Compactness theorems for the Bakry-Emery Ricci tensor on semi-Riemannian manifolds

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Abstract. In this manuscript we provide new extensions for the Myers theorem in weighted Riemannian and Lorentzian manifolds. As application we obtain a closure theorem for spatial hypersurfaces immersed in some time-like manifolds.

Keywords: Bakry-Emery Ricci curvature tensor; closure theorem; Riccati equation

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1. Introduction

Given a semi-Riemannian manifold (M^n, g) and a smooth function f on M^n, the *weighted manifold* M^n_f associated to M^n and f is the triple $(M^n, g, d\mu = e^{-f} d\nu)$, where $d\nu$ is the volume element of M. In the literature it is also called *smooth metric space*. The analysis of the weighted geometry in M_f is deeply related with a family of Ricci tensors introduced by Bakry and Emery in [1] given by

$$\operatorname{Ric}^k_f = \operatorname{Ric} + \operatorname{Hess} f - \frac{1}{k} df \otimes df,$$

where $k \in (0, \infty)$.

In the Riemannian context, the classical Myers compactness theorem [15] was extended for weighted manifolds in many ways. See for instance, [2], [5], [11], [12], [13], [14], [16], [17], [21], [24]. Exploring a Riccati inequality obtained from the Bochner formula we get the following improved version of weighted Myers theorem in the spirit of the Sprouse theorem in [20].

Theorem 1.1. *Let M^n_f be a weighted complete Riemannian manifold. Then for any $\delta > 0$, and $a > 0$, there exists an $\epsilon = \epsilon(n, a, \delta)$ satisfying the following:*

If there is a point p such that along each geodesic γ emanating from p, the Ric^k_f curvature satisfies

$$\int_0^\infty \max\left\{(n-1)a - \operatorname{Ric}^k_f(\gamma', \gamma'), 0\right\} dt < \epsilon(n, a, \delta) \tag{1.1}$$

then M is compact with $\operatorname{diam}(M) \leq \frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}} + \delta$.

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Recently, the geometry of weighted Lorentzian manifold has been subject of great interest. For instance in [4], J. Case has shown that certain aspects of the Bakry-Emery comparison theory can be adapted to a Lorentzian manifold, allowing him to prove a Bakry-Emery version of the Hawking-Penrose singularity theorem for general relativity. This result leads to a Hawking-Penrose theorem for scalar-tensor gravitation theories. In this same context, we can mention Rupert and Woolgar in [19], Woolgar in [22] and Galloway and Woolgar in [9].

Motivated by the ideas of Yun in [23] we obtain an extension of Theorem 1.1 for *weighted globally hyperbolic space-time* (see Section 2.2). The main tool in our approach is a Raychaudhuri type inequality raised by Case in [4]. Precisely, we obtain the following result:

Theorem 1.2. *Let M^n_f be a weighted globally hyperbolic space-time. Then for any $\delta > 0$, $a > 0$, there exists an $\epsilon = \epsilon(n, a, \delta)$ satisfying the following:*

If there is a point p such that along each future directed time-like geodesic γ emanating from p, with $l(\gamma) = \sup\{t \geq 0, d(p, \gamma(t)) = t\}$ the Ric^k_f curvature satisfies

$$\int_0^{l(\gamma)} \max\left\{(n-1)a - \mathrm{Ric}^k_f(\gamma', \gamma'), 0\right\} dt < \epsilon(n, a, \delta)$$

then the time-like diameter satisfies $\mathrm{diam}\,(M) \leq \frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}} + \delta$.

Theorem 1.1 can be applied in the context of space-time manifolds. Namely, suppose that $\bar{M}^{n+1}$ is a space-time and M^n is a spatial hypersurface of $\bar{M}$. It is an interesting problem to inquire under which conditions M is compact. In this sense, the works of Galloway in [8] and Galloway and Frankel in [7] are very successful. Recently, Cavalcante, Oliveira and the author [5] generalized the closure theorems of [7] and [8] in the weighted case. Here, using Theorem 1.1 we are able to improve Theorem 4.3 of [5] as follows:

Theorem 1.3. *Let M^n be a spatial hypersurface in $\bar{M}^{n+1}_f$ and assume that M is complete in the induced metric. Then for any $\delta > 0$, $a > 0$, there exists an $\epsilon = \epsilon(n, a, \delta)$ satisfying the following:*

If there is a point $p \in M$ such that along each geodesic γ in M emanating from p, the condition

$$\int_0^{\infty} \max\left\{(n-1)a - \overline{\mathrm{Ric}}^k_f(X, X) - \langle v(X), X\rangle H_f + \langle a(X), X\rangle, 0\right\} dt < \epsilon(n, a, \delta)$$

is satisfied, where $X = \gamma'$, then M is compact and $\mathrm{diam}\,(M) \leq \frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}} + \delta$.

2. Weighted Myers theorems

2.1 The Riemannian case. Given a weighted manifold M^n_f the *weighted Laplacian operator* can be defined by $\Delta_f u = \Delta u - \langle \nabla f, \nabla u\rangle$, for functions u of class

C^2 on M. It is a remarkable fact that the weighted Laplacian satisfies a Bochner type inequality (see [10] or [21]).

Fixed $p \in M$ let $r(x) = d(x,p)$ denote the distance function and let $m_f(r)$ be the weighted Laplacian of the distance function. From the Bochner inequality for the weighted Laplacian one can verify that m_f satisfies the following Riccati inequality (see Appendix A of [21]):

$$m_f' + \frac{m_f^2}{n+k-1} \le -\mathrm{Ric}\,_f^k(\gamma', \gamma'), \quad \text{for all} \quad t \ge 0, \tag{2.1}$$

where γ is a geodesic emanating from p and m_f' stands for the derivative of m_f with respect to r. The inequality (2.1) is the main tool in the proof of our first result.

Proof of Theorem 1.1: For any small positive $\epsilon < a^2$ to be determined later, consider the following sets

$$E_1 = \left\{t \in [0,\infty); \mathrm{Ric}\,_f^k(\gamma'(t), \gamma'(t)) \ge (n-1)(a - \sqrt{\epsilon})\right\}$$

and

$$E_2 = \left\{t \in [0,\infty); \mathrm{Ric}\,_f^k(\gamma'(t), \gamma'(t)) < (n-1)(a - \sqrt{\epsilon})\right\}.$$

From the inequality (2.1) we have on E_1

$$\frac{\frac{m_f'}{n+k-1}}{(\frac{m_f}{n+k-1})^2 + \frac{(n-1)(a-\sqrt{\epsilon})}{n+k-1}} \le -1. \tag{2.2}$$

On the other hand, on E_2 we have

$$\frac{\frac{m_f'}{n+k-1}}{(\frac{m_f}{n+k-1})^2 + \frac{(n-1)(a-\sqrt{\epsilon})}{n+k-1}} \le \frac{a - \sqrt{\epsilon} - \mathrm{Ric}\,_f^k(\gamma', \gamma')/(n-1)}{a - \sqrt{\epsilon}}. \tag{2.3}$$

Now, using the assumption (1.1) on the Bakry-Emery Ricci tensor we get

$$\begin{aligned}
\epsilon &> \int_0^\infty \max\left\{(n-1)a - \mathrm{Ric}\,_f^k(\gamma'(t), \gamma'(t)), 0\right\} \\
&> \int_{E_2} \left\{(n-1)a - \mathrm{Ric}\,_f^k(\gamma'(t), \gamma'(t))\right\} \, dt \\
&> \int_{E_2} \left\{(n-1)a - (n-1)(a - \sqrt{\epsilon})\right\} \, dt \\
&= \mu(E_2)(n-1)\sqrt{\epsilon}.
\end{aligned}$$

That is, we have

$$\mu(E_2) < \frac{\sqrt{\epsilon}}{n-1}, \tag{2.4}$$

where μ is the Lebesgue measure on $\mathbb{R}$.

Using inequalities (2.2)−(2.4) we obtain

$$\begin{aligned}
\int_0^r \frac{\frac{m_f'}{n+k-1}}{(\frac{m_f}{n+k-1})^2+\frac{(n-1)(a-\sqrt{\epsilon})}{n+k-1}}\,dt &\leq \int_{[0,r]\cap E_1} \frac{\frac{m_f'}{n+k-1}}{(\frac{m_f}{n+k-1})^2+\frac{(n-1)(a-\sqrt{\epsilon})}{n+k-1}}\,dt \\
&\quad + \int_{[0,r]\cap E_2} \frac{\frac{m_f'}{n+k-1}}{(\frac{m_f}{n+k-1})^2+\frac{(n-1)(a-\sqrt{\epsilon})}{n+k-1}}\,dt \\
&\leq -\mu\{[0,r]\cap E_1\} + \frac{\epsilon}{(n-1)(a-\sqrt{\epsilon})} \\
&\leq -r+\mu\{[0,r]\cap E_2\} + \frac{\epsilon}{(n-1)(a-\sqrt{\epsilon})} \\
&\leq -r+\frac{\sqrt{\epsilon}}{n-1} + \frac{\epsilon}{(n-1)(a-\sqrt{\epsilon})}\,.
\end{aligned}$$

Define $\tau(\epsilon)=\frac{\sqrt{\epsilon}}{n-1}+\frac{\epsilon}{(n-1)(a-\sqrt{\epsilon})}$. The integral of the left hand side can be computed explicitly and therefore we get

$$\arctan\left(\frac{m_f(r)}{(n+k-1)a(\epsilon)}\right) \leq a(\epsilon)(-r+\tau(\epsilon))+\frac{\pi}{2}\,,$$

where $a(\epsilon)=\sqrt{\frac{(n-1)(a-\sqrt{\epsilon})}{n+k-1}}$.

So we get

$$m_f(r) \leq -(n+k-1)a(\epsilon)\cot(a(\epsilon)(-r+\tau(\epsilon))),$$

for any r such that $\tau(\epsilon)<r<\frac{\pi}{a(\epsilon)}+\tau(\epsilon)$.

In particular, $m_f(\gamma(r))$ goes to $-\infty$ as $r\to(\frac{\pi}{a(\epsilon)}+\tau(\epsilon))^+$. It implies that γ cannot be minimal beyond $\frac{\pi}{a(\epsilon)}+\tau(\epsilon)$. Otherwise m_f would be a smooth function at $r=\frac{\pi}{a(\epsilon)}+\tau(\epsilon)$. Taking ϵ explicitly so that $\frac{\pi}{a(\epsilon)}+\tau(\epsilon)=\frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}}+\delta$ and using the completeness of M we have the desired result. $\square$

2.2 The Lorentzian case. Now let us discuss the Lorentzian version of the Theorem 1.1. Let M be a time-oriented Lorentzian manifold. Given $p\in M$ we set

$$J^+(p)=\{q\in M:\ \text{there exist a future pointing causal curve from } p \text{ to } q\},$$

called the *causal future* of p. The *causal past* $J^-(p)$ is defined similarly. We say that M is *globally hyperbolic* if the set $J(p,q):=J^+(p)\cap J^-(q)$ is compact for all p and q joined by a causal curve (see [3]). Mathematically, global hyperbolicity often plays a role analogous to geodesic completeness in Riemannian geometry.

Let $\gamma:[a,b]\longrightarrow M$ be a future-directed time-like unit-speed geodesic. Given $\{E_1,E_2,\ldots,E_n\}$ an orthonormal frame field along γ, and for each $i\in\{1,\ldots,n\}$

we let J_i be the unique Jacobi field along c such that $J_i(a) = 0$ and $J_i'(0) = E_i$. Denote by A the matrix $A = [J_1 J_2 \dots J_n]$, where each column is just the vector for J_i in the basis defined by $\{E_i\}$. In this situation, we have that $A(t)$ is invertible if and only if $\gamma(t)$ is not conjugate to $\gamma(a)$.

Now we define $B_f = A'A^{-1} - \frac{1}{n-1}(f \circ \gamma)'E$ wherever A is invertible, where $E(t)$ is the identity map on $(\gamma'(t))^{\perp}$. The *f-expansion function* is a smooth function defined by $\theta_f = \operatorname{tr} B_f$ (see [4, Definition 2.6]). Note that, if $|\theta_f| \to \infty$ as $t \to t_0$, where $t_0 \in [a, b]$, then $\gamma(t_0)$ is conjugate to $\gamma(a)$.

Recently, Case in [4] obtained the following relation between the Bakry-Emery Ricci tensor and the *f-expansion function* θ_f:

Lemma 2.1. *Under the above notations,*

$$\theta_f' \leq -\operatorname{Ric}_f^k(\gamma', \gamma') - \frac{\theta_f^2}{k+n-1}. \tag{2.5}$$

The inequality (2.5) is called (k, f)-*Raychaudhuri inequality*. This inequality is a generalization of the well known Raychaudhuri inequality (see for instance [6]).

The *distance* between two time-like related points is the supremum of lengths of causal curves joining the points. It follows that the distance between any two time-like related points in a globally hyperbolic space-time is the length of such a maximal time-like geodesic. The *time-like diameter*, $\operatorname{diam}(M)$, of a Lorentzian manifold is defined to be the supremum of distances $d(p, q)$ between points of M.

Proof of Theorem 1.2: Let $\epsilon(n, a, \delta)$ be the explicit constant in the previous theorem. Assume by contradiction that there are two points p and q with $d(p, q) > \frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}} + \delta$. On the other hand, since M is globally hyperbolic, there exists a maximal time-like geodesic γ joining p and q such that $\ell(\gamma) = d(p, q)$. Following the steps of the proof of Theorem 1.1 using the (k, f)-Raychaudhuri inequality (2.5) we get

$$\lim_{t \to t_0^+} \theta_f(t) = -\infty,$$

where $t_0 = \frac{\pi}{a(\epsilon)} + \tau(\epsilon)$. So, we conclude that γ cannot be maximal beyond $\frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}} + \delta$ which is a contradiction. $\square$

An immediate consequence of Theorem 1.2 is the Lorentzian version of the original weighted Myers theorem obtained by Qian in [16]. Namely:

Corollary 2.2. Let M_f^n be a weighted globally hyperbolic space-time. Let a be a positive constant and assume that

$$\operatorname{Ric}_f^k(v, v) \geq (n-1)a,$$

for all unit time-like vector field $v \in TM$. Then the time-like diameter satisfies $\operatorname{diam}(M) \leq \frac{\pi}{\sqrt{\frac{(n-1)a}{n+k-1}}}$.

3. Application

Let $\bar{M}^{n+1}$ be an $(n+1)$-dimensional space-time manifold. Let M^n be a spatial hypersurface immersed in $\bar{M}^{n+1}$. That is, the metric induced in M^n by $\bar{M}^{n+1}$ is a Riemannian metric. Then the unit tangent vectors to the future directed time-like geodesics orthogonal to M define a smooth unit time-like vector field $\mathbf{u}$ in a neighborhood of M. For the sake of simplicity we will omit the dimension super index and we always use a *bar* for geometric objects related to $\bar{M}$.

Let X be a vector field tangent to M. Extend X along the flow lines generated by $\mathbf{u}$, and suppose that X is invariant under the flow generated by $\mathbf{u}$, i.e, $[X, \mathbf{u}] = 0$, where $[,]$ is the Lie bracket. The *velocity* and *acceleration* of X along the flow generated by $\mathbf{u}$ are given respectively by

$$v(X) = \overline{\nabla}_{\mathbf{u}} X \quad \text{and} \quad a(X) = \overline{\nabla}_{\mathbf{u}} \overline{\nabla}_{\mathbf{u}} X.$$

The second fundamental form of M as a hypersurface of $\bar{M}$ is defined by $b(X) = -\overline{\nabla}_X \mathbf{u}$. Let H denote its *mean curvature* function. We point out that $H = -\overline{\mathrm{div}}\mathbf{u}$, that is, the averaged *Hubble expansion parameter* at points of M in relativistic cosmology (see [18, §3.3.1] or [6, p. 161]).

The *f-mean curvature* or *weighted mean curvature* of M is defined by

$$H_f = H - (\overline{\nabla} f)^{\perp},$$

where $(\overline{\nabla} f)^{\perp}$ is a normal projection of $\overline{\nabla} f$ on M.

We will use the following technical lemma obtained in [7] (see also [5]).

Lemma 3.1. *Under the above conditions, let $\gamma(s)$ be a normalized geodesic in M and set $X = \gamma'$. Then*

$$\begin{aligned} \mathrm{Ric}(X, X) = \overline{\mathrm{Ric}}(X, X) &- \langle a(X), X\rangle + \langle v(X), X\rangle H + \langle v(X), X\rangle^2 \\ &+ \sum_{j=2}^{n} \langle v(X), e_j\rangle^2, \end{aligned} \tag{3.1}$$

where $\{e_1 = X, e_2, \ldots, e_n\}$ is an orthonormal basis of T_pM.

Finally, we are in position to prove our closure theorem.

Proof of Theorem 1.3: Taking into account that X is invariant under the flow, we get $v(X) = -b(X)$. Then

$$\begin{aligned} \overline{\mathrm{Hess}}\, f(X, X) &= \langle \overline{\nabla}_X \overline{\nabla} f, X\rangle \\ &= \mathrm{Hess}\, f(X, X) + \langle \overline{\nabla} f, \mathbf{u}\rangle \langle b(X), X\rangle \\ &= \mathrm{Hess}\, f(X, X) - \langle \overline{\nabla} f, \mathbf{u}\rangle \langle v(X), X\rangle. \end{aligned} \tag{3.2}$$

From equations (1), (3.1) and (3.2) we get easily

$$\mathrm{Ric}_f^k(X, X) \geq \overline{\mathrm{Ric}}_f^k(X, X) - \langle a(X), X\rangle + \langle v(X), X\rangle H_f,$$

for all $k > 0$. Thus

$$(n-1)a - \operatorname{Ric}_f^k(X,X) \leq (n-1)a - \overline{\operatorname{Ric}}_f^k(X,X) + \langle a(X), X\rangle - \langle v(X), X\rangle H_f.$$

Applying Theorem 1.1 we have the desired result. $\square$

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