

Order-theoretic properties of some sets of quasi-measures

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Abstract. Let \mathfrak{M} and \mathfrak{R} be algebras of subsets of a set Ω with $\mathfrak{M} \subset \mathfrak{R}$, and denote by $E(\mu)$ the set of all quasi-measure extensions of a given quasi-measure μ on \mathfrak{M} to \mathfrak{R} . We show that $E(\mu)$ is order bounded if and only if it is contained in a principal ideal in $ba(\mathfrak{R})$ if and only if it is weakly compact and $\text{extr } E(\mu)$ is contained in a principal ideal in $ba(\mathfrak{R})$. We also establish some criteria for the coincidence of the ideals, in $ba(\mathfrak{R})$, generated by $E(\mu)$ and $\text{extr } E(\mu)$.

Keywords: linear lattice; ideal; order bounded; ideal dominated; order unit; Banach lattice; AM -space; convex set; extreme point; weakly compact; additive set function; quasi-measure; atomic; extension

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1. Introduction

By a *quasi-measure* we mean a positive additive function on an algebra of sets. Let \mathfrak{M} and \mathfrak{R} be algebras of subsets of a set Ω with $\mathfrak{M} \subset \mathfrak{R}$ and let μ be a quasi-measure on \mathfrak{M} . The ‘sets’ appearing in the title of the paper¹ are the convex set $E(\mu)$ of all quasi-measure extensions of μ to \mathfrak{R} and the set $\text{extr } E(\mu)$ of its extreme points. These sets have been studied in many earlier papers by the author, including [4]–[7]. So far, their topological and linear-topological properties as subsets of the dual Banach lattice $ba(\mathfrak{R})$ have been of main concern. A systematic presentation of most of the results obtained is given in the memoir [8].

This paper is a continuation of [9]. Its starting point is the following consequence of classical results: if $E(\mu)$ is order bounded, then it is weakly compact ([9, Proposition 2(c)]). To fill the gap between order boundedness and weak compactness, we introduce, in Section 2, a property of subsets of a general linear lattice X , which we call ideal domination. (By definition, $V \subset X$ is *ideal dominated* if it is contained in a principal ideal in X .) This property is weaker than order boundedness, in general, but coincides with it for compact convex subsets of X , the topology involved being compatible with the linear structure and order of X (Theorem 1 in Section 2). Compactness alone does not suffice here; see the passage introducing Proposition 1 in Section 2. It follows from Theorem 1

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that $E(\mu)$ is order bounded if and only if it is ideal dominated (Theorem 4 in Section 6). A further equivalent condition is the following one: $E(\mu)$ is weakly compact and $\text{extr } E(\mu)$ is ideal dominated (Theorem 6 in Section 6).

Order boundedness of $E(\mu)$ is equivalent to that of $\text{extr } E(\mu)$, according to [9, Theorem 2, (i) \Leftrightarrow (ii)]. This is still true for ideal domination provided μ is nonatomic (see Theorem 5 in Section 6), but not in general. Therefore, we establish some criteria for ideal domination of $\text{extr } E(\mu)$. One of them is concerned with the case where μ has finite range or, more generally, is atomic (Proposition 3 and Theorem 3 in Section 5). Another one applies in the situation where \mathfrak{M} , \mathfrak{R} and μ are related in a special way (Theorem 7 in Section 6).

Finally, Section 7 is concerned with the question when the ideals, in $ba(\mathfrak{R})$, generated by $E(\mu)$ and $\text{extr } E(\mu)$ coincide. A necessary and sufficient condition is provided in the case where μ has finite range or is atomic (Proposition 4 and Theorem 8). The answer is, moreover, affirmative when \mathfrak{R} is generated, as an algebra, by \mathfrak{M} and a finite family of subsets of Ω , and μ is arbitrary (Proposition 6(b)).

Many results of [8] and [9] are applied extensively in the paper. We also frequently appeal to the Baire category theorem, both for compact topological spaces and complete metric spaces (see the proofs of Theorem 1 in Section 2, and of Lemma 2 in Section 4 and Theorem 5 in Section 6, respectively).

The measure-theoretic notation and terminology we use are explained in Section 3. They are mostly standard and coincide with those of [8]. Section 3 also contains a few auxiliary results on $E(\mu)$ and $\text{extr } E(\mu)$. More auxiliary results on these sets are presented in Section 4.

2. Ideal domination in linear lattices and in Banach lattices

Let X be a real linear lattice (= Riesz space in the terminology of [2]), with the order and lattice operations denoted by \leq and \wedge , \vee , respectively. As usual, $|x|$ stands for the modulus or absolute value of $x \in X$ and X_+ for the positive cone of X .

The *order interval* $[x, y]$, where $x, y \in X$ and $x \leq y$, is the set

$$\{z \in X : x \leq z \leq y\}.$$

Let V be a subset of X . Recall that V is *order bounded* if $V \subset [x, y]$ for some x and y as above. We denote by A_V the ideal in X generated by V . The notation $A_{\{x\}}$, where $x \in X$, is abbreviated to A_x . Such ideals are called *principal*. We have

$$A_x = \bigcup_{n \in \mathbb{N}} [-n|x|, n|x|].$$

We shall tacitly make use of this simple formula, combined with the Baire category theorem, in some proofs.

We call V *ideal dominated* if $V \subset A_x$ for some $x \in X$. Clearly, every principal ideal is ideal dominated. Also, every order interval $[x, y]$ is ideal dominated since

$x, y \in A_{|x| \vee |y|}$. Note that X is itself ideal dominated if and only if it has a (*strong*) order unit e , i.e., $e \in X$ and for every $x \in X$ there exists $n \in \mathbb{N}$ with $|x| \leq ne$.

We start by a result which will be used in establishing Proposition 2 in this section and Theorem 4 in Section 6. For a special case see [2, Chapter 6, Exercise 5].

Theorem 1. *Let τ be a linear topology on X with X_+ closed. For a compact convex subset W of X the following two conditions are equivalent:*

- (i) W is order bounded;
- (ii) W is ideal dominated.

PROOF: Suppose (ii) holds. The order intervals in X being τ -closed, the Baire category theorem for compact spaces applied in W yields a nonempty relatively τ -open and order bounded subset U of W . A translation argument allows us to assume that $0 \in U$. Using the continuity of the mapping

$$[0, 1] \ni t \longmapsto tx \in W, \quad x \in W,$$

we can find for each $x \in W$ some $0 < \varepsilon_x < 1$ with $\varepsilon_x x \in U$. We then have

$$W = \bigcup_{x \in W} \left(\frac{1}{\varepsilon_x} U \right) \cap W.$$

The sets $(tU) \cap W$, where $t > 1$, being relatively τ -open in W , there exist $x_1, \dots, x_n \in W$ with

$$W \subset \bigcup_{i=1}^n \frac{1}{\varepsilon_{x_i}} U.$$

This yields (i). □

Clearly, the compactness assumption in Theorem 1 cannot be dispensed with. That this is also the case for the convexity assumption is seen from Theorem 2, (ii) \Rightarrow (i), and Proposition 1 below. The latter will be also used in establishing Proposition 2 in this section and Lemma 1 in Section 3. Needless to say, Proposition 1 is surely known.

Proposition 1. *Every countable subset V of a Banach lattice X is ideal dominated.*

PROOF: Let $V = \{x_1, x_2, \dots\}$. We may assume that $0 \notin V$. Set

$$x = \sum_{i=1}^{\infty} 2^{-i} \frac{|x_i|}{\|x_i\|}.$$

Clearly, we then have $V \subset A_x$. □

The next result is also known. The equivalence of conditions (i) and (ii) thereof is due to V. Schlotterbeck (see [14, Theorem IV.2.8]). The original proof of the

implication (ii) \Rightarrow (i) is somewhat involved. Therefore, we shall present below a simple and elementary proof based on a known idea (see [1, proof of Theorem 1]; cf. also [11, proof of Theorem 5]). The implication (i) \Rightarrow (iii) is a consequence of standard results (see [15, Lemma 9.23]). The implication (ii) \Rightarrow (i) will be used in the proof of Proposition 2 in this section. For the definition of an *AM-space* we refer the reader to [14, Definition II.7.1] or [15, Definition 9.1(i)].

Theorem 2. *For a Banach lattice X the following three conditions are equivalent:*

- (i) *X is isomorphic to an AM-space;*
- (ii) *every sequence (x_n) in X such that $\|x_n\| \rightarrow 0$ is order bounded;*
- (iii) *every relatively compact subset of X is order bounded.*

PROOF OF THE IMPLICATION (ii) \Rightarrow (i): Set

$$M = \sup \left\{ \| |x_1| \vee \dots \vee |x_k| \| : x_1, \dots, x_k \in X, \max_{1 \leq i \leq k} \|x_i\| \leq 1, \text{ and } k \in \mathbb{N} \right\}.$$

We claim that $M < \infty$. Otherwise, for each $s \in \mathbb{N}$, we could find $x_1^s, \dots, x_{k_s}^s \in X$ with $\| |x_1^s| \vee \dots \vee |x_{k_s}^s| \| > s^2$ and $\|x_i^s\| \leq 1$ for all $1 \leq i \leq k_s$. Considering the sequence

$$x_1^1, \dots, x_{k_1}^1, \frac{1}{2}x_1^2, \dots, \frac{1}{2}x_{k_2}^2, \dots,$$

we then see that (ii) fails, and so the claim is established. Set, for $x \in X$,

$$\|x\|' = \inf \left\{ \max_{1 \leq i \leq k} \|x_i\| : x_1, \dots, x_k \in X_+, |x| \leq x_1 \vee \dots \vee x_k, \text{ and } k \in \mathbb{N} \right\}.$$

As easily seen, $\|\cdot\|'$ is an M -norm in X and

$$\|x\|' \leq \|x\| \leq M\|x\|' \quad \text{for all } x \in X,$$

and so we are done. □

We note that there are straightforward examples showing that condition (ii) of Theorem 2 fails for $X = l_p$, where $1 \leq p < \infty$. Indeed, set $x_n = t_n e_n$ for $n \in \mathbb{N}$, where (e_n) is the standard basis of l_p and t_n are real numbers with $t_n \rightarrow 0$ and $(t_n) \notin l_p$.

For a result related to Theorem 2, (i) \Leftrightarrow (iii), see [16, Exercise 122.8].

Part (a) of the next result is in contrast with [9, Proposition 1], which implies that a compact convex set W of a linear lattice X equipped with a locally convex topology τ such that X_+ is closed is order bounded if and only if so is $\text{extr } W$. Similarly, part (b) thereof shows that an analogue of [9, Lemma 1(b)] for ideal domination does not hold.

Proposition 2. *Let X be a Banach lattice nonisomorphic to an AM-space.*

- (a) *There exists a compact convex subset W of X which is not ideal dominated but $\overline{\text{extr } W}$ is ideal dominated. In particular, $A_W \neq A_{\overline{\text{extr } W}}$.*

(b) *There exists an ideal dominated convex subset V of X such that \overline{V} is compact but not ideal dominated.*

PROOF: By Theorem 2, (ii) \Rightarrow (i), there exists a sequence (x_n) in X such that $\|x_n\| \rightarrow 0$ but $\{x_n : n \in \mathbb{N}\}$ is not order bounded. Set

$$V = \text{conv}\{x_n : n \in \mathbb{N}\} \quad \text{and} \quad W = \overline{V}.$$

According to Mazur's theorem [13, II.4.3], W is compact. It follows from Milman's theorem [13, II.10.5] that

$$\text{extr } W \subset \{x_n : n \in \mathbb{N}\} \cup \{0\}.$$

Therefore, $\overline{\text{extr } W}$ and V are both ideal dominated, by Proposition 1. Since W is not order bounded, Theorem 1 implies that it is not ideal dominated either. \square

Proposition 2(a) is in contrast with Theorem 6, (ii) \Rightarrow (i), in Section 6.

3. Further notation and measure-theoretic preliminaries

The set of nonzero $\{0, 1\}$ -valued additive functions on a Boolean algebra A is denoted by $\text{ult}(A)$.

For a set Ω we denote by 2^Ω the family of all subsets of Ω and by $|\Omega|$ the cardinality of Ω .

Throughout the rest of the paper, Ω stands for a nonempty set and \mathfrak{M} for an algebra of subsets of Ω .

Given $\mathfrak{E} \subset 2^\Omega$, we denote by \mathfrak{E}_b the algebra of subsets of Ω generated by \mathfrak{E} .

We denote by $ba(\mathfrak{M})$ the Banach lattice of all real-valued bounded additive functions on \mathfrak{M} (see [3, Section 2.2]). By definition, $\|\varphi\| = |\varphi|(\Omega)$ for $\varphi \in ba(\mathfrak{M})$. In addition to the strong topology, $ba(\mathfrak{M})$ is equipped with its weak and weak* topologies; see [3, Section 4.7] for the canonical Banach-lattice predual of $ba(\mathfrak{M})$.

Let $\mu \in ba_+(\mathfrak{M})$. Adapting a general linear-lattice-theoretical terminology (see [2, p. 13]), we say that $\nu \in ba(\mathfrak{M})$ is a *component* of μ if

$$\nu \wedge (\mu - \nu) = 0.$$

We denote by \mathcal{U}_μ the set of all components of μ which take at most two values. As easily seen (cf. [3, Proposition 5.2.2]), for different $\nu_1, \nu_2 \in \mathcal{U}_\mu$ we have $\nu_1 \wedge \nu_2 = 0$. Therefore, \mathcal{U}_μ is countable.

We say that $\mu \in ba_+(\mathfrak{M})$ is *nonatomic* provided for every $\varepsilon > 0$ there exists an \mathfrak{M} -partition $\{M_1, \dots, M_n\}$ of Ω with $\mu(M_i) < \varepsilon$ for all i (see [3, Definition 5.1.4], where the term *strongly continuous* is used). We say that μ is (*purely*) *atomic* provided $\mu \wedge \nu = 0$ for every nonatomic $\nu \in ba_+(\mathfrak{M})$. According to the Sobczyk–Hammer decomposition theorem [3, Theorem 5.2.7], μ is atomic if and only if $\mu = \sum_{\nu \in \mathcal{U}_\mu} \nu$, while μ is nonatomic if and only if $\mathcal{U}_\mu = \{0\}$. Moreover, $\mu = \mu_1 + \mu_2$, where $\mu_1, \mu_2 \in ba_+(\mathfrak{M})$, μ_1 is atomic and μ_2 is nonatomic. We shall use this decomposition in the proofs of Theorems 6 and 7 in Section 6.

As usual, we associate with $\mu \in ba_+(\mathfrak{M})$ the outer quasi-measure μ^* , defined, for all $E \subset \Omega$, by the formula:

$$\mu^*(E) = \inf\{\mu(M) : E \subset M \in \mathfrak{M}\}.$$

Throughout the rest of the paper, \mathfrak{A} stands for an algebra of subsets of Ω with $\mathfrak{M} \subset \mathfrak{A}$. Given $\mu \in ba_+(\mathfrak{M})$, we set

$$E(\mu) = \{\varrho \in ba_+(\mathfrak{A}) : \varrho|_{\mathfrak{M}} = \mu\}.$$

It is a classical result that $E(\mu)$ is always nonempty (see [3, Chapter 3]). Moreover, it is, clearly, convex. In some other papers by the author, including [8], the more comprehensive notation $E(\mu, \mathfrak{A})$ instead of $E(\mu)$ is occasionally used.

We shall also need the following notation (see [8, p. 18]). Given $\mu \in ba_+(\mathfrak{M})$, we set

$$\mathfrak{J}_\mu = \{R \in \mathfrak{A} : \text{there exists } M \in \mathfrak{M} \text{ with } R \subset M \text{ and } \mu(M) = 0\}.$$

Clearly, \mathfrak{J}_μ is an ideal in \mathfrak{A} .

The following result will be often applied below.

(D)' For $\mu \in ult(\mathfrak{M})$ we have $\text{extr } E(\mu) = E(\mu) \cap ult(\mathfrak{A})$.

See [8, p. 19] or [5, p. 396].

We shall also make frequent use of the following two formulas:

- (1)
$$E\left(\sum_{j=1}^n \mu_j\right) = \sum_{j=1}^n E(\mu_j) \quad \text{for } \mu_1, \dots, \mu_n \in ba_+(\mathfrak{M});$$
- (2)
$$\text{extr } E\left(\sum_{j=1}^n \mu_j\right) = \sum_{j=1}^n \text{extr } E(\mu_j) \quad \text{for } \mu_1, \dots, \mu_n \in ba_+(\mathfrak{M})$$

with $\mu_j \wedge \mu_{j'} = 0$ whenever $j \neq j'$.

They are immediate consequences of the corresponding parts of [8, Theorem 6.1] or [5, Theorem 1].

Formulas (1) and (2) imply, in view of [2, Theorem 1.2] applied in $ba(\mathfrak{A})$, the next two formulas, respectively:

- (3)
$$A_{E\left(\sum_{j=1}^n \mu_j\right)} = \sum_{j=1}^n A_{E(\mu_j)} \quad \text{for } \mu_1, \dots, \mu_n \in ba_+(\mathfrak{M});$$
- (4)
$$A_{\text{extr } E\left(\sum_{j=1}^n \mu_j\right)} = \sum_{j=1}^n A_{\text{extr } E(\mu_j)} \quad \text{for } \mu_1, \dots, \mu_n \in ba_+(\mathfrak{M})$$

with $\mu_j \wedge \mu_{j'} = 0$ whenever $j \neq j'$.

They will be used in the proofs of Lemmas 2 and 4 in Section 4 and of Proposition 3 and Theorem 3 in Section 5.

The next assertion will be used in the proofs of Lemma 4 in Section 4 and Theorem 8 in Section 7.

(5) If $\mu_1, \mu_2 \in ba(\mathfrak{M})$ and $\mu_1 \wedge \mu_2 = 0$, then $\varrho_1 \wedge \varrho_2 = 0$ whenever $\varrho_1 \in E(\mu_1)$ and $\varrho_2 \in E(\mu_2)$.

This holds, since $\mu_1 \wedge \mu_2 = 0$ if and only if for every $\varepsilon > 0$ there exists $M \in \mathfrak{M}$ with $\mu_1(M) + \mu_2(M^c) < \varepsilon$ (see [3, Theorem 2.2.1(7)]).

4. Auxiliary results on $E(\mu)$ and $\text{extr } E(\mu)$

The following lemma will be used in establishing Proposition 3 and Theorem 3 in Section 5.

Lemma 1. *Let $\mu \in \text{ult}(\mathfrak{M})$. Then the following two conditions are equivalent:*

- (i) $\text{extr } E(\mu)$ is ideal dominated;
- (ii) $\text{extr } E(\mu)$ is countable.

PROOF: The implication (ii) \Rightarrow (i) holds, by Proposition 1.

To get a contradiction, suppose that (ii) fails, but (i) holds. Then there exists an uncountable subset \mathcal{E} of $\text{extr } E(\mu)$ and $\tau \in ba_+(\mathfrak{A})$ such that $\tau \geq \pi$ for each $\pi \in \mathcal{E}$. In view of (D)', this implies $\tau \geq \sum_{\pi \in \mathcal{F}} \pi$ whenever \mathcal{F} is a finite subset of \mathcal{E} . Hence $\tau(\Omega) = \infty$, which is impossible. \square

We continue with a lemma which will be used in the proof of Theorem 3 in Section 5.

Lemma 2. *Suppose $\mu, \mu_j \in ba_+(\mathfrak{M})$ are such that $\sum_{j=1}^\infty \mu_j = \mu$ and $\mu_j \wedge \mu_{j'} = 0$ whenever $j \neq j'$. Then the following two conditions are equivalent:*

- (i) $\text{extr } E(\mu)$ is ideal dominated;
- (ii) $\text{extr } E(\mu_j)$ is ideal dominated for each $j \in \mathbb{N}$ and there exists $n \in \mathbb{N}$ such that $\text{extr } E(\sum_{j=n+1}^\infty \mu_j)$ is order bounded.

PROOF: That (ii) implies (i) is clear, in view of formula (4). By the same formula, (i) implies the first part of condition (ii). According to [8, Proposition 4.4(b)] or [4, Proposition 1(b)], $\text{extr } E(\mu)$ is closed in $ba(\mathfrak{A})$. Thus, by an application of the Baire category theorem combined with (i), there exist $\pi \in \text{extr } E(\mu)$, $\varepsilon > 0$ and $\tau \in ba_+(\mathfrak{A})$ such that

$$\{\pi' \in \text{extr } E(\mu) : \|\pi - \pi'\| < \varepsilon\} \subset [0, \tau].$$

Fix $n \in \mathbb{N}$ with $\sum_{j=n+1}^\infty \mu_j(\Omega) < \varepsilon/2$. To establish the second part of condition (ii), it is enough to prove the following claim:

$$\text{extr } E\left(\sum_{j=n+1}^\infty \mu_j\right) \subset [0, \tau].$$

According to [8, Theorem 6.1(b)] or [5, Theorem 1(b)], there exist (unique) $\pi_j \in \text{extr } E(\mu_j)$, $j \in \mathbb{N}$, such that $\sum_{j=1}^\infty \pi_j = \pi$. By the same result, given

$\pi'_j \in \text{extr } E(\mu_j)$, $j = n + 1, n + 2, \dots$, we have

$$\pi' := \sum_{j=1}^n \pi_j + \sum_{j=n+1}^{\infty} \pi'_j \in \text{extr } E(\mu).$$

In addition, $\|\pi' - \pi\| < \varepsilon$. It follows that π' is in $[0, \tau]$, and the same is true for $\sum_{j=n+1}^{\infty} \pi'_j$. Thus, the claim holds, by one more application of [8, Theorem 6.1(b)] or [5, Theorem 1(b)]. □

The next two lemmas will be used in establishing Proposition 4 and Theorem 8 in Section 7.

Lemma 3. *Let $\mu \in \text{ult}(\mathfrak{M})$. Then the following two conditions are equivalent:*

- (i) $A_{E(\mu)} = A_{\text{extr } E(\mu)}$;
- (ii) $\text{extr } E(\mu)$ is finite.

PROOF: Suppose (ii) holds. Since $E(\mu)$ is weak* compact (see [8, Proposition 4.4(a)] or [4, Proposition 1(a)]), the Krein–Milman theorem implies that

$$E(\mu) = \text{conv extr } E(\mu).$$

Hence (i) holds.

Suppose (ii) fails, and let π_1, π_2, \dots be different elements of $\text{extr } E(\mu)$. Setting $\varrho = \sum_{n=1}^{\infty} 2^{-n} \pi_n$, we have $\varrho \in E(\mu)$. On the other hand, it follows from (D)' that $\varrho \notin A_{\text{extr } E(\mu)}$. Thus, (i) fails, too. □

Lemma 4. *Let $\mu_1, \dots, \mu_n \in \text{ba}_+(\mathfrak{M})$ and $\mu_j \wedge \mu_{j'} = 0$ whenever $j \neq j'$. Then the following two conditions are equivalent:*

- (i) $A_{E(\sum_{j=1}^n \mu_j)} = A_{\text{extr } E(\sum_{j=1}^n \mu_j)}$;
- (ii) $A_{E(\mu_j)} = A_{\text{extr } E(\mu_j)}$ for each $j = 1, \dots, n$.

PROOF: In view of formulas (3) and (4), (ii) implies (i).

Suppose (i) holds. Using formula (4), we then get

$$E(\mu_{j'}) \subset \sum_{j=1}^n A_{\text{extr } E(\mu_j)} \quad \text{for } j' = 1, \dots, n.$$

By (5), it follows that $E(\mu_{j'}) \subset A_{\text{extr } E(\mu_{j'})}$, and so (ii) holds. □

The next lemma is an essential tool in establishing Theorem 7 in Section 6 and Proposition 5 in Section 7. Both results assume condition (*), which is intermediate between the condition of independence and that of almost independence of algebras of sets considered by E. Marczewski (see [10, p. 220]). For other uses of (*) see [8, Proposition 12.4] or [6, Proposition 2] as well as [7, Theorem 7] and [9, Corollaries 2 and 3].

Lemma 5. *Let \mathfrak{N} be an algebra of subsets of Ω with $\mathfrak{R} = (\mathfrak{M} \cup \mathfrak{N})_b$ and let $\mu \in \text{ult}(\mathfrak{M})$. Then $\mathfrak{R}/\mathfrak{J}_\mu$ is homomorphic image of \mathfrak{N} . If, in addition,*

$$(*) \quad M \cap N \neq \emptyset \text{ for all } M \in \mathfrak{M} \text{ with } \mu(M) > 0 \text{ and nonempty } N \in \mathfrak{N}$$

holds, then $\mathfrak{R}/\mathfrak{J}_\mu$ and \mathfrak{N} are isomorphic. In particular,

$$|\text{ult}(\mathfrak{R}/\mathfrak{J}_\mu)| = |\text{ult}(\mathfrak{N})|.$$

PROOF: Denote by h the canonical mapping from \mathfrak{R} onto $\mathfrak{R}/\mathfrak{J}_\mu$. For $M \in \mathfrak{M}$ and $N \in \mathfrak{N}$ we have

$$h(M \cap N) = \begin{cases} 0 & \text{if } \mu(M) = 0, \\ h(N) & \text{if } \mu(M) = 1. \end{cases}$$

It follows that $h(\mathfrak{N}) = \mathfrak{R}/\mathfrak{J}_\mu$. Condition $(*)$ implies that $\mathfrak{N} \cap \mathfrak{J}_\mu = \{\emptyset\}$, and so the injectivity of $h|_{\mathfrak{N}}$. □

5. $\text{extr } E(\mu)$ for atomic μ

We start by an extension of Lemma 1. It is worth-while to compare it with [9, Proposition 5], which, under the same assumption, asserts that $\text{extr } E(\mu)$ is order bounded if and only if it is finite.

Proposition 3. *Let $\mu \in \text{ba}_+(\mathfrak{M})$ have finite range. Then the following three conditions are equivalent:*

- (i) $\text{extr } E(\mu)$ is ideal dominated;
- (ii) $\text{extr } E(\mu)$ is countable;
- (iii) $\text{ult}(\mathfrak{R}/\mathfrak{J}_\nu)$ is countable for each $\nu \in \mathcal{U}_\mu$.

PROOF: The assumption implies that μ is atomic and \mathcal{U}_μ is finite (see [8, Lemma 3.2] and [3, Lemma 11.1.3]). Therefore, it follows from formula (2) that (ii) holds if and only if $\text{extr } E(\nu)$ is countable for each $\nu \in \mathcal{U}_\mu$. Thus, (ii) and (iii) are equivalent, by [8, Proposition 7.1, 4°] or [6, Proposition 1]. The equivalence of (i) and (ii) follows from Lemma 1 and formula (4). □

The next result is a partial generalization of Proposition 3.

Theorem 3. *Let $\mu \in \text{ba}_+(\mathfrak{M})$ be atomic, and set*

$$\mathcal{D} = \{\nu \in \mathcal{U}_\mu : \text{ult}(\mathfrak{R}/\mathfrak{J}_\nu) \text{ is infinite}\}.$$

Then the following three conditions are equivalent:

- (i) $\text{extr } E(\mu)$ is ideal dominated;
- (ii) $\text{ult}(\mathfrak{R}/\mathfrak{J}_\nu)$ is countable for each $\nu \in \mathcal{U}_\mu$, \mathcal{D} is finite, and

$$\text{extr } E\left(\sum_{\nu \in \mathcal{U}_\mu \setminus \mathcal{D}} \nu\right) \text{ is order bounded;}$$

(iii) $ult(\mathfrak{R}/\mathfrak{J}_\nu)$ is countable for each $\nu \in \mathcal{U}_\mu$, \mathcal{D} is finite, and

$$\sum_{\nu \in \mathcal{U}_\mu \setminus \mathcal{D}} \nu(\Omega) |ult(\mathfrak{R}/\mathfrak{J}_\nu)| < \infty.$$

PROOF: Using [8, Proposition 7.1, 4°] or [6, Proposition 1], and Lemma 1, we can reword the first part of condition (ii) as follows: $\text{extr } E(\nu)$ is ideal dominated for each $\nu \in \mathcal{U}_\mu$. Moreover, in view of [9, Proposition 5, (i) \Rightarrow (iii)], $\text{extr } E(\nu)$ is not order bounded for each $\nu \in \mathcal{D}$. Thus, (i) and (ii) are equivalent by formula (4) and Lemma 2.

The equivalence of (ii) and (iii) is a direct consequence of [9, Theorem 3]. \square

Remark 1. Condition (i) of Theorem 3 neither implies nor is implied by the condition that $E(\mu)$ be weakly compact (equivalently, $\text{extr } E(\mu)$ be relatively weakly compact; see [8, Theorem 5.1]), even for atomic $\mu \in ba_+(\mathfrak{M})$. Indeed, in Example 1 of [4] μ is two-valued, $\text{extr } E(\mu)$ has cardinality \aleph_0 , and so is ideal dominated, by Proposition 1, but $E(\mu)$ is not weakly compact (cf. [9, Proposition 5, (ii) \Rightarrow (iii)]). On the other hand, in Example 1 of [9] $E(\mu)$ is weakly compact, but not order bounded. Therefore, $\text{extr } E(\mu)$ is not ideal dominated, by Theorem 6 in the next section.

6. $E(\mu)$ and $\text{extr } E(\mu)$ for arbitrary μ and $\text{extr } E(\mu)$ for nonatomic μ

The functionals

$$ba(\mathfrak{R}) \ni \varphi \longmapsto \varphi(R) \in \mathbb{R}, \quad \text{where } R \in \mathfrak{R},$$

can be identified with elements of the predual of $ba(\mathfrak{R})$. Consequently, the positive cone of $ba(\mathfrak{R})$ is weak* closed. In fact, the positive cone of an arbitrary dual Banach lattice is weak* closed, in view of a classical result (see [14, Proposition II.5.5]). Therefore, the following result is a direct consequence of Theorem 1 above, and [8, Proposition 4.4(a)] or [4, Proposition 1(a)].

Theorem 4. For $\mu \in ba_+(\mathfrak{M})$ the following two conditions are equivalent:

- (i) $E(\mu)$ is order bounded;
- (ii) $E(\mu)$ is ideal dominated.

The next result is a partial strengthening of Theorem 4.

Theorem 5. Let $\mu \in ba_+(\mathfrak{M})$ be nonatomic. Then the following three conditions are equivalent:

- (i) $E(\mu)$ is order bounded;
- (ii) $E(\mu)$ is ideal dominated;
- (iii) $\text{extr } E(\mu)$ is ideal dominated.

PROOF: Clearly (i) \Rightarrow (ii) \Rightarrow (iii). Suppose (iii) holds. To derive (i), note that there exist $\pi_0 \in \text{extr } E(\mu)$, $\varepsilon > 0$ and $\tau \in ba_+(\mathfrak{R})$ such that

$$\{\pi \in \text{extr } E(\mu) : \|\pi_0 - \pi\| < \varepsilon\} \subset [0, \tau].$$

Indeed, $\text{extr } E(\mu)$ being closed in $ba(\mathfrak{A})$ (see [8, Proposition 4.4(b)] or [4, Proposition 1(b)]), this is a consequence of (iii) and the Baire category theorem. We shall show that $\mu^* \mathfrak{A} \leq \tau$, which is equivalent to $E(\mu) \subset [0, \tau]$ (see [8, p. 19, (C)*]). To this end, fix an \mathfrak{M} -partition $\{M_1, \dots, M_n\}$ of Ω with $\mu(M_i) < \varepsilon/2$ for each i and $R_0 \in \mathfrak{A}$. Appealing to [8, p. 19, (C)*] again, we find, for each $i = 1, \dots, n$,

$$\pi_i \in \text{extr } E(\mu) \quad \text{with} \quad \pi_i(R_0 \cap M_i) = \mu^*(R_0 \cap M_i).$$

Set

$$\tilde{\pi}_i(R) = \pi_i(R \cap M_i) + \pi_0(R \cap M_i^c) \quad \text{for } R \in \mathfrak{A} \text{ and } i = 1, \dots, n.$$

By [8, Lemma 4.5(d)] or [5, Lemma 4(d)], $\tilde{\pi}_i \in \text{extr } E(\mu)$. Moreover, we have

$$\|\pi_0 - \tilde{\pi}_i\| < \varepsilon, \quad \text{and so} \quad \tilde{\pi}_i \leq \tau, \quad i = 1, \dots, n.$$

It follows that

$$\mu^*(R_0) = \sum_{i=1}^n \mu^*(R_0 \cap M_i) = \sum_{i=1}^n \tilde{\pi}_i(R_0 \cap M_i) \leq \sum_{i=1}^n \tau(R_0 \cap M_i) = \tau(R_0).$$

□

Remark 2. The nonatomicity assumption is essential for the validity of the implications (iii) \Rightarrow (i), (ii) of Theorem 5. Indeed, in Example 1 of [4] $\text{extr } E(\mu)$ is countable, and so ideal dominated, by Proposition 1, but $E(\mu)$ is seen not to be order bounded (cf. Remark 1). In view of Theorem 4, nor is $E(\mu)$ ideal dominated.

According to Remark 1, neither part of condition (ii) of Theorem 6 below implies the other part thereof, in general.

Theorem 6. For $\mu \in ba_+(\mathfrak{M})$ the following two conditions are equivalent:

- (i) $E(\mu)$ is order bounded;
- (ii) $E(\mu)$ is weakly compact and $\text{extr } E(\mu)$ is ideal dominated.

PROOF: The nontrivial part of the implication (i) \Rightarrow (ii) coincides with [9, Proposition 2(c)].

Suppose (ii) holds. Let μ_1 and μ_2 stand for the atomic and nonatomic components of μ , respectively. Then $E(\mu_i)$ is weakly compact and $\text{extr } E(\mu_i)$ is ideal dominated for $i = 1, 2$, by [8, Corollary 6.3] and formula (2), respectively. Thus, $E(\mu_2)$ is order bounded, according to Theorem 5, (iii) \Rightarrow (i). From [8, Theorem 7.7, (ii) \Rightarrow (iii)] we infer that $\mathfrak{A}/\mathfrak{J}_\nu$ is finite for each $\nu \in \mathcal{U}_{\mu_1}$, and so Theorem 3 yields that $\text{extr } E(\mu_1)$ is order bounded. By [9, Theorem 2, (ii) \Rightarrow (i)], $E(\mu_1)$ is also order bounded. An application of formula (1) completes the proof of (i). □

Theorem 7. Let \mathfrak{A} be an algebra of subsets of Ω with $\mathfrak{A} = (\mathfrak{M} \cup \mathfrak{N})_b$, let $\mu \in ba_+(\mathfrak{M})$ and let μ_1 and μ_2 stand for the atomic and nonatomic components of μ , respectively. Suppose

- (*) $M \cap N \neq \emptyset$ for all $M \in \mathfrak{M}$ with $\mu(M) > 0$ and nonempty $N \in \mathfrak{N}$.

Then the following two conditions are equivalent:

- (i) $\text{extr } E(\mu)$ is ideal dominated;
- (ii) μ_1 has finite range, $\mu_2 = 0$ and $\text{ult}(\mathfrak{N})$ is countable, or \mathfrak{N} is finite or $\mu = 0$.

PROOF: Suppose (i) holds. By formula (2), $\text{extr } E(\mu_i)$ is then ideal dominated for $i = 1, 2$. Hence $E(\mu_2)$ is order bounded, by Theorem 5, (iii) \Rightarrow (i). If $\mu_2 \neq 0$, it follows by [9, Corollary 2, (i) \Rightarrow (iii)], that \mathfrak{N} is finite, and so (ii) holds. Suppose $\mu_2 = 0$ and, moreover, $\mu_1 \neq 0$ and \mathfrak{N} is infinite. According to Lemma 5, we have $\text{ult}(\mathfrak{N}) = \text{ult}(\mathfrak{R}/\mathfrak{J}_\nu)$ for $\nu \in \mathcal{U}_{\mu_1}$ with $\nu \neq 0$. It follows from Theorem 3 that $\text{ult}(\mathfrak{N})$ is countable and \mathcal{U}_{μ_1} is finite, and so $\mu_1(\mathfrak{M})$ is also finite. Thus, the implication (i) \Rightarrow (ii) is established.

Plainly, (i) holds if $\mu = 0$. It also holds if \mathfrak{N} is finite, by [4, Theorem 1(a)]; see also Proposition 6(a) in Section 7. Suppose the first part of condition (ii) holds and $\mu_1 \neq 0$. By Lemma 5 again, $\text{ult}(\mathfrak{R}/\mathfrak{J}_\nu)$ is countable for each $\nu \in \mathcal{U}_{\mu_1}$. Proposition 3, (iii) \Rightarrow (i), now yields (i). Thus, the implication (ii) \Rightarrow (i) is also established. □

7. Coincidence of $A_{E(\mu)}$ and $A_{\text{extr } E(\mu)}$

The following result extends Lemma 3.

Proposition 4. *Let $\mu \in \text{ba}_+(\mathfrak{M})$ have finite range. Then the following three conditions are equivalent:*

- (i) $A_{E(\mu)} = A_{\text{extr } E(\mu)}$;
- (ii) $\text{extr } E(\mu)$ is finite;
- (iii) $\text{ult}(\mathfrak{R}/\mathfrak{J}_\nu)$ is finite for each $\nu \in \mathcal{U}_\mu$.

PROOF: As in the proof of Proposition 3, the assumption implies that μ is atomic and \mathcal{U}_μ is finite. Now, formula (2) shows that (ii) is equivalent to the condition that $\text{extr } E(\nu)$ is finite for each $\nu \in \mathcal{U}_\mu$. Therefore, the equivalence of (i) and (ii) is a consequence of Lemmas 3 and 4, while the equivalence of (ii) and (iii) follows from [8, Proposition 7.1, 4°] or [6, Proposition 1]. Indeed, according to those results, $\text{extr } E(\nu)$ and $\text{ult}(\mathfrak{R}/\mathfrak{J}_\nu)$ are equipotent for $\nu \in \text{ult}(\mathfrak{M})$. □

The next result is a partial extension of Proposition 4.

Theorem 8. *Let $\mu \in \text{ba}_+(\mathfrak{M})$ be atomic. Then the following two conditions are equivalent:*

- (i) $A_{E(\mu)} = A_{\text{extr } E(\mu)}$;
- (ii) there exists $n \in \mathbb{N}$ such that $|\text{ult}(\mathfrak{R}/\mathfrak{J}_\nu)| \leq n$ for each $\nu \in \mathcal{U}_\mu$.

Under these conditions, $E(\mu)$ is order bounded.

PROOF: We shall consider below an equivalent version of condition (ii) with “ $\text{ult}(\mathfrak{R}/\mathfrak{J}_\nu)$ ” replaced by “ $\text{extr } E(\nu)$ ” (see [8, Proposition 7.1, 4°] or [6, Proposition 1]).

Suppose (i) holds. We first show that $\text{extr } E(\nu)$ is then finite for each $\nu \in \mathcal{U}_\mu$. Indeed, fix $\nu \in \mathcal{U}_\mu$. Applying Lemma 4 to ν and $\mu - \nu$, we get $A_{E(\nu)} = A_{\text{extr } E(\nu)}$. Lemma 3 now shows that $\text{extr } E(\nu)$ is, in fact, finite.

Suppose, moreover, that (ii) fails. By what we have proved so far, there exist different ν_1, ν_2, \dots in \mathcal{U}_μ such that $\text{extr } E(\nu_n)$ contains different elements $\pi_1^{\nu_n}, \dots, \pi_n^{\nu_n}, n = 1, 2, \dots$. Fix $\varrho^\nu \in E(\nu)$ for $\nu \in \mathcal{U}_\mu$ with $\nu \neq \nu_1, \nu_2, \dots$, and set

$$\varrho = \sum_{n=1}^{\infty} \frac{1}{n} (\pi_1^{\nu_n} + \dots + \pi_n^{\nu_n}) + \sum_{\substack{\nu \in \mathcal{U}_\mu \\ \nu \neq \nu_1, \nu_2, \dots}} \varrho^\nu.$$

Clearly, $\varrho \in E(\mu)$. We claim that $\varrho \notin A_{\text{extr } E(\mu)}$, which contradicts (i). To establish the claim, fix $\pi_1, \dots, \pi_p \in \text{extr } E(\mu)$. In view of [8, Theorem 6.1(b)] or [5, Theorem 1(b)], we have

$$\pi_j = \sum_{\nu \in \mathcal{U}_\mu} \sigma_j^\nu, \quad \text{where } j = 1, \dots, p \text{ and } \sigma_j^\nu \in \text{extr } E(\nu) \text{ for } \nu \in \mathcal{U}_\mu.$$

It follows that for $n > p$ and some $1 \leq j_n \leq n$ we have $\pi_{j_n}^{\nu_n} \wedge \pi_j = 0, j = 1, \dots, p$ (see (D)' and (5)). Thus, the claim is established.

Suppose (ii) holds. Let, for $\nu \in \mathcal{U}_\mu$,

$$\text{extr } E(\nu) = \{\pi_1^\nu, \dots, \pi_n^\nu\},$$

repetitions being allowed. Set

$$\pi_j = \sum_{\nu \in \mathcal{U}_\mu} \pi_j^\nu, \quad j = 1, \dots, n.$$

In view of [8, Theorem 6.1(b)] or [5, Theorem 1(b)], we have $\pi_j \in \text{extr } E(\mu)$. To establish (i), it is enough to show that

$$\varrho \leq \sum_{j=1}^n \pi_j \quad \text{for every } \varrho \in E(\mu).$$

Fix $\varrho \in E(\mu)$, and choose, for $\nu \in \mathcal{U}_\mu$,

$$\varrho^\nu \in E(\nu) \quad \text{with} \quad \sum_{\nu \in \mathcal{U}_\mu} \varrho^\nu = \varrho$$

(see [8, Theorem 6.1(a)] or [5, Theorem 1(a)]). As in the proof of Lemma 3, we have

$$E(\nu) = \text{conv}\{\pi_1^\nu, \dots, \pi_n^\nu\}.$$

Consequently, $\varrho^\nu \leq \sum_{j=1}^n \pi_j^\nu$. It follows that $\varrho \leq \sum_{j=1}^n \pi_j$, and so (i) is established.

The final assertion is now an immediate consequence of [9, Theorem 3]. □

The final assertion of Theorem 8 is not equivalent to its conditions (i) and (ii), as the following example shows.

Example 1 (cf. [9, Example 1]). Set $\Omega = \mathbb{N}$, and let $\{M_1, M_2, \dots\}$ be a partition of Ω with $|M_i| = i$ for each i . Define

$$\mathfrak{M} = \{M_1, M_2, \dots\}_b \quad \text{and} \quad \mathfrak{R} = \{\{n\} : n \in \Omega\}_b.$$

Set, for $i \in \Omega$ and $M \in \mathfrak{M}$,

$$\nu_i(M) = 1/i^3 \text{ if } M \cap M_i \neq \emptyset \quad \text{and} \quad \nu_i(M) = 0 \text{ otherwise.}$$

Define $\mu = \sum_{i=1}^{\infty} \nu_i$. Clearly, $\mu \in ba_+(\mathfrak{M})$. Moreover, μ is atomic and $\mathcal{U}_\mu = \{0, \nu_1, \nu_2, \dots\}$. As easily seen, μ does not satisfy condition (ii) of Theorem 8. On the other hand, $\sum_{n \in M_i} \mu^*(\{n\}) = 1/i^2$ for each i , and so $\sum_{n \in \Omega} \mu^*(\{n\}) < \infty$. Hence $E(\mu)$ is order bounded, by [9, Corollary 4].

The author does not know whether condition (i) of Theorem 8 implies that $E(\mu)$ is order bounded for arbitrary $\mu \in ba_+(\mathfrak{M})$.

Proposition 5. *Let \mathfrak{R} be an algebra of subsets of Ω with $\mathfrak{R} = (\mathfrak{M} \cup \mathfrak{R})_b$ and let $\mu \in ba_+(\mathfrak{M})$ be atomic. Suppose*

$$(*) \quad M \cap N \neq \emptyset \text{ for all } M \in \mathfrak{M} \text{ with } \mu(M) > 0 \text{ and nonempty } N \in \mathfrak{R}.$$

Then the following two conditions are equivalent:

- (i) $A_{E(\mu)} = A_{\text{extr } E(\mu)}$;
- (ii) $\mu = 0$ or \mathfrak{R} is finite.

This is a direct consequence of Theorem 8 and Lemma 5. The implication (ii) \Rightarrow (i) of Proposition 5 holds, in fact, in general (see Proposition 6(b) below).

Part (a) of our next result is an improvement of [4, Theorem 1(a)]. It is established by a slight modification of the original argument.

Proposition 6. *Let $\mathfrak{R} = (\mathfrak{M} \cup \{E_1, \dots, E_n\})_b$, where $\{E_1, \dots, E_n\}$ is a partition of Ω , and let $\mu \in ba_+(\mathfrak{M})$. Then*

- (a) *there exist $\pi_1, \dots, \pi_n \in \text{extr } E(\mu)$ with $\varrho \leq \sum_{i=1}^n \pi_i$ for each $\varrho \in E(\mu)$;*
- (b) $A_{E(\mu)} = A_{\text{extr } E(\mu)}$.

PROOF: Let $\tilde{\pi}_i \in \text{extr}\{\varrho \in ba_+(\mathfrak{M} \cup \{E_i\})_b : \varrho|_{\mathfrak{M}} = \mu\}$ be such that

$$\tilde{\pi}_i(M \cap E_i) = \mu^*(M \cap E_i) \quad \text{for all } M \in \mathfrak{M} \text{ and } i = 1, \dots, n$$

(see [12, Example 1]). Continuing in the same way, we get, after $n - 1$ more steps, $\pi_i \in \text{extr } E(\mu)$ such that

$$\pi_i|_{(\mathfrak{M} \cup \{E_i\})_b} = \tilde{\pi}_i, \quad i = 1, \dots, n.$$

Fix $R \in \mathfrak{R}$ and $\varrho \in E(\mu)$. We then have

$$R = \bigcup_{i=1}^n M_i \cap E_i, \quad \text{where } M_1, \dots, M_n \in \mathfrak{M}.$$

It follows that

$$\varrho(R) = \sum_{i=1}^n \varrho(M_i \cap E_i) \leq \sum_{i=1}^n \mu^*(M_i \cap E_i) = \sum_{i=1}^n \pi_i(M_i \cap E_i) \leq \sum_{i=1}^n \pi_i(R).$$

Thus, (a) holds.

Part (b) is a direct consequence of (a). \square

In Proposition 6 we cannot replace a finite partition by a countable one, even if μ is atomic (see Example 1). In fact, part (a) of Proposition 6 may then fail in a stronger sense. Namely, in the example below $\text{extr} E(\mu)$ is not even ideal dominated.

Example 2. Set $\Omega = \mathbb{N}$ and let $\{M_1, M_2, \dots\}$ be a partition of Ω with M_i infinite for each i . Define \mathfrak{M} and \mathfrak{R} as in Example 1. Let $\mu \in ba_+(\mathfrak{M})$ satisfy $\mu(M_i) > 0$ for each i . Then, as easily seen, $\text{ult}(\mathfrak{R}/\mathfrak{J}_\nu)$ is infinite whenever $\nu \in \mathcal{U}_\mu$ and $\nu \neq 0$. Therefore, $\text{extr} E(\mu)$ is not ideal dominated, by Theorem 3.

Postscript. Related results on the sets $E(\mu)$ and $\text{extr} E(\mu)$ are presented in another paper by the author, *Order-theoretic properties and separability of some sets of quasi-measures* (preprint).

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REFERENCES

- [1] Abramovič Ju. A., *Some theorems on normed lattices*, Vestnik Leningrad. Univ. **13** (1971), 5–11 (in Russian); English transl.: Vestnik Leningrad Univ. Math. **4** (1977), 153–159.
- [2] Aliprantis C.D., Burkinshaw O., *Locally Solid Riesz Spaces*, Academic Press, Orlando, 1978.
- [3] Bhaskara Rao K.P.S., Bhaskara Rao M., *Theory of Charges. A Study of Finitely Additive Measures*, Academic Press, London, 1983.
- [4] Lipecki Z., *On compactness and extreme points of some sets of quasi-measures and measures*, Manuscripta Math. **86** (1995), 349–365.
- [5] Lipecki Z., *On compactness and extreme points of some sets of quasi-measures and measures. II*, Manuscripta Math. **89** (1996), 395–406.
- [6] Lipecki Z., *Cardinality of the set of extreme extensions of a quasi-measure*, Manuscripta Math. **104** (2001), 333–341.
- [7] Lipecki Z., *Cardinality of some convex sets and of their sets of extreme points*, Colloq. Math. **123** (2011), 133–147.
- [8] Lipecki Z., *Compactness and extreme points of the set of quasi-measure extensions of a quasi-measure*, Dissertationes Math. (Rozprawy Mat.) **493** (2013), 59 pp.
- [9] Lipecki Z., *Order boundedness and weak compactness of the set of quasi-measure extensions of a quasi-measure*, Comment. Math. Univ. Carolin. **56** (2015), 331–345.
- [10] Marczewski E., *Measures in almost independent fields*, Fund. Math. **38** (1951), 217–229; reprinted in: Marczewski E., *Collected Mathematical Papers*, Institute of Mathematics, Polish Academy of Sciences, Warszawa, 1996, 413–425.

- [11] de Pagter B., Wnuk W., *Some remarks on Banach lattices with non-atomic duals*, Indag. Math. (N.S.) **1** (1990), 391–395.
- [12] Plachky D., *Extremal and monogenic additive set functions*, Proc. Amer. Math. Soc. **54** (1976), 193–196.
- [13] Schaefer H.H., *Topological Vector Spaces*, Macmillan, New York, 1966.
- [14] Schaefer H.H., *Banach Lattices and Positive Operators*, Springer, Berlin and New York, 1974.
- [15] Schwarz H.-U., *Banach Lattices and Operators*, Teubner, Leipzig, 1984.
- [16] Zaanen A.C., *Riesz Spaces II*, North-Holland, Amsterdam, 1983.

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