Spaces with property $(DC(\omega_1))$

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Abstract. We prove that if X is a first countable space with property $(DC(\omega_1))$ and with a G_δ-diagonal then the cardinality of X is at most $\mathfrak{c}$. We also show that if X is a first countable, DCCC, normal space then the extent of X is at most $\mathfrak{c}$.

Keywords: G_δ-diagonal; property $(DC(\omega_1))$; cardinal; DCCC

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1. Introduction

Diagonal property is useful in estimating the cardinality of a space. For example, Ginsburg and Woods in [6] proved that the cardinality of a space with countable extent and a G_δ-diagonal is at most $\mathfrak{c}$. Therefore, if X is Lindelöf and has a G_δ-diagonal then $|X| \leq \mathfrak{c}$. However, the cardinality of a regular space with the countable Souslin number and a G_δ-diagonal need not have an upper bound [11], [12]. Buzyakova in [4] proved that if a space X with the countable Souslin number has a regular G_δ-diagonal then the cardinality of X does not exceed $\mathfrak{c}$. Recently, Xuan and Shi in [13] show that if X is a DCCC space with a rank 3-diagonal then the cardinality of X is at most $\mathfrak{c}$.

In this paper, we prove that if X is a first countable space with property $(DC(\omega_1))$ (defined below) and with a G_δ-diagonal then the cardinality of X is at most $\mathfrak{c}$. We also show that if X is a first countable, DCCC, normal space then the extent of X is at most $\mathfrak{c}$.

2. Notation and terminology

All the spaces are assumed to be Hausdorff unless otherwise stated.

The cardinality of a set X is denoted by $|X|$, and $[X]^2$ will denote the set of two-element subsets of X. We write ω for the first infinite cardinal and $\mathfrak{c}$ for the cardinality of the continuum.

Definition 2.1. We say that a topological space X has a G_δ-diagonal if there exists a sequence$\{G_n : n \in \omega\}$ of open sets in X^2 such that $\Delta_X = \bigcap\{G_n : n < \omega\}$, where $\Delta_X = \{(x, x) : x \in X\}$.

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Definition 2.2. A space X has a strong rank 1-diagonal [2] if there exists a sequence $\{\mathcal{U}_n : n < \omega\}$ of open covers of X such that for each $x \in X$, $\{x\} = \bigcap\{\overline{\mathrm{St}(x,\mathcal{U}_n)} : n < \omega\}$.

Note that a space having a strong rank 1-diagonal always has a G_δ-diagonal.

Definition 2.3. A topological space X is pracompact if it has a dense subspace every infinite subset of which has a limit point in X.

Clearly, every countably compact space is pracompact and it is easy to see that every countably pracompact space is DFCC, i.e., every infinite family ξ of open sets of X has an accumulation point in X. It should be pointed out that for Tychonoff spaces DFCC is equivalent to pseudocompactness, i.e., every continuous real-valued function on X is bounded.

$(DC(\omega_1))$ is a property which is the analog of countable pracompactness.

Definition 2.4. A topological space X has property $(DC(\omega_1))$ if it has a dense subspace every uncountable subset of which has a limit point in X.

This notion was first introduced and studied in [7] by Ikenaga. It is clear that every countably pracompact space and every space with a dense subspace of countable extent is $(DC(\omega_1))$.

Definition 2.5. We say that a space X satisfies the Discrete Countable Chain Condition (DCCC for short) if every discrete family of non-empty open subsets of X is countable.

All notation and terminology not explained here is given in [5].

3. Results

We will use the following countable version of a set-theoretic theorem due to Erdös and Radó.

Lemma 3.1 ([8, p. 8]). *Let X be a set with $|X| > \mathfrak{c}$ and suppose $[X]^2 = \bigcup\{P_n : n \in \omega\}$. Then there exist $n_0 < \omega$ and a subset S of X with $|S| > \omega$ such that $[S]^2 \subset P_{n_0}$.*

Theorem 3.2. *Let X be a first countable space with property $(DC(\omega_1))$ and with a G_δ-diagonal. Then the cardinality of X is at most $\mathfrak{c}$.*

Proof: Since X has a G_δ-diagonal, there exists a sequence $\{G_k : k < \omega\}$ of open sets of X^2 such that $\Delta_X = \bigcap\{G_k : k < \omega\}$. For each $k \in \omega$ and $x \in X$, there exists an open subset $V_k(x)$ of X such that $(x, x) \in V_k(x) \times V_k(x) \subset G_k$. Thus without loss of generality, we assume that $G_k = \bigcup\{V_k(x) \times V_k(x) : x \in X\}$ and $G_{k+1} \subset G_k$.

Assume that Y is a dense subspace of X which witnesses that X has property $(DC(\omega_1))$. We shall show that $|Y| \leq \mathfrak{c}$. Suppose not. For $n < \omega$, let

$$P_n = \Big\{\{x, y\} \in [Y]^2 : (x, y) \notin G_n\Big\}.$$

Clearly, for any $\{x,y\} \in [Y]^2$, there exists $n < \omega$ such that $\{x,y\} \in P_n$. Thus, $[Y]^2 = \bigcup\{P_n : n < \omega\}$. Then by Lemma 3.1 there exists a subset S of Y with $|S| > \omega$ and $[S]^2 \subset P_{n_0}$ for some $n_0 < \omega$. Since X has property $(DC(\omega_1))$, it follows that S has a limit point $x \in X$. Since X is T_1 each neighborhood of x meets infinitely many members of S. In particular, there exist distinct points y and z in $S \cap V_{n_0}(x)$. Thus $(y,z) \in V_{n_0}(x) \times V_{n_0}(x) \subset G_{n_0}$. However, since $\{y,z\} \in P_{n_0}$, $(y,z) \notin G_{n_0}$, which is a contradiction. This shows $|Y| \leq \mathfrak{c}$.

From Theorem 4.4 of [8, p. 55] that every first countable Hausdorff space with a dense subset of cardinality $\leq \mathfrak{c}$ has cardinality $\leq \mathfrak{c}$, we conclude that $|X| \leq \mathfrak{c}$. This completes the proof. □

The authors do not know whether the condition "first countable" is necessary in Theorem 3.2. However, we know that if we drop the condition "property $(DC(\omega_1))$" or "G_δ-diagonal", the conclusion will be no longer true, as can be seen in the following examples.

Example 3.3. Let D be the discrete space with $|D| = 2^{\mathfrak{c}}$. It is evident that D is first countable and has a G_δ-diagonal. However it does not have property $(DC(\omega_1))$.

Example 3.4. Let X be the subspace of $[0, 2^{\mathfrak{c}}]$, consisting of all ordinals of countable cofinality, equipped with the ordered topology. Then X has cardinality $2^{\mathfrak{c}}$. Moreover X is first countable and countably compact, and hence X is $(DC(\omega_1))$. However, it does not have a G_δ-diagonal.

Clearly, every point of any space with a G_δ-diagonal is a G_δ-point. By applying Lemma 2 of [3] that if every point of a regular DFCC space X is a G_δ-point then X is first countable, we can conclude the following conclusion.

Corollary 3.5. *Let X be a regular countably pracompact space with a G_δ-diagonal. Then the cardinality of X is at most $\mathfrak{c}$.*

However, a Tychonoff pseudocompact space with a G_δ-diagonal can have arbitrarily big cardinality.

Example 3.6. For every cardinal τ, there exists a pseudocompact space of cardinality $> \tau$ but having a G_δ-diagonal [9, p. 34].

Theorem 3.7. *Let X be a regular DFCC space with a strong rank 1-diagonal. Then the cardinality of X is at most $\mathfrak{c}$.*

Proof: By Theorem 3.7 of [2], it is easy to deduce that X is a Moore space, and hence X is perfect. Moreover, every DFCC, perfect space has countable chain condition (short for CCC) [10, Proposition 2.3]. Since the cardinality of a first countable, CCC space is at most $\mathfrak{c}$, it follows that $|X| \leq \mathfrak{c}$. □

Recall that the extent $e(X)$ of X is the supremum of the cardinalities of closed discrete subsets of X. A space X is star countable if whenever $\mathcal{U}$ is an open cover of X, there is a countable subset $A \subset X$ such that $\mathrm{St}(A, \mathcal{U}) = X$. It is clear that a space with countable extent is star countable.

Theorem 3.8. *Let X be a first countable, DCCC, normal space. Then the extent $e(X)$ of X is at most $\mathfrak{c}$.*

Proof: Suppose that $e(X) > \mathfrak{c}$. Then there exists a closed and discrete subset Y of X such that $|Y| > \mathfrak{c}$. Let $\mathcal{B}(x) = \{B_n(x) : n < \omega\}$ be a local base for x. Assume $B_{n+1}(x) \subset B_n(x)$ for any $n < \omega$. For each $n < \omega$ let

$$P_n = \left\{\{x, y\} \in [Y]^2 : B_n(x) \cap B_n(y) = \emptyset\right\}.$$

Thus, $[Y]^2 = \bigcup\{P_n : n \in \omega\}$. Then by Lemma 3.1 there exists a subset S of Y with $|S| > \omega$ and $[S]^2 \subset P_{n_0}$ for some $n_0 < \omega$. Since $S \subset Y$, one easily sees that S is closed and discrete. Besides, it is evident that for any two distinct points $x, y \in S$, $B_{n_0}(x) \cap B_{n_0}(y) = \emptyset$.

Since X is normal, there exists an open subset U of X such that $S \subset U \subset \overline{U} \subset \bigcup\{B_{n_0}(x) : x \in S\}$. Let $\mathcal{U} = \{B_{n_0}(x) \cap U : x \in S\}$. It must have a cluster point $y \in X$, since X is DCCC. Since $\overline{B_{n_0}(x) \cap U} \subset \overline{U} \subset \bigcup\{B_{n_0}(x) : x \in S\}$, we can conclude that $y \in \bigcup\{B_{n_0}(x) : x \in S\}$. Now we assume that $y \in B_{n_0}(x_0)$ for some $x_0 \in S$. It is clear to see that $B_{n_0}(x_0) \cap B_{n_0}(x) = \emptyset$ for any $x \in S \setminus \{x_0\}$. This shows that y is not a cluster point of $\{B_{n_0}(x) \cap U : x \in S\}$. A contradiction! This proves that $e(X) \leq \mathfrak{c}$. $\square$

Remark 3.9. Theorem 3.8 would be compared to a recent result of [1]: if X is first countable and $e(X) > \mathfrak{c}$, then X is not star countable. It can be proved by using our method in the proof of Theorem 3.8 that, clearly, X has an uncountable closed discrete subset S whose points can be separated by pairwise disjoint open sets. For each $x \in S$, let $U_x \subset X$ be an open set containing x such that for each $y \in S \setminus \{x\}$, $U_x \cap U_y = \emptyset$. Then $\mathcal{U} = \{U_x : x \in S\} \cup \{X \setminus S\}$ is an open cover for which there is no countable A of X such that $\mathrm{St}(A, \mathcal{U}) = X$. This shows that X is not star countable.

We finish this paper by the following question.

Question 3.10. Must a first countable, DCCC, normal space be star countable?

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