Property of being semi-Kelley for the cartesian products and hyperspaces

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Abstract. In this paper we construct a Kelley continuum X such that $X \times [0,1]$ is not semi-Kelley, this answers a question posed by J.J. Charatonik and W.J. Charatonik in *A weaker form of the property of Kelley*, Topology Proc. **23** (1998), 69–99. In addition, we show that the hyperspace $C(X)$ is not semi-Kelley. Further we show that small Whitney levels in $C(X)$ are not semi-Kelley, answering a question posed by A. Illanes in *Problemas propuestos para el taller de Teoría de continuos y sus hiperespacios*, Queretaro, 2013.

Keywords: continuum; property of Kelley; semi-Kelley; cartesian products; hyperspaces; Whitney levels

Classification: Primary 54F15, 54B20, 54G20

1. Introduction

A *continuum* is a nonempty compact connected metric space. A map is a continuous function. Given a continuum X with metric d, $p \in X$ and $A \subset X$, we put $B(p,\varepsilon) = \{x \in X : d(p,x) < \varepsilon\}$ and $N(A,\varepsilon) = \bigcup\{B(a,\varepsilon) : a \in A\}$.

Given a continuum X and $p, q \in X$, we say that a subcontinuum A of X is *irreducible between p and q* provided that $p, q \in A$, and not proper subcontinuum of A contains p and q.

Given a continuum X, we let 2^X denote the hyperspace of all nonempty closed subsets of X equipped with the Hausdorff metric. Furthermore, we denote by $C(X)$ the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X. Let X and Y be continua and let $f : X \to Y$ be a map, the *induced map* $C(f) : C(X) \to C(Y)$ is given by $C(f)(A) = f(A)$, for each $A \in C(X)$.

A map $\mu : C(X) \to [0,\infty)$ is called a *Whitney map for $C(X)$* provided that:

(1) $\mu(\{x\}) = 0$ for each $x \in X$,
(2) $\mu(A) < \mu(B)$ for every $A, B \in C(X)$ such that $A \subsetneq B$.

DOI 10.14712/1213-7243.2015.217

This paper was partially supported by the project Teoría de Continuos e Hiperespacios (0221413)" of CONACYT, 2014; and the project "Teoría de Continuos, Hiperespacios y Sistemas Dinámicos II" (IN101216) of PAPIIT, DGAPA, UNAM and by COMECYT by agreement to implement the *Séptimo Taller de Investigación de Continuos e Hiperespacios*.

If μ is a Whitney map for $C(X)$ and $t \in [0, \mu(X)]$, then $\mu^{-1}(t)$ is called a *Whitney level*. It is known that each Whitney level is a continuum [6, p. 1032]. A topological property P is said to be a *Whitney property* provided that whenever a continuum X has property P, so does $\mu^{-1}(t)$ for each Whitney map μ for $C(X)$ and each t with $0 < t < \mu(X)$.

A continuum X is said to be *Kelley* provided that for each point $x \in X$, for each subcontinuum K of X containing x and for each sequence of points $\{x_n\}_{n=1}^{\infty}$ of X converging to x there exists a sequence of subcontinua $\{K_n\}_{n=1}^{\infty}$ of X such that for each $n \in \mathbb{N}$, $x_n \in K_n$ and $\lim_{n\to\infty} K_n = K$. This property introduced by J. L. Kelley in [8, p. 26], is an important tool in investigation of various properties of continua and hyperspaces (see [5]).

Let K be a subcontinuum of a continuum X. A continuum $M \subset K$ is called *maximal limit continuum in* K provided that there exists a sequence of subcontinua $\{M_n\}_{n=1}^{\infty}$ of X converging to M such that for each convergent sequence of subcontinua $\{M'_n\}_{n=1}^{\infty}$ of X with $M_n \subset M'_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} M'_n = M' \subset K$ we have that $M' = M$.

A continuum X is said to be *semi-Kelley* provided that for each subcontinuum K and for every two maximal limit continua M and L in K either $M \subset L$ or $L \subset M$. The property of semi-Kelley is a weaker form of the property of Kelley, this property has been introduced and studied in [3] by J.J. Charatonik and W.J. Charatonik (see [2], [1]).

In particular in [3, Theorem 4.1, p. 80] J.J. Charatonik and W.J. Charatonik proved that, if the cartesian product of two nondegenerate continua is semi-Kelley then each factor is Kelley (so, semi-Kelley). Also they constructed a Kelley continuum X, [3, Example 4.3, p. 81], such that $X \times X$ and 2^X are not semi-Kelly. In connection with this, in [3] they extend Kato's question [7, Problem 3.4, p. 1148] to the following.

Question ([3, Question 4.4, p. 82]). Is it true that if a continuum X is Kelley, then the cartesian product $X \times [0, 1]$ is semi-Kelley?

In this paper, we answer this question in negative form. The continuum X of the Example 2.1 is Kelley, however $X \times [0, 1]$ is not semi-Kelley.

With respect to hyperspaces in [3, Theorem 4.5 and Theorem 4.7, p. 83-84] they proved that, if the hyperspace $C(X)$ (or 2^X) is semi-Kelley then X is Kelley. In this paper, the continuum X of the Example 2.1 is Kelley but the hyperspace $C(X)$ is not semi-Kelley.

A. Illanes posed the following problem, see Problem 5.5 in *Problemas propuestos para el taller de Teoría de continuos y sus hiperespacios*, Queretaro, 2013.

Problem Is the property of being semi-Kelley a Whitney property?

In this paper, we prove that if X is as in the Example 2.1, for each Whitney map μ for $C(X)$ there exists a number $0 < t_0 < \mu(X)$ such that for each $t \in (0, t_0)$ the Whitney level $\mu^{-1}(t)$ is not semi-Kelley, therefore being semi-Kelley is not a Whitney property.

2. The example

Given Y the example defined by J.J. Charatonik and W.J. Charatonik in [4], the continuum X of the Example 2.1 is homeomorphic to the union of two copies of Y with a common point.

Example 2.1. *In the polar coordinates (r,φ) in the plane, we consider the following circles*

$$R = \{(r,\varphi) : r = 1\} \text{ and } S = \{(r,\varphi) : r = 3\},$$

for each $n \in \mathbb{N}$,

$$R_n = \{(r,\varphi) : r = 1 + \frac{1}{2n\pi}\} \text{ and } S_n = \{(r,\varphi) : r = 3 - \frac{1}{2n\pi}\},$$

four spirals

$$\Sigma_R = \{(r,\varphi) : r = 1 + \frac{1}{\varphi} \text{ and } \varphi \in [2\pi,\infty)\},$$

$$\Sigma_S = \{(r,\varphi) : r = 3 - \frac{1}{\varphi} \text{ and } \varphi \in [2\pi,\infty)\},$$

$$\Sigma_1 = \{(r,\varphi) : r = 1 - \frac{1}{\varphi} \text{ and } \varphi \in [2\pi,\infty)\},$$

$$\Sigma_2 = \{(r,\varphi) : r = 3 + \frac{1}{\varphi} \text{ and } \varphi \in [2\pi,\infty)\},$$

and an arc

$$\Lambda = \{(r,\varphi) : r = \frac{1-2\pi}{2\pi^2}\varphi + 3 - \frac{1}{2\pi} \text{ and } \varphi \in [0,2\pi]\}.$$

Define the following continua

$$X_1 = R \cup (\bigcup_{n\in\mathbb{N}} R_n) \cup \Sigma_R \cup \Sigma_1,$$

see Figure 1,

$$X_2 = S \cup (\bigcup_{n\in\mathbb{N}} S_n) \cup \Sigma_S \cup \Sigma_2$$

see Figure 2, and finally define the continuum $X = X_1 \cup X_2 \cup \Lambda$, see Figure 3.

Furthermore, for each $n \in \mathbb{N}$ define $p_n = (1 + \frac{1}{2n\pi}, 0)$, $p'_n = (1 - \frac{1}{2n\pi}, 0)$, $q_n = (3 - \frac{1}{2n\pi}0)$ and $q'_n = (3 + \frac{1}{2n\pi}, 0)$, also define $p = (1,0)$, $q = (3,0)$. Observe that, for each $n \in \mathbb{N}$, $R_n \cap \Sigma_R = \{p_n\}$, $S_n \cap \Sigma_S = \{q_n\}$, moreover $\lim_{n\to\infty} p_n = p = \lim_{n\to\infty} p'_n$ and $\lim_{n\to\infty} q_n = q = \lim_{n\to\infty} q'_n$.

Additionally, for each $n \in \mathbb{N}$, define the following subcontinua of X

$$\Lambda_R^n = \{(r,\varphi) : r = 1 + \frac{1}{\varphi} \text{ and } \varphi \in [2n\pi, 2(n+1)\pi]\},$$

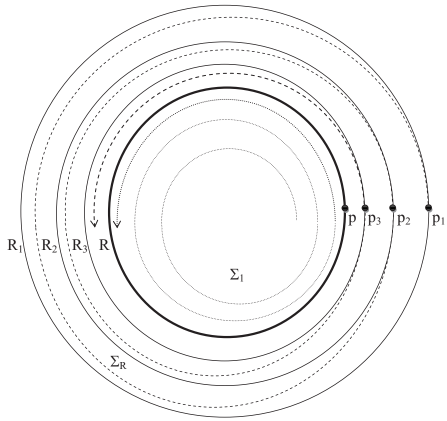

Figure 1. X_1

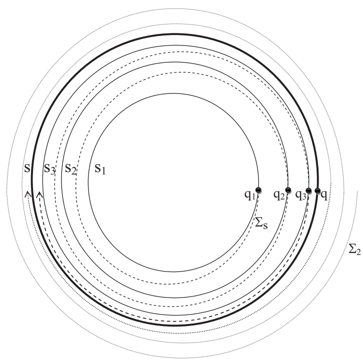

Figure 2. X_2

$$\Lambda_S^n = \{(r, \varphi) : r = 3 - \frac{1}{\varphi} \text{ and } \varphi \in [2n\pi, 2(n+1)\pi)]\},$$

$$\Lambda_1^n = \{(r, \varphi) : r = 1 - \frac{1}{\varphi} \text{ and } \varphi \in [2n\pi, 2(n+1)\pi]\},$$

$$\Lambda_2^n = \{(r, \varphi) : r = 3 + \frac{1}{\varphi} \text{ and } \varphi \in [2n\pi, 2(n+1)\pi)]\}.$$

Notice that Λ_R^n, Λ_S^n, Λ_1^n and Λ_2^n are arcs with end points p_n, p_{n+1}; q_n, q_{n+1}; p'_n, p'_{n+1} and q'_n, q'_{n+1}, respectively. Moreover $\lim_{n\to\infty} \Lambda_R^n = R = \lim_{n\to\infty} \Lambda_1^n$ and $\lim_{n\to\infty} \Lambda_S^n = S = \lim_{n\to\infty} \Lambda_2^n$.

Additionally, denote by $\varrho_1 : X \to R$, $\varrho_2 : X \to S$ the projections defined by $\varrho_1((r, \varphi)) = (1, \varphi)$ and $\varrho_2((r, \varphi)) = (3, \varphi)$.

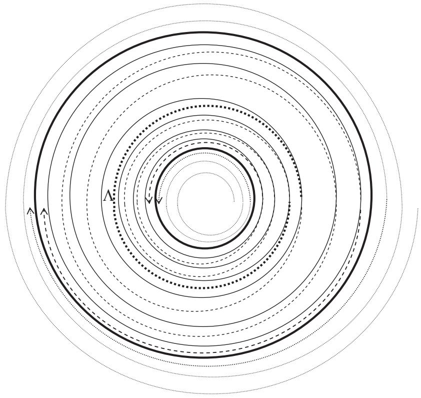

FIGURE 3. X

Theorem 2.2. *The continuum X of the Example 2.1 has the following properties:*

(1) *X is Kelley,*
(2) *$X \times [0,1]$ is not semi-Kelley,*
(3) *the hyperspace $C(X)$ is not semi-Kelley,*
(4) *for each Whitney map $\mu : C(X) \to [0,\infty)$ there exists a number $0 < t_0 < \mu(X)$ such that for each $t \in (0,t_0)$ the Whitney level $\mu^{-1}(t)$ is not semi-Kelley.*

PROOF: (1) To show that X is Kelley we consider a point $x \in X$, a sequence of points $\{x_n\}_{n=1}^{\infty}$ of X converging to x and a continuum $K \subset X$ containing the point x. We have to show that there exists a sequence of continua $\{K_n\}_{n=1}^{\infty}$ such that for each $n \in \mathbb{N}$, $x_n \in K_n$ and $\lim_{n\to\infty} K_n = K$.

If $x \in X \setminus (R \cup S)$, then X is locally connected at x, thus there exists $m \in \mathbb{N}$ such that x_n belongs to the arc component of X containing x for every $n \geq m$. We may take K_n as the union of K and the smallest arc in X joining x_n and x if $n \geq m$, and $K_n = \{x_n\}$ if $n < m$.

Now, if $x \in R \cup S$, without lost of generality suppose that $x \in S$, thus there exists $m \in \mathbb{N}$ such that for every $n \geq m$, x_n belong to X_2. We have two cases:

Case 1. $K \subsetneq S$. For each $n \in \mathbb{N}$, let P_n be the smallest arc that is irreducible between x and $\varrho_2(x_n)$. Note that $\lim_{n\to\infty}(\operatorname{diam}(P_n)) = 0$ and $\lim_{n\to\infty}(K \cup P_n) = K$. Then it is enough to define K_n as the component of $\varrho_2^{-1}(K \cup P_n)$ containing x_n.

Case 2. $S \subset K$. Then for each $n \geq m$ there is a spiral Σ_S^n having x_n as its end point and approaching S. Indeed, if $x_n \in \Sigma_S$ then Σ_S^n can be chosen as a subspiral of Σ_S; if $x_n \in \Sigma_2$ then Σ_S^n is a subspiral of Σ_2; and if $x_n \in S_k$ for some $k \in \mathbb{N}$, then Σ_S^n is the union of an arc joining x_n to q_k and a subspiral of Σ_S with end point q_k. Finally put $K_n = K \cup \Sigma_S^n$ if $n \geq m$ and $K_n = \{x_n\}$ if $n < m$. Since the spirals Σ_S^n converges to S, we have that $\lim_{n\to\infty} K_n = K$.

Thus we have X is Kelley. By [3, Statement 3.17, p. 79], we have that X is semi- Kelley.

(2) We consider $X \times [0,1]$ with cylindrical coordinates (r, φ, z).

To show that $X \times [0,1]$ is not semi-Kelley, define the following subcontinua of $X \times [0,1]$,

$$M = \{(1, 2\pi z, z) : z \in [0,1]\} \subset R \times [0,1].$$

Thus M is an arc from $(p, 0)$ to $(p, 1)$. Furthermore, for each $n \in \mathbb{N}$, define

$$A_n = \{(r, \varphi, z) : r = 1 + \frac{1}{\varphi},\ \varphi = 2(n+z)\pi,\ \text{and } z \in [0,1]\} \subset \Lambda_R^n \times [0,1],$$

$$B_n = \{(r, \varphi, z) : r = 1 + \frac{1}{2n\pi},\ \varphi = 2\pi z,\ \text{and } z \in [0,1]\} \subset R_n \times [0,1].$$

Notice that A_n and B_n are arcs with end points $(p_n, 0)$, $(p_{n+1}, 1)$ and $(p_n, 0)$, $(p_n, 1)$, respectively. Additionally, observe that $A_n \cap B_n = \{(p_n, 0)\}$ and $A_n \cap B_{n+1} = \{(p_{n+1}, 1)\}$. Similarly, define an arc from $(q, 0)$ to $(q, 1)$ by

$$L = \{(3, 2\pi z, z) : z \in [0,1]\} \subset S \times [0,1].$$

And for each $n \in \mathbb{N}$, define

$$D_n = \{(r, -\varphi, z) : r = 3 - \frac{1}{\varphi},\ \varphi = 2(n+z)\pi,\ \text{and } z \in [0,1]\} \subset \Lambda_S^n \times [0,1],$$

$$E_n = \{(r, -\varphi, z) : r = 3 - \frac{1}{2n\pi},\ \varphi = 2\pi z,\ \text{and } z \in [0,1]\} \subset S_n \times [0,1].$$

In this case D_n and E_n are arcs with end points $(q_n, 0)$, $(q_{n+1}, 1)$ and $(q_n, 0)$, $(q_n, 1)$ respectively. Furthermore, $D_n \cap E_n = \{(q_n, 0)\}$ and $D_n \cap E_{n+1} = \{(q_{n+1}, 1)\}$. Also define

$$K_M = M \cup (\bigcup_{n \in \mathbb{N}} A_n) \cup (\bigcup_{n \in \mathbb{N}} B_n),$$

$$K_L = L \cup (\bigcup_{n \in \mathbb{N}} D_n) \cup (\bigcup_{n \in \mathbb{N}} E_n).$$

Notice that K_M and K_L are homeomorphic to a sinoidal curve.

Furthermore, define $\Lambda_0 = \Lambda \times \{0\} \subset \Lambda \times [0,1]$. Thus Λ_0 is an arc with end points $(q_1, 0)$ and $(p_1, 0)$. Finally, define the continuum

$$K = K_L \cup K_M \cup \Lambda_0.$$

Notice that K is homeomorphic to the union of two sinoidal curves with a common point (see Figure 4) and by construction $K \subset X \times [0,1]$.

We will show that M and L are maximal limit continua in K.

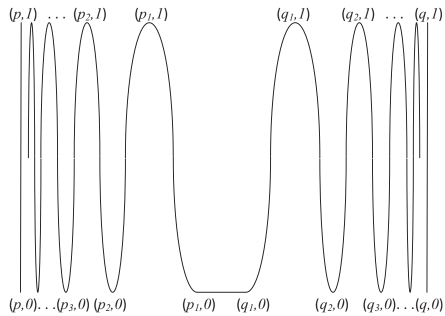

FIGURE 4. K

In order to show that M and L are maximal limit continua in K, for each $n \in \mathbb{N}$, define

$$M_n = (\varrho_1 \times id)^{-1}(M) \cap (\Lambda_1^n \times [0,1]),$$

$$L_n = (\varrho_2 \times id)^{-1}(L) \cap (\Lambda_2^n \times [0,1]).$$

It is clear that $\lim_{n\to\infty} M_n = M$ and $\lim_{n\to\infty} L_n = L$. Suppose that there exists a convergent sequence of subcontinua $\{M'_n\}_{n=1}^{\infty}$ of $X \times [0,1]$ such that $M_n \subset M'_n$, $\lim_{n\to\infty} M'_n = M' \subset K$, and $M \neq M'$.

As $M' \neq M$ and $M \subset M' \subset K$ the set $P = \{r \in \mathbb{N} : (p_r, 0) \in M'\}$ is nonempty, define $r_0 = \min P$.

Let $0 < \varepsilon < 1$, as $(\varrho_1 \times id)(K) = M$, then $(\varrho_1 \times id)(M') = M$, it follows that $M' \subset (\varrho_1 \times id)^{-1}(N(M,\varepsilon))$, therefore there exists $n_0 \in \mathbb{N}$ such that $M'_n \subset (\varrho_1 \times id)^{-1}(N(M,\varepsilon))$ for every $n > n_0$.

Notice that the component of $(\varrho_1 \times id)^{-1}(N(\varepsilon, M))$ that contains M_n is a subset of $(\Lambda_1^{n-1} \cup \Lambda_1^n \cup \Lambda_1^{n+1} \times [0,1])$ so $M'_n \subset (\Sigma_1 \times [0,1])$.

Hence, if d denotes the metric in $X \times [0,1]$ and H denotes the Hausdorff metric in $C(X \times [0,1])$, we have that $H(M', M'_n) \geq d((p,0),(p_{r_0},0)) = \frac{1}{2r_0\pi}$ for each $n \in \mathbb{N}$; it follows that M' is not the limit of continua M'_n, this is a contradiction.

Therefore, M is a maximal limit continuum in K. Similarly L is a maximal limit continuum in K. Notice that $M \cap L = \emptyset$ therefore $X \times [0,1]$ is not semi-Kelley.

(3) To show that the hyperspace $C(X)$ is not semi-Kelley. Let $\mu : C(X) \to [0,\infty)$ be a Whitney map and define $r = \mu(R)$, $s = \mu(S)$. Suppose that $r \leq s$.

Define

$$\mathbf{M} = \{A \in C(R) : A \in \mu^{-1}(\frac{r}{2}),\ p \notin Int_R(A)\},$$

$$\mathbf{C} = \{A \in C(X) : C(\varrho_1)(A) \in \mathbf{M}\}$$

and $t_0 = \min\{\mu(A) : A \in \mathbf{C}\}$ as $\mathbf{C}$ is a nonempty closed subset of $C(X)$ and μ is a map, it follows that t_0 is well defined and there exists $A_0 \in \mathbf{C}$ such that $\mu(A_0) = t_0$, moreover as $A_0 \in \mathbf{C}$, then $t_0 > 0$ and as $\mathbf{M} \subset \mathbf{C}$, then $t_0 \leq \frac{r}{2} < r$; therefore $0 < t_0 < r$.

Let $0 < t < t_0$, notice that $\mu(R), \mu(S) > t$, and $\mu(R_n), \mu(\Lambda_n^R), \mu(S_n), \mu(\Lambda_n^S) > t$ for each $n \in \mathbb{N}$, then we can define the following continua:

$$\mathcal{M} = \{A \in C(R) : A \in \mu^{-1}(t),\ p \notin Int_R(A)\},$$

$$\mathcal{L} = \{A \in C(S) : A \in \mu^{-1}(t),\ q \notin Int_S(A)\}.$$

Notice that $\mathcal{M}$ and $\mathcal{L}$ are arcs in $C(R)$ and $C(S)$ respectively. Denote the end points of $\mathcal{M}$ and $\mathcal{L}$ by M_0, M_1 and L_0, L_1 respectively. It is easy to see that $p \in M_0$, $p \in M_1$, $q \in L_0$, $q \in L_1$. Furthermore, for each $n \in \mathbb{N}$, define

$$\mathcal{A}_n = \{A \in C(R_n) : A \in \mu^{-1}(t),\ p_n \notin Int_{R_n}(A)\},$$

$$\mathcal{B}_n = \{A \in C(\Lambda_n^R) : A \in \mu^{-1}(t)\}.$$

Notice that $\mathcal{A}_n$ is an arc in $C(R_n)$ and $\mathcal{B}_n$ is an arc in $C(\Lambda_n^R)$. Moreover $\lim_{n\to\infty} \mathcal{A}_n = \mathcal{M} = \lim_{n\to\infty} \mathcal{B}_n$. Denote the end points of $\mathcal{A}_n$ and $\mathcal{B}_n$ by A_n^0, A_n^1 and B_n^0, B_n^1, respectively. It is easy to see that $p_n \in A_n^0$, $p_n \in A_n^1$, $p_n \in B_n^0$, $p_{n+1} \in B_n^1$ and $\mu(A_n^0 \cup B_n^0), \mu(B_n^1 \cup A_{n+1}^1) > t$.

Also, for each $n \in \mathbb{N}$, define

$$\mathcal{C}_n = \{A \in C(A_n^0 \cup B_n^0) : A \in \mu^{-1}(t)\},$$

$$\mathcal{D}_n = \{A \in C(B_n^1 \cup A_{n+1}^1) : A \in \mu^{-1}(t)\}.$$

Thus $\mathcal{C}_n$ and $\mathcal{D}_n$ are arcs with end points A_n^0, B_n^0 and B_n^1, A_{n+1}^1, respectively. Furthermore, $\lim_{n\to\infty} \mathcal{C}_n = \{M_0\}$ and $\lim_{n\to\infty} \mathcal{D}_n = \{M_1\}$. Moreover, observe that $\mathcal{A}_n \cap \mathcal{C}_n = \{A_n^0\}$, $\mathcal{C}_n \cap \mathcal{B}_n = \{B_n^0\}$, $\mathcal{B}_n \cap \mathcal{D}_n = \{B_n^1\}$, $\mathcal{D}_n \cap \mathcal{A}_{n+1} = \{A_{n+1}^1\}$.

Similarly, for each $n \in \mathbb{N}$, define

$$\mathcal{E}_n = \{A \in C(S_n) : A \in \mu^{-1}(t),\ q_n \notin Int_{S_n}(A)\},$$

$$\mathcal{F}_n = \{A \in C(\Lambda_n^S) : A \in \mu^{-1}(t)\}.$$

Notice that $\mathcal{E}_n$ is an arc in $C(S_n)$ and $\mathcal{F}_n$ is an arc in $C(\Lambda_n^S)$. Moreover $\lim_{n\to\infty} \mathcal{E}_n = \mathcal{L} = \lim_{n\to\infty} \mathcal{F}_n$. Denote the end points of $\mathcal{E}_n$ and $\mathcal{F}_n$ by E_n^0, E_n^1 and F_n^0, F_n^1, respectively. It is easy to see that $q_n \in E_n^0$, $q_n \in E_n^1$, $q_n \in F_n^0$, $q_{n+1} \in F_n^1$ and $\mu(E_n^1 \cup F_n^1), \mu(F_n^0 \cup E_{n+1}^0) > t$.

Also, for each $n \in \mathbb{N}$, define

$$\mathcal{G}_n = \{A \in C(E_n^1 \cup F_n^1) : A \in \mu^{-1}(t)\},$$

$$\mathcal{H}_n = \{A \in C(F_n^0 \cup E_{n+1}^0) : A \in \mu^{-1}(t)\}.$$

Thus $\mathcal{G}_n$ and $\mathcal{H}_n$ are arcs with end points E_n^1, F_n^1 and F_n^0, E_{n+1}^0, respectively. Furthermore, $\lim_{n\to\infty} \mathcal{G}_n = \{L_1\}$ and $\lim_{n\to\infty} \mathcal{H}_n = \{L_0\}$.

Additionally, observe that $\mathcal{E}_n \cap \mathcal{G}_n = \{E_n^1\}$, $\mathcal{G}_n \cap \mathcal{F}_n = \{F_n^1\}$, $\mathcal{F}_n \cap \mathcal{H}_n = \{F_n^0\}$, $\mathcal{H}_n \cap \mathcal{E}_{n+1} = \{E_{n+1}^0\}$. Furthermore, define

$$\mathcal{I} = \{A \in C(\Lambda) : A \in \mu^{-1}(t)\}.$$

In this case, $\mathcal{I}$ is an arc. Denote the end points of $\mathcal{I}$ by I^0 and I^1. It is easy to see that $q_1 \in I^0$, $p_1 \in I^1$ and $\mu(I^0 \cup E_1^0), \mu(I^1 \cup A_1^1) > t$.

Also define

$$\mathcal{I}_0 = \{A \in C(I^0 \cup E_1^0) : A \in \mu^{-1}(t)\},$$

$$\mathcal{I}_1 = \{A \in C(I^1 \cup A_1^1) : A \in \mu^{-1}(t)\}.$$

Notice that $\mathcal{I}_0$ and $\mathcal{I}_1$ are arcs with end points I^0, E_1^0 and I^1, A_1^1, respectively. Moreover, observe that $\mathcal{I}_0 \cap \mathcal{E}_1 = \{E_1^0\}$, $\mathcal{I}_0 \cap \mathcal{I} = \{I^0\}$, $\mathcal{I} \cap \mathcal{I}_1 = \{I^1\}$, $\mathcal{I}_1 \cap \mathcal{A}_1 = \{A_1^1\}$. Define the following subcontinua of $C(X)$

$$\mathcal{K}_M = \mathcal{M} \cup (\bigcup_{n\in\mathbb{N}} \mathcal{A}_n) \cup (\bigcup_{n\in\mathbb{N}} \mathcal{B}_n) \cup (\bigcup_{n\in\mathbb{N}} \mathcal{C}_n) \cup (\bigcup_{n\in\mathbb{N}} \mathcal{D}_n),$$

$$\mathcal{K}_L = \mathcal{L} \cup (\bigcup_{n\in\mathbb{N}} \mathcal{E}_n) \cup (\bigcup_{n\in\mathbb{N}} \mathcal{F}_n) \cup (\bigcup_{n\in\mathbb{N}} \mathcal{G}_n) \cup (\bigcup_{n\in\mathbb{N}} \mathcal{H}_n).$$

Notice that $\mathcal{K}_M$ and $\mathcal{K}_L$ are homeomorphic to a sinoidal curve.

Define $\Lambda_0 = \mathcal{I}_0 \cup \mathcal{I} \cup \mathcal{I}_1$. Thus Λ_0 is an arc with end points A_1^1 and E_1^0. Finally, define the continuum

$$\mathcal{K} = \mathcal{K}_M \cup \Lambda_0 \cup \mathcal{K}_L.$$

Notice that $\mathcal{K}$ is homeomorphic to the union of two sinoidal curves with a common point (see Figure 5), by construction $\mathcal{K} \subset \mu^{-1}(t) \subset C(X)$.

Let $0 < \varepsilon < \frac{r}{2}$, and $\delta_1 > 0$ given by the uniform continuity of μ for ε and $0 < \delta < \delta_1$ given by the uniform continuity of $C(\varrho_1)$ for δ_1. Denote by H the Hausdorff metric in $C(X)$.

Claim 1. For each $A \in \mathcal{K}$,

(i) $\mu(C(\varrho_1)(A)) \in [0, \frac{r}{2}]$,
(ii) for each $B \in C(X)$ such that $H(A, B) < \delta$, $\mu(C(\varrho_1)(B)) < r$.

(i) For each $A \in \mathcal{K}$, there exists $D \in \mathbf{C}$ such that $A \subset D$, then $C(\varrho_1)(A) \subseteq C(\varrho_1)(D)$, hence $\mu(C(\varrho_1)(A)) \le \mu(C(\varrho_1)(D)) = \frac{r}{2}$.

(ii) If $H(A, B) < \delta$, then $H(C(\varrho_1)(A), C(\varrho_1)(B)) < \delta_1$, it follows that $|\mu(C(\varrho_1)(A)) - \mu(C(\varrho_1)(B))| < \varepsilon$, so $\mu(C(\varrho_1)(B)) \in [0, \frac{r}{2} + \varepsilon]$, therefore $\mu(C(\varrho_1)(B)) < r$.

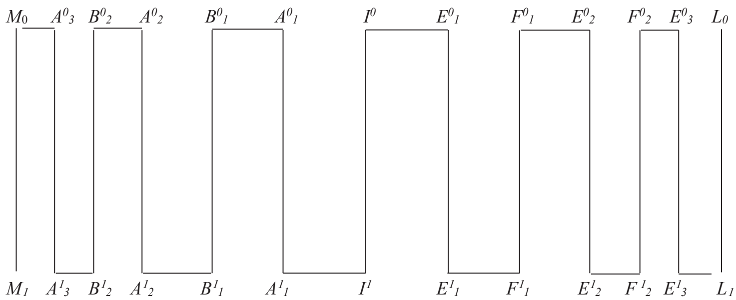

FIGURE 5. $\mathcal{K}$

We will show that $\mathcal{M}$ and $\mathcal{L}$ are maximal limit continua in $\mathcal{K}$. In order to show that $\mathcal{M}$ and $\mathcal{L}$ are maximal limit continua in $\mathcal{K}$, for each $n \in \mathbb{N}$, define

$$\mathcal{M}_n = \{A \in C(\Lambda_1^n) : A \in \mu^{-1}(t)\},$$
$$\mathcal{L}_n = \{A \in C(\Lambda_2^n) : A \in \mu^{-1}(t)\}.$$

Notice that $\mathcal{M}_n$ is an arc in $C(\Lambda_1^n)$ and $\mathcal{L}_n$ is an arc in $C(\Lambda_2^n)$. Denote the end points of $\mathcal{M}_n$ and $\mathcal{L}_n$ by M_n^0, M_n^1 and L_n^0, L_n^1, respectively. It is easy to see that $p_n' \in M_n^0$, $p_{n+1}' \in M_n^1$, $q_n' \in L_n^0$, $q_{n+1}' \in L_n^1$.

It is clear that $\lim_{n\to\infty} \mathcal{M}_n = \mathcal{M}$ and $\lim_{n\to\infty} \mathcal{L}_n = \mathcal{L}$. Suppose that, there exists a sequence of subcontinua $\{\mathcal{M}_n'\}_{n=1}^{\infty}$ of $C(X)$ with $\mathcal{M}_n \subset \mathcal{M}_n'$, $\lim_{n\to\infty} \mathcal{M}_n' = \mathcal{M}' \subset \mathcal{K}$ and $\mathcal{M} \neq \mathcal{M}'$.

As $\mathcal{M}' \subset N(\mathcal{K}, \delta)$ and $\lim_{n\to\infty} \mathcal{M}_n' = \mathcal{M}'$, there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$, $\mathcal{M}_n' \subset N(\mathcal{K}, \delta)$. Notice that for each $B \in \mathcal{M}_n'$, there exists $A \in \mathcal{K}$ such that $H(A, B) < \delta$, by Claim 1, $\mu(C(\varrho_1)(B)) < r$, so $C(\varrho_1)(B) \subsetneq R$. It follows that $\mathcal{M}_n' \subset C(\Lambda_1^{n-1} \cup \Lambda_1^n \cup \Lambda_1^{n+1}) \subset C(\Sigma_1)$, therefore $\mathcal{M}_n' \in C(C(\Sigma_1))$.

Moreover as $\mathcal{M}' \neq \mathcal{M}$ and $\mathcal{M}' \subset \mathcal{K}$ the set $P = \{m \in \mathbb{N} : A_m^0 \in \mathcal{M}'\}$ is nonempty, define $m_0 = \min P$. Hence, if d denotes the metric in X and $\mathbf{H}$ denotes the Hausdorff metric in $C(C(X))$, for each $n > n_0$, $\mathbf{H}(\mathcal{M}', \mathcal{M}_n') \geq H(A_{m_0}^0, M_n^0) \geq d(p_{m_0}, p_n') > d(p_{m_0}, p) = \frac{1}{2m_0\pi}$, this contradicts that $\lim_{n\to\infty} \mathcal{M}_n' = \mathcal{M}'$.

Therefore, $\mathcal{M}$ is maximal limit continuum in $\mathcal{K}$. Similarly $\mathcal{L}$ is maximal limit continuum in $\mathcal{K}$. Since $\mathcal{M} \cap \mathcal{L} = \emptyset$, $C(X)$ is not semi-Kelley. Similarly if we suppose that $s \leq r$, $C(X)$ is not semi-Kelley.

(4) Let t_0 as in (3) and $0 < t < t_0$ we consider the continua defined in (3). Since $\mu^{-1}(t) \subset C(X)$ in particular we can take the sequence of subcontinua $\{\mathcal{M}_n'\}_{n=1}^{\infty}$

of $\mu^{-1}(t)$, and conclude that $\mathcal{M}$ is maximal limit continuum in $\mathcal{K}$; similarly $\mathcal{L}$ is maximal limit continuum in $\mathcal{K}$.

As $\mathcal{M}, \mathcal{L}, \mathcal{K} \subset \mu^{-1}(t)$ and $\mathcal{M} \cap \mathcal{L} = \emptyset$, it follows that $\mu^{-1}(t)$ is not semi-Kelley. □

To finish this paper we propose the following problems.

Problem 5. Does there exist a hereditarily unicoherent continuum X such that $X \times [0,1]$ or $C(X)$ is not semi-Kelley?

Problem 6. Classify the continua for which being semi-Kelley is a Whitney property.

Problem 7 (A. Illanes)**.** Is the property of being semi-Kelley a Whitney reversible property?

Acknowledgment. The authors wish to thank the participants in *Séptimo Taller de Investigación de Continuos e Hiperespacios*, celebrated in Queretaro City during the Summer of 2013, for useful discussions. Particularly, M. Bernal Romero, M. Chacón-Tirado, L. E. García Hernández, M. Flores González, C. Islas Moreno E. Márquez Rodríguez, J. A. Martínez Cortez and C. Solís Said.

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(*Received* January 21, 2017, *revised* March 22, 2017)