *OD-characterization of alternating and symmetric groups of degree* 106 *and* 112, Proceedings of the International Conference on Algebra 2010, World Sci. Publ., Hackensack, NJ, 2012, pp. 690–696.

[38] Yan Y., Chen G.Y., Zhang L.C., Xu H., *Recognizing finite groups through order and degree patterns*, Chin. Ann. Math. Ser. B **34** (2013), no. 5, 777–790.

[39] Zavarnitsine A.V., *Exceptional action of the simple groups* $\mathrm{L}_4(q)$ *in the defining characteristic*, Sib. Elektron. Mat. Izv. **5** (2008), 68–74.

[40] Zavarnitsine A.V., *Finite simple groups with narrow prime spectrum*, Sib. Elektron. Mat. Izv. **6** (2009), 1–12.

[41] Zhang L., Shi W., *OD-characterization of simple* K_4*-groups*, Algebra Colloq. **16** (2009), 275–282.

[42] Zhang L., Shi W., *OD-characterization of almost simple groups related to* $U_6(2)$, Acta Math. Sci. Ser. B Engl. Ed. **31** (2011), no. 2, 441–450.

[43] Zhang L., Shi W., *OD-characterization of the projective special linear groups* $L_2(q)$, Algebra Colloq. **19** (2012), no. 3, 509–524.

[44] Zinov'eva M.R., Kondrat'ev A.S., *An example of a double Frobenius group with order components as in the simple group* $S_4(3)$, Vladikavkaz. Mat. Zh. **10** (2008), no. 1, 35–36 (Russian).

[45] Zinov'eva M.R., Mazurov V.D., *On finite groups with disconnected prime graph*, Proceedings of the Steklov Institute of Mathematics **283** (2013), no. 1, 139–145.

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