Applications of limited information strategies in Menger's game

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Abstract. As shown by Telgársky and Scheepers, winning strategies in the Menger game characterize σ-compactness amongst metrizable spaces. This is improved by showing that winning Markov strategies in the Menger game characterize σ-compactness amongst regular spaces, and that winning strategies may be improved to winning Markov strategies in second-countable spaces. An investigation of 2-Markov strategies introduces a new topological property between σ-compact and Menger spaces.

Keywords: Menger property; Menger game; σ-compact spaces; limited information strategies

Classification: 03E35, 54D20, 54D45, 91A44

1. The Menger property and game

Definition 1.1. A space X is Menger if for every sequence $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$ of open covers of X there exists a sequence $\langle \mathcal{F}_0, \mathcal{F}_1, \dots \rangle$ such that $\mathcal{F}_n \subseteq \mathcal{U}_n$, $|\mathcal{F}_n| < \omega$, and $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

Note that many authors refer to this property as $S_{fin}(\mathcal{O}, \mathcal{O})$ [6], where $\mathcal{O}$ is the collection of open covers of X, and $S_{fin}(A, B)$ denotes the selection property such that for each sequence in A^ω, there are finite subsets of each entry for which the union of these subsets belongs in B.

Proposition 1.2. *X is σ-compact $\Rightarrow$ X is Menger $\Rightarrow$ X is Lindelöf.*

None of these implications may be reversed; the irrationals are a simple example of a Lindelöf space which is not Menger, and we'll see several examples of Menger spaces which are not σ-compact.

It will be convenient to consider subsets of X rather than subsets of the open covers.

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Definition 1.3. For each cover $\mathcal{U}$ of X, $S \subseteq X$ is *$\mathcal{U}$-finite* if there exists a finite subcollection of $\mathcal{U}$ which covers S.

Of course, a compact space is $\mathcal{U}$-finite for all open covers $\mathcal{U}$.

Proposition 1.4. *A space X is Menger if and only if for every sequence $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$ of open covers of X there exists a sequence $\langle F_0, F_1, \ldots \rangle$ such that $F_n \subseteq X$, F_n is $\mathcal{U}_n$-finite, and $X = \bigcup_{n<\omega} F_n$.*

This is the characterization we will use in this paper. A game version of the Menger property is also often considered.

Game 1.5. Let $Men(X)$ denote the *Menger game* with players $\mathscr{C}$, $\mathscr{F}$. In round n, $\mathscr{C}$ chooses an open cover $\mathcal{U}_n$, followed by $\mathscr{F}$ choosing a $\mathcal{U}_n$-finite subset F_n of X.

$\mathscr{F}$ wins the game if $X = \bigcup_{n<\omega} F_n$, and $\mathscr{C}$ wins otherwise.

As with the Menger property, authors usually characterize this game using finite subsets $\mathcal{F}_n$ of $\mathcal{U}_n$ instead (and often refer to it as the selection game $G_{fin}(\mathcal{O}, \mathcal{O})$). This is obviously equivalent in the case of perfect information, and is also equivalent in the case of limited information, provided $\mathscr{F}$ knows $\mathcal{U}_n$ during round n. However, we make this change as we will investigate 0-Markov strategies which consider no moves of the opponent, and instead rely only on the current round number.

This game may be used to characterize the Menger property.

Definition 1.6. If $\mathscr{A}$ has a winning strategy for a game G (which defeats every possible counterattack by her opponent), then we write $\mathscr{A} \uparrow G$.

Theorem 1.7 (Hurewicz [2])**.** *A space X is Menger if and only if $\mathscr{C} \not\uparrow Men(X)$.*

2. Limited information strategies

Definition 2.1. A k-tactical strategy for a game G with moveset M is a function $\sigma : M^{\leq k} \to M$; intuitively, it is a strategy which only considers the previous k or less moves of the opponent $t \in M^{\leq k}$, and yields the appropriate move $\sigma(t)$ for the player using it. If a winning k-tactical strategy exists for $\mathscr{P}$ in the game G, then we write $\mathscr{P} \underset{k\text{-tact}}{\uparrow} G$.

Definition 2.2. A k-Markov strategy for a game G with moveset M is a function $\sigma : M^{\leq k} \times \omega \to M$; intuitively, it is a strategy which only considers the previous k or less moves of the opponent $t \in M^{\leq k}$ and the round number $n < \omega$, and yields the appropriate move $\sigma(t, n)$ for the player using it. If a winning k-Markov strategy exists for $\mathscr{P}$ in the game G, then we write $\mathscr{P} \underset{k\text{-mark}}{\uparrow} G$.

We will call k-tactical strategies "k-tactics" and k-Markov strategies "k-marks". If the k is omitted then it is assumed that $k = 1$. In addition, note that some authors refer to tactics as *stationary strategies*.

Tactics will be of interest in a game discussed later; proving the following is an easy exercise.

Proposition 2.3. *X is compact if and only if $\mathscr{F} \underset{tact}{\uparrow} Men\,(X)$ if and only if $\mathscr{F} \underset{k+1\text{-}tact}{\uparrow} Men\,(X)$ for some $k < \omega$.*

Essentially, because $\mathscr{C}$ may repeat the same finite sequence of open covers, $\mathscr{F}$ needs to be seeded with knowledge of the round number to prevent being trapped in a loop.

If $\mathscr{F}$'s memory of $\mathscr{C}$'s past moves is bounded, then there is no need to consider more than the two most recent moves. The intuitive reason is that $\mathscr{C}$ could simply play the same cover repeatedly until $\mathscr{F}$'s memory is exhausted, in which case $\mathscr{F}$ would only ever see the change from one cover to another.

Theorem 2.4. *For each $k < \omega$, $F \underset{(k+2)\text{-}mark}{\uparrow} Men\,(X)$ if and only if $F \underset{2\text{-}mark}{\uparrow} Men\,(X)$.*

Proof: Let σ be a winning $(k+2)$-mark. We define the 2-mark τ as follows:

$$\tau(\langle \mathcal{U} \rangle, 0) = \bigcup_{m<k+2} \sigma(\langle \underbrace{\mathcal{U}, \dots, \mathcal{U}}_{m+1} \rangle, m),$$

$$\tau(\langle \mathcal{U}, \mathcal{V} \rangle, n+1) = \bigcup_{m<k+2} \sigma(\langle \underbrace{\mathcal{U}, \dots, \mathcal{U}}_{k+1-m}, \underbrace{\mathcal{V}, \dots, \mathcal{V}}_{m+1} \rangle, (n+1)(k+2)+m).$$

Let $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$ be an attack by $\mathscr{C}$ against τ. Then consider the attack

$$\langle \underbrace{\mathcal{U}_0, \dots, \mathcal{U}_0}_{k+2}, \underbrace{\mathcal{U}_1, \dots, \mathcal{U}_1}_{k+2}, \dots \rangle$$

by $\mathscr{C}$ against σ. Since σ is a winning $(k+2)$-mark,

$$\begin{aligned} X = & \bigcup_{m<k+2} \sigma(\langle \underbrace{\mathcal{U}_0, \dots, \mathcal{U}_0}_{m+1} \rangle, m) \\ & \cup \bigcup_{n<\omega} \bigcup_{m<k+2} \sigma(\langle \underbrace{\mathcal{U}_n, \dots, \mathcal{U}_n}_{k+1-m}, \underbrace{\mathcal{U}_{n+1}, \dots, \mathcal{U}_{n+1}}_{m+1} \rangle, (n+1)(k+2)+m) \\ & = \tau(\langle \mathcal{U}_0 \rangle, 0) \cup \bigcup_{n<\omega} \tau(\langle \mathcal{U}_n, \mathcal{U}_{n+1} \rangle, n+1). \end{aligned}$$

Thus τ is a winning 2-mark. Of course, it is trivial to define a winning $(k+2)$-mark from a 2-mark by simply ignoring the extra k moves. $\square$

It is worth noting that the above proof holds for any selection game of the form $G_{fin}(A, B)$.

One might wonder if a 2-mark may always be improved to a 1-mark as well; this is not the case.

Definition 2.5. For any cardinal κ, let $L(\kappa) = \kappa \cup \{\infty\}$ denote the *one-point Lindelöfication* of discrete κ, where points in κ are isolated, and the neighborhoods of ∞ are the co-countable sets containing it.

One may prove directly that $\mathscr{F} \underset{\text{mark}}{\not\uparrow} Men\,(L(\omega_1))$ without much effort; however, we postpone stating this until we have shown that being Markov Menger is equivalent to being σ-compact in regular spaces. Likewise, we postpone proving $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men\,(L(\omega_1))$ in order to consider an equivalent set-theoretic game.

Essentially, the greatest advantage of a strategy which has knowledge of two or more previous moves of the opponent, versus only knowledge of the most recent move, is the ability to react to changes from one round to the next. It is this ability to react that often allows a player to wield a winning 2-Markov strategy when a winning 1-Markov strategy does not exist.

3. Scheepers' countable-finite games

We now turn to a related game whose k-tactics were studied by Marion Scheepers in [4].

Game 3.1. Let $Sch^{\cup,\subsetneq}(\kappa)$ denote *Scheepers' strict countable-finite union game* with two players $\mathscr{C}$, $\mathscr{F}$. In round 0, $\mathscr{C}$ chooses $C_0 \in [\kappa]^{\leq\omega}$, followed by $\mathscr{F}$ choosing $F_0 \in [\kappa]^{<\omega}$. In round $n+1$, $\mathscr{C}$ chooses $C_{n+1} \in [\kappa]^{\leq\omega}$ such that $C_{n+1} \supsetneq C_n$, followed by $\mathscr{F}$ choosing $F_{n+1} \in [\kappa]^{<\omega}$.

$\mathscr{F}$ wins the game if $\bigcup_{n<\omega} F_n \supseteq \bigcup_{n<\omega} C_n$; otherwise, $\mathscr{C}$ wins.

In $Men\,(L(\kappa))$, $\mathscr{C}$ essentially chooses a countable set not included in her neighborhood of ∞, followed by $\mathscr{F}$ choosing a finite subset of this complement to cover during each round. Thus, $\mathscr{F}$ need only be concerned with the *intersection* of the countable sets chosen by $\mathscr{C}$ in $Men\,(L(\kappa))$, rather than the union as in $Sch^{\cup,\subsetneq}(\kappa)$.

Another difference between these games: Scheepers required that $\mathscr{C}$ always choose strictly growing countable sets. If the goal is to study tactics, then $\mathscr{C}$ cannot be allowed to trap $\mathscr{F}$ in a loop by repeating the same moves. But by eliminating this requirement, the study can then turn to Markov strategies, bringing the game further in line with the Menger game played upon $L(\kappa)$.

We introduce a few games to make the relationship between Scheepers' $Sch^{\cup,\subsetneq}(\kappa)$ and $Men(L(\kappa))$ more precise.

Game 3.2. Let $Sch^{\cup,\subseteq}(\kappa)$ denote the *Scheepers' countable-finite union game* which proceeds analogously to $Sch^{\cup,\subsetneq}(\kappa)$, except that $\mathscr{C}$'s restriction in round $n+1$ is reduced to $C_{n+1} \supseteq C_n$.

Game 3.3. Let $Sch^{1,\subseteq}(\kappa)$ denote the *Scheepers' countable-finite initial game* which proceeds analogously to $Sch^{\cup,\subseteq}(\kappa)$, except that $\mathscr{F}$ wins whenever $\bigcup_{n<\omega} F_n \supseteq C_0$.

Game 3.4. Let $Sch^{\cap}(\kappa)$ denote the *Scheepers' countable-finite intersection game* which proceeds analogously to $Sch^{1,\subseteq}(\kappa)$, except that $\mathscr{C}$ may choose any $C_n \in [\kappa]^{\leq\omega}$ each round, and $\mathscr{F}$ wins whenever $\bigcup_{n<\omega} F_n \supseteq \bigcap_{n<\omega} C_n$.

$$
\begin{array}{ccccc}
 & & \mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Men(L(\kappa)) & \Longleftrightarrow & \mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Sch^{\cap}(\kappa) \\
 & & \Downarrow & & \Downarrow \\
\mathscr{F} \underset{(k+1)\text{-tact}}{\uparrow} Sch^{\cup,\subsetneq}(\kappa) & \Longrightarrow & \mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Sch^{\cup,\subseteq}(\kappa) & \Longleftrightarrow & \mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Sch^{1,\subseteq}(\kappa) \\
 & & \Downarrow & \swarrow & \\
 & & \mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Sch^{\cup,\subsetneq}(\kappa) & &
\end{array}
$$

FIGURE 1. Diagram of Scheepers/Menger game implications

Theorem 3.5. *For any cardinal $\kappa \geq \omega$ and integer $k < \omega$, Figure 1 holds.*

PROOF: We show $\mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Men(L(\kappa)) \Rightarrow \mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Sch^{\cap}(\kappa)$. Let σ be a winning $(k+1)$-mark for $\mathscr{F}$ in $Men(L(\kappa))$. Let $\mathcal{U}(C)$ (resp. $\mathcal{U}(s)$) convert each countable subset C of κ (resp. finite sequence s of such subsets) into the open cover $[C]^1 \cup \{L(\kappa) \setminus C\}$ (resp. finite sequence of such open covers). Then τ defined by

$$\tau(s^\frown\langle C\rangle, n) = C \cap \sigma(\mathcal{U}(s^\frown\langle C\rangle), n)$$

where $|s| \leq k$ is a winning $(k+1)$-mark for $\mathscr{F}$ in $Sch^{\cap}(\kappa)$.

We show $\mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Sch^{\cap}(\kappa) \Rightarrow \mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Men(L(\kappa))$. Let σ be a winning $(k+1)$-mark for $\mathscr{F}$ in $Sch^{\cap}(\kappa)$. Let $C(\mathcal{U})$ (resp. $C(s)$) convert each open cover $\mathcal{U}$ of $L(\kappa)$ (resp. finite sequence s of such covers) into a countable set C which is the complement of some neighborhood of ∞ in $\mathcal{U}$ (resp. finite sequence of such countable sets). Then τ defined by

$$\tau(s^\frown\langle\mathcal{U}\rangle, n) = (L(\kappa) \setminus C(\mathcal{U})) \cup \sigma(C(s^\frown\langle\mathcal{U}\rangle), n)$$

where $|s| \leq k$ is a winning $(k+1)$-mark for $\mathscr{F}$ in $Men\,(L(\kappa))$.

We show $\mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Sch^{\cap}(\kappa) \Rightarrow \mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Sch^{1,\subseteq}(\kappa)$. Let σ be a winning $(k+1)$-mark for $\mathscr{F}$ in $Sch^{\cap}(\kappa)$. Then σ is also a winning $(k+1)$-mark for $\mathscr{F}$ in $Sch^{1,\subseteq}(\kappa)$.

We show $\mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Sch^{1,\subseteq}(\kappa) \Rightarrow \mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Sch^{\cup,\subseteq}(\kappa)$. Let σ be a winning $(k+1)$-mark for $\mathscr{F}$ in $Sch^{1,\subseteq}(\kappa)$. For each finite sequence s, let $t \preceq s$ mean t is a final subsequence of s. Then τ defined by

$$\tau(s^\frown\langle C\rangle, n) = \bigcup_{t \preceq s, m \leq n} \sigma(t^\frown\langle C\rangle, m)$$

where $|s| \leq k$ is a winning $(k+1)$-mark for $\mathscr{F}$ in $Sch^{\cup,\subseteq}(\kappa)$, considering that σ ensures that C_n is covered when attacked by $\langle C_n, C_{n+1}, \ldots\rangle$ for all $n < \omega$.

We show $\mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Sch^{\cup,\subseteq}(\kappa) \Rightarrow \mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Sch^{1,\subseteq}(\kappa)$. Let σ be a winning $(k+1)$-mark for $\mathscr{F}$ in $Sch^{\cup,\subseteq}(\kappa)$. σ is also a winning $(k+1)$-mark for $\mathscr{F}$ in $Sch^{1,\subseteq}(\kappa)$.

We show $\mathscr{F} \underset{(k+1)\text{-tact}}{\uparrow} Sch^{\cup,\subsetneq}(\kappa) \Rightarrow \mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Sch^{\cup,\subseteq}(\kappa)$. Let σ be a winning $(k+1)$-tactic for $\mathscr{F}$ in $Sch^{\cup,\subsetneq}(\kappa)$. For each countable subset C of κ, let $C+n$ be the union of C with the n least ordinals in $\kappa \setminus C$. Then for every valid attack $\langle C_0, C_1, C_2, \ldots\rangle$ by $\mathscr{C}$ in $Sch^{\cup,\subseteq}(\kappa)$, it follows that $\langle C_0, C_1+1, C_2+2, \ldots\rangle$ is a valid attack by $\mathscr{C}$ in $Sch^{\cup,\subsetneq}(\kappa)$ as $C_{n+1} \supseteq C_n$ ensures that $C_{n+1}+(n+1)$ includes all the n added elements in C_n+n in addition to at least one more. Thus τ defined by

$$\tau(\langle C_n, \ldots, C_{n+i}\rangle, n+i) = \sigma(\langle C_n+n, \ldots, C_{n+i}+(n+i)\rangle)$$

where $i \leq k$ is a winning $(k+1)$-mark for $\mathscr{F}$ in $Sch^{\cup,\subseteq}(\kappa)$, considering that σ ensures that $\bigcup_{n<\omega}(C_n+n) \supseteq \bigcup_{n<\omega} C_n$ is covered when attacked by $\langle C_0, C_1+1, C_2+2, \ldots\rangle$.

We show $\mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Sch^{\cup,\subseteq}(\kappa) \Rightarrow \mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Sch^{\cup,\subsetneq}(\kappa)$. Let σ be a winning $(k+1)$-mark for $\mathscr{F}$ in $Sch^{\cup,\subseteq}(\kappa)$. Then σ is also a winning $(k+1)$-mark for $\mathscr{F}$ in $Sch^{\cup,\subsetneq}(\kappa)$. □

While we have not shown a direct implication between the Menger game and Scheepers' original countable-finite game, Scheepers introduced the statement $\mathcal{A}(\kappa)$ relating to the almost-compatibility of injective functions from countable subsets of κ into ω which may be applied to both.

Definition 3.6. For two functions f, g we say f is **almost compatible** with g ($f \|^* g$) if $|\{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : f(x) \neq g(x)\}| < \omega$.

Definition 3.7. $\mathcal{A}(\kappa)$ states that there exist injective functions $f_A : A \to \omega$ for each $A \in [\kappa]^{\leq\omega}$ such that $f_A \|^* f_B$ for all $A, B \in [\kappa]^\omega$.

Scheepers went on to show that $\mathcal{A}(\kappa)$ implies $\mathscr{F} \underset{\text{2-tact}}{\uparrow} Sch^{\cup,\subsetneq}(\kappa)$. This proof, along with the following facts, gives us inspiration for finding a winning 2-Markov strategy in the Menger game played on $L(\kappa)$.

Theorem 3.8. *$\mathcal{A}(\omega_1)$ and $\neg\mathcal{A}(\mathfrak{c}^+)$ are theorems of ZFC. $\mathcal{A}(\mathfrak{c})$ is a theorem of $ZFC + CH$ and is consistent with $ZFC + \neg CH$.*

Proof: The construction of an Aronzajn tree in [3, Theorem 5.9] produced a witness for $\mathcal{A}(\omega_1)$; this of course implies $\mathcal{A}(\mathfrak{c})$ under CH. $\neg\mathcal{A}(\mathfrak{c}^+)$ is shown by a cardinality argument in [4]. The consistency result under $ZFC + \neg CH$ is a lemma for the main theorem in [4]. □

$$\begin{array}{ccccc}
\mathcal{A}(\kappa) & \Longrightarrow & \mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Men(L(\kappa)) & \Longleftrightarrow & \mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Sch^{\cap}(\kappa) \\
\Downarrow & & \Downarrow & & \Downarrow \\
\mathscr{F} \underset{(k+1)\text{-tact}}{\uparrow} Sch^{\cup,\subsetneq}(\kappa) & \Longrightarrow & \mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Sch^{\cup,\subseteq}(\kappa) & \Longleftrightarrow & \mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Sch^{1,\subseteq}(\kappa) \\
 & & \Downarrow & \swarrow & \\
 & & \mathscr{F} \underset{(k+1)\text{-mark}}{\uparrow} Sch^{\cup,\subsetneq}(\kappa) & &
\end{array}$$

Figure 2. Diagram of Scheepers/Menger game implications with $\mathcal{A}(\kappa)$

Theorem 3.9. *$\mathcal{A}(\kappa)$ implies the game-theoretic results in Figure 2.*

Proof: Since $\mathcal{A}(\kappa) \Rightarrow \mathscr{F} \underset{\text{2-tact}}{\uparrow} Sch^{\cup,\subsetneq}(\kappa)$ was a main result of [4], we need only show that $\mathcal{A}(\kappa) \Rightarrow \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cap}(\kappa)$. Let f_A for $A \in [\kappa]^{\leq\omega}$ witness $\mathcal{A}(\kappa)$. We define the 2-mark σ as follows:

$$\sigma(\langle C \rangle, 0) = \{\alpha \in C : f_C(\alpha) \leq 0\}$$

$$\sigma(\langle C, D \rangle, n+1) = \{\alpha \in C \cap D : f_D(\alpha) \leq n+1 \text{ or } f_C(\alpha) \neq f_D(\alpha)\}.$$

For any attack $\langle C_0, C_1, \ldots \rangle$ by $\mathscr{C}$ and $\alpha \in \bigcap_{n<\omega} C_n$, either $f_{C_n}(\alpha)$ is constant for all n, or $f_{C_n}(\alpha) \neq f_{C_{n+1}}(\alpha)$ for some n; either way, α is covered. □

Corollary 3.10. $\mathscr{F} \underset{\textit{2-mark}}{\uparrow} Men(L(\omega_1))$.

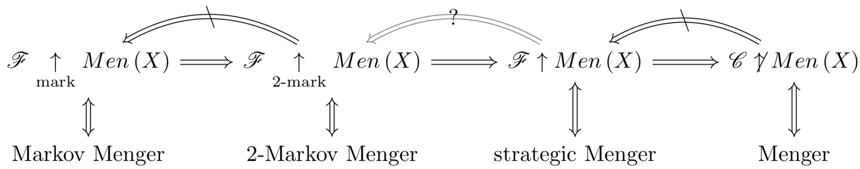

FIGURE 3. Diagram of covering properties related to the Menger game

4. Menger game derived covering properties

Limited information strategies for the Menger game naturally define the spectrum of covering properties shown in Figure 3. However, we do not know if the middle two properties are actually distinct.

Question 4.1. Does there exist a space X such that $\mathscr{F} \uparrow Men(X)$ but $\mathscr{F} \underset{\text{2-mark}}{\not\uparrow} Men(X)$?

We are also interested in non-game-theoretic characterizations of these covering properties. It has been known for some time that metrizable strategic Menger spaces are exactly the metrizable σ-compact spaces, shown first by Telgársky in [8] and later directly by Scheepers in [5]. It is Scheepers' technique which gives inspiration to the following.

In the interest of generality, we will first characterize the Markov Menger spaces without any separation axioms.

Definition 4.2. A subset Y of X is *relatively compact* to X, or Y is *compact in* X, if for every open cover of X, there exists a finite subcollection which covers Y.

Definition 4.3. A subset Y of X is *precompact* in X if $\mathrm{cl}_X(Y)$ is compact.

Some authors use the term relative compactness to denote precompactness, which causes no confusion in the context of regular spaces.

Proposition 4.4. *For regular spaces, Y is relatively compact to X if and only if Y is precompact in X.*

PROOF: Shown by Arhangel'skii in [1]; a proof is provided for the convenience of the reader.

Let $\mathcal{U} = \{U_y : y \in \mathrm{cl}_X(Y)\}$ where each U_y is an open set in X containing y; apply regularity to get $y \in V_y \subseteq \mathrm{cl}_X(V_y) \subseteq U_y$ with V_y open in X. Then $\{X \setminus \mathrm{cl}_X(Y)\} \cup \{V_y : y \in \mathrm{cl}_X(Y)\}$ is an open cover of X, so use relative compactness to choose a finite subset F of $\mathrm{cl}_X(Y)$ such that $\{V_y : y \in F\}$ is a finite cover of Y.

Then since $\bigcup\{\mathrm{cl}_X(V_y) : y \in F\}$ is a closed set in X containing Y, it contains $\mathrm{cl}_X(Y)$, and thus $\{U_y : y \in F\}$ is a finite subcollection of $\mathcal{U}$ covering $\mathrm{cl}_X(Y)$.

For the reverse direction, simply take a finite subcover of $\mathrm{cl}_X(Y)$ to obtain a finite subcover of Y. □

Lemma 4.5. *Let σ be a Markov strategy for $\mathscr{F}$ in $Men(X)$, and let $\mathcal{O}$ collect all open covers of X. Then the set*

$$R_n = \bigcap_{\mathcal{U} \in \mathcal{O}} \sigma(\mathcal{U}, n)$$

is relatively compact to X. If σ is a winning Markov startegy, then $\bigcup_{n<\omega} R_n = X$.

PROOF: First, for every open cover $\mathcal{U} \in \mathcal{O}$, $R_n \subseteq \sigma(\mathcal{U}, n)$ is covered by a finite subcollection of $\mathcal{U}$.

Suppose that $x \notin R_n$ for any $n < \omega$. Then for each n, pick $\mathcal{U}_n \in \mathcal{O}$ such that $x \notin \sigma(\mathcal{U}_n, n)$. Then $\mathscr{C}$ may counter σ with the attack $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$. □

Definition 4.6. A *σ-relatively-compact* space is the countable union of relatively compact subsets.

Corollary 4.7. *The following are equivalent:*

- *X is σ-relatively-compact,*
- $\mathscr{F} \underset{0\text{-}mark}{\uparrow} Men(X)$,
- $\mathscr{F} \underset{mark}{\uparrow} Men(X)$.

PROOF: If $X = \bigcup_{n<\omega} R_n$ for R_n relatively compact, then $\sigma(n) = R_n$ defines a winning 0-mark σ, which of course gives a winning 1-mark. The previous lemma finishes the proof. □

Corollary 4.8. *Let X be a regular space. The following are equivalent:*

- *X is σ-compact,*
- *X is σ-relatively-compact,*
- $\mathscr{F} \underset{0\text{-}mark}{\uparrow} Men(X)$,
- $\mathscr{F} \underset{mark}{\uparrow} Men(X)$.

Corollary 4.9. $\mathscr{F} \underset{mark}{\not\uparrow} Men(L(\omega_1))$.

Note that for Lindelöf spaces, metrizability is characterized by regularity and second-countability.

Lemma 4.10. *Let X be a second-countable space. Then $\mathscr{F} \uparrow Men(X)$ if and only if $\mathscr{F} \underset{mark}{\uparrow} Men(X)$.*

Proof: Let σ be a strategy for $\mathscr{F}$, and note that it is sufficient to consider playthroughs with only basic open covers.

We proceed by constructing basic open covers indexed by $\omega^{<\omega}$. This will turn out to emulate the behavior of $\mathscr{C}$ well enough that $\mathscr{F}$ will be able to define a Markov strategy from a perfect information strategy by substituting these covers in the place of perfect information. So if $\mathcal{U}_t$ is a basic open cover for $t < s \in \omega^{<\omega}$, and $\mathcal{V}$ is any basic open cover, we may choose a finite subcollection $\mathcal{F}(s, \mathcal{V})$ of $\mathcal{V}$ such that

$$\sigma(\langle \mathcal{U}_{s \restriction 1}, \ldots, \mathcal{U}_s, \mathcal{V} \rangle) \subseteq \bigcup \mathcal{F}(s, \mathcal{V}).$$

Note that there are only countably-many finite collections of basic open sets. Thus we may choose basic open covers $\mathcal{U}_{s^\frown \langle n \rangle}$ for $n < \omega$ such that for any basic open cover $\mathcal{V}$, there exists $n < \omega$ where $\mathcal{F}(s, \mathcal{V}) = \mathcal{F}(s, \mathcal{U}_{s^\frown \langle n \rangle})$.

Let $t : \omega \to \omega^{<\omega}$ be a bijection. We define the Markov strategy τ as follows:

$$\tau(\langle \mathcal{V} \rangle, n) = \bigcup \mathcal{F}(t(n), \mathcal{V}).$$

Suppose there exists a counter-attack $\langle \mathcal{V}_0, \mathcal{V}_1, \ldots \rangle$ of basic open covers which defeats τ. Then there exists $f : \omega \to \omega$ such that, letting $t(m_n) = f \restriction n$:

$$\begin{array}{rcl} x & \notin & \tau(\langle \mathcal{V}_{m_n} \rangle, m_n) \\ & = & \bigcup \mathcal{F}(f \restriction n, \mathcal{V}_{m_n}) \\ & = & \bigcup \mathcal{F}(f \restriction n, \mathcal{U}_{f \restriction (n+1)}) \\ & \supseteq & \sigma(\langle \mathcal{U}_{f \restriction 1}, \ldots, \mathcal{U}_{f \restriction (n+1)} \rangle). \end{array}$$

Thus $\langle \mathcal{U}_{f \restriction 1}, \mathcal{U}_{f \restriction 2}, \ldots \rangle$ is a successful counter-attack by $\mathscr{C}$ against the perfect information strategy σ. □

Corollary 4.11. *Let X be a second-countable space. The following are equivalent:*

- *X is σ-relatively-compact,*
- $F \underset{0\text{-}mark}{\uparrow} Men(X)$,
- $F \underset{mark}{\uparrow} Men(X)$,
- $F \uparrow Men(X)$.

Corollary 4.12. *Let X be a metrizable space. The following are equivalent:*

- *X is σ-compact,*
- *X is σ-relatively-compact,*
- $F \underset{0\text{-}mark}{\uparrow} Men(X)$,
- $F \underset{mark}{\uparrow} Men(X)$,
- $F \uparrow Men(X)$.

Proof: Each bullet implies X is Lindelöf, so X may be assumed to be regular and second-countable. □

5. Robustly Menger

To help instigate the topological property $\mathscr{F} \underset{\text{2-mark}}{\uparrow} \mathit{Men}\,(X)$, we introduce a slight variant of the Menger game and a related covering property.

Game 5.1. Let $\mathit{Men}\,(X, Y)$ denote the *Menger subspace game* which proceeds analogously to the Menger game, except that $\mathscr{F}$ wins whenever $Y \subseteq \bigcup_{n<\omega} F_n$.

Note of course that $\mathit{Men}\,(X, X) = \mathit{Men}\,(X)$.

Definition 5.2. A subset Y of X is *relatively robustly Menger* if there exist functions $r_{\mathcal{V}} : Y \to \omega$ for each open cover $\mathcal{V}$ of X such that for all open covers $\mathcal{U}, \mathcal{V}$ and numbers $n < \omega$, the following sets are $\mathcal{V}$-finite:

$$c(\mathcal{V}, n) = \{x \in Y : r_{\mathcal{V}}(x) \leq n\},$$
$$p(\mathcal{U}, \mathcal{V}, n+1) = \{x \in Y : n < r_{\mathcal{U}}(x) < r_{\mathcal{V}}(x)\}.$$

Definition 5.3. A space X is *robustly Menger* if it is relatively robustly Menger to itself.

Proposition 5.4. *All σ-relatively-compact spaces are robustly Menger.*

Proof: If $X = \bigcup_{n<\omega} R_n$ for R_n relatively compact to X, then for all $\mathcal{U}$, let $r_{\mathcal{U}}(x)$ be the least n such that $x \in R_n$. Then $c(\mathcal{V}, n) = \bigcup_{m \leq n} R_m$ and $p(\mathcal{U}, \mathcal{V}, n+1) = \emptyset$. □

Theorem 5.5. *If $Y \subseteq X$ is relatively robustly Menger, then $\mathscr{F} \underset{\text{2-mark}}{\uparrow} \mathit{Men}\,(X, Y)$.*

Proof: We define the Markov strategy σ as follows. Let $\sigma(\langle \mathcal{U} \rangle, 0) = c(\mathcal{U}, 0)$, and let $\sigma(\langle \mathcal{U}, \mathcal{V} \rangle, n+1) = c(\mathcal{V}, n+1) \cup p(\mathcal{U}, \mathcal{V}, n+1)$.

For any attack $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$ by $\mathscr{C}$ and $x \in Y$, one of the following must occur:

- $r_{\mathcal{U}_0}(x) = 0$ and thus $x \in c(\mathcal{U}_0, 0) \subseteq \sigma(\langle \mathcal{U}_0 \rangle, 0)$;
- $r_{\mathcal{U}_0}(x) = N + 1$ for some $N \geq 0$ and
  - (i) for all $n \leq N$,
  $$r_{\mathcal{U}_{n+1}}(x) \leq N + 1$$
  and thus $x \in c(\mathcal{U}_{N+1}, N+1) \subseteq \sigma(\langle U_N, U_{N+1} \rangle, N+1)$;
  - (ii) for some $n \leq N$,
  $$r_{\mathcal{U}_n}(x) \leq n$$
  and thus $x \in c(\mathcal{U}_{n+1}, n+1) \subseteq \sigma(\langle U_n, U_{n+1} \rangle, n+1)$;

(iii) for some $n \leq N$,

$$n < r_{\mathcal{U}_n}(x) \leq N + 1 < r_{\mathcal{U}_{n+1}}(x)$$

and thus $x \in p(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1) \subseteq \sigma(\langle U_n, U_{n+1} \rangle, n+1)$. □

Theorem 5.6. *$\mathcal{A}(\kappa)$ implies $L(\kappa)$ is robustly Menger.*

Proof: Let f_A for $A \in [\kappa]^{\leq\omega}$ witness $\mathcal{A}(\kappa)$ and fix $A(\mathcal{U}) \in [\kappa]^{\leq\omega}$ for each open cover $\mathcal{U}$ such that $L(\kappa) \setminus A(\mathcal{U})$ is contained in some element of $\mathcal{U}$. Then let $r_{\mathcal{U}}(x) = 0$ for $x \in L(\kappa) \setminus A(\mathcal{U})$, and $r_{\mathcal{U}}(\alpha) = f_{A(\mathcal{U})}(\alpha)$ for $\alpha \in A(\mathcal{U})$.

It follows that

$$c(\mathcal{U}, n) = (L(\kappa) \setminus A(\mathcal{U})) \cup \{\alpha \in A(\mathcal{U}) : f_{A(\mathcal{U})}(\alpha) \leq n\}$$

is $\mathcal{U}$-finite, $\bigcup_{n<\omega} c(\mathcal{U}, n) = X$, and

$$p(\mathcal{U}, \mathcal{V}, n+1) = \{\alpha \in A(\mathcal{U}) \cap A(\mathcal{V}) : n < f_{A(\mathcal{U})}(\alpha) < f_{A(\mathcal{V})}(\alpha)\}$$

is finite. □

We may also consider common counterexamples (specifically, Examples 67 and 63 of [7]) which are finer than the usual Euclidean line.

Definition 5.7. Let $R_{\mathbb{Q}}$ be the real line with the topology generated by open intervals with or without the rationals removed.

Example 5.8. $R_{\mathbb{Q}}$ is non-regular and non-σ-compact, but is second-countable and σ-relatively-compact.

Proof: Note that π is a point that cannot be separated from the closed set $\mathbb{Q}$ by open sets, demonstrating non-regularity. Compact sets in $R_{\mathbb{Q}}$ cannot contain open intervals, and thus are nowhere dense in nonmeager $\mathbb{R}$, so $R_{\mathbb{Q}}$ is not σ-compact. The usual base of intervals with rational endpoints (with or without rationals removed) witnesses second-countability.

We now will show that $[-n, n] \setminus \mathbb{Q}$ is relatively compact. Let $\mathcal{U}$ be a cover of $R_{\mathbb{Q}}$ by open intervals with the rationals removed, and let $\mathcal{V}$ be the corresponding cover of open intervals. There exists a finite subcollection of $\mathcal{V}$ which covers $[-n, n]$, so there exists a corresponding subcollection of $\mathcal{U}$ which covers $[-n, n] \setminus \mathbb{Q}$. Thus $R_{\mathbb{Q}} = \mathbb{Q} \cup \bigcup_{n<\omega}([-n, n] \setminus \mathbb{Q})$ is a countable union of relatively compact subsets. □

Definition 5.9. Let R_ω be the real line with the topology generated by open intervals with countably many points removed.

Lemma 5.10. *For each compact subset K of $\mathbb{R}$ and open cover $\mathcal{U}$ of R_ω, there exists a countable set C such that $K \setminus C$ is $\mathcal{U}$-finite.*

Proof: Let $\mathcal{V}$ be the corresponding open cover of $\mathbb{R}$ where $\mathcal{U}$ may be obtained by removing countable sets from members of $\mathcal{V}$. Choose a finite subcollection $\mathcal{G}$ of $\mathcal{V}$ which covers K, and then note that the corresponding finite subcollection $\mathcal{F}$ of $\mathcal{U}$ covers $K \setminus C$ where C is the finite union of the countable sets removed from sets in $\mathcal{G}$ to obtain the sets in $\mathcal{F}$. □

Example 5.11. R_ω is non-regular, non-second-countable, and non-σ-relatively-compact, but $\mathscr{F} \uparrow Men\,(R_\omega)$.

Proof: If $S \supseteq \{s_n : n < \omega\}$ for distinct s_n, then $U_m = R_\omega \setminus \{s_n : m < n < \omega\}$ yields an infinite cover $\{U_m : m < \omega\}$ with no finite subcollection covering S, showing that all relatively compact sets are finite, and R_ω is not σ-relatively-compact. The set $R_\omega \setminus \mathbb{Q}$ is open, but the closure of any open subset must contain a rational, so this space is not regular. Finally, the space is not second-countable, since for any countable collection of nonempty open sets $\{U_n : n < \omega\}$, we may choose $p_n \in U_n$ and note $R_\omega \setminus \{p_n : n < \omega\}$ is an open set not containing any U_n.

Define the winning strategy σ for $\mathscr{F}$ in $Men\,(R_\omega)$ as follows: let $\sigma(\langle \mathcal{U}_0, \ldots, \mathcal{U}_{2n}\rangle) = [-n, n] \setminus C_n$ where $C_n = \{c_{i,n} : i < \omega\}$ witnesses Lemma 5.10 for $[-n, n]$ and $\mathcal{U}_{2n}$, and let $\sigma(\langle \mathcal{U}_0, \ldots, \mathcal{U}_{2n+1}\rangle) = \{c_{i,j} : i, j < n\}$. □

We will soon see that, assuming $\mathcal{A}(\mathfrak{c})$, $\mathscr{F}$ has a winning 2-Markov strategy for $Men\,(R_\omega)$ as well.

Proposition 5.12. *If $X = \bigcup_{i<\omega} X_i$ and $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men\,(X, X_i)$ for $i < \omega$, then $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men\,(X)$.*

Proof: Let σ_i be a 2-Markov strategy for $\mathscr{F}$ in $Men\,(X, X_i)$.

We define the 2-Markov strategy σ for $Men\,(X)$ as follows:

$$\sigma(\langle \mathcal{U}\rangle, 0) = \sigma_0(\langle \mathcal{U}\rangle, 0),$$

$$\sigma(\langle \mathcal{U}, \mathcal{V}\rangle, n+1) = \bigcup_{i,j \le n} \sigma_i(\langle \mathcal{V}\rangle, 0) \cup \sigma_i(\langle \mathcal{U}, \mathcal{V}\rangle, j+1).$$

Let $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots\rangle$ be a successful counter-attack by $\mathscr{C}$ against σ. Then there exists $x \in X_i$ for some $i < \omega$ such that x is not contained in

$$\sigma(\langle \mathcal{U}_0\rangle, 0) \cup \bigcup_{n<\omega} \sigma(\langle \mathcal{U}_n, \mathcal{U}_{n+1}\rangle, n+1).$$

It then follows that x is not contained in

$$\sigma_i(\langle \mathcal{U}_i \rangle, 0) \cup \bigcup_{n<\omega} \sigma_i(\langle \mathcal{U}_{i+n}, \mathcal{U}_{i+n+1} \rangle, n+1)$$

and $\langle \mathcal{U}_i, \mathcal{U}_{i+1}, \ldots \rangle$ is a successful counter-attack by $\mathscr{C}$ against σ_i. □

Theorem 5.13. *Assuming CH or just* $\mathcal{A}(\mathfrak{c})$, $\mathscr{F} \underset{2\text{-mark}}{\uparrow} Men(R_\omega)$.

Proof: It is sufficient to show that $[0,1] \subseteq R_\omega$ is relatively robustly Menger. Let f_A witness $\mathcal{A}(\mathfrak{c})$ for $A \in [[0,1]]^{\leq\omega}$. For each open cover $\mathcal{U}$, let $A_\mathcal{U}$ witness Lemma 5.10 for $[0,1]$ and $\mathcal{U}$. Let $r_\mathcal{U}(x) = 0$ if $x \in [0,1] \setminus A_\mathcal{U}$ and $r_\mathcal{U}(x) = f_{A_\mathcal{U}}(x)$ otherwise.

It follows then that

$$c(\mathcal{U}, n) = [0,1] \setminus \{x \in A_\mathcal{U} : f_{\mathcal{A}_\mathcal{U}}(x) > n\}$$

is $\mathcal{U}$-finite and

$$p(\mathcal{U}, \mathcal{V}, n+1) = \{x \in A_\mathcal{U} \cap A_\mathcal{V} : n < f_{\mathcal{A}_\mathcal{U}}(x) < f_{\mathcal{A}_\mathcal{V}}(x)\}$$

is finite. □

The following is left open:

Question 5.14. Are all 2-Markov Menger spaces robustly Menger?

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