On graph associated to co-ideals of commutative semirings

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Abstract. Let R be a commutative semiring with non-zero identity. In this paper, we introduce and study the graph $\Omega(R)$ whose vertices are all elements of R and two distinct vertices x and y are adjacent if and only if the product of the co-ideals generated by x and y is R. Also, we study the interplay between the graph-theoretic properties of this graph and some algebraic properties of semirings. Finally, we present some relationships between the zero-divisor graph $\Gamma(R)$ and $\Omega(R)$.

Keywords: semiring; co-ideal; maximal co-ideal

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1. Introduction

The concept of the *zero-divisor graph* of a commutative ring R was first introduced by Beck [3]. He defined this graph as a simple graph where all elements of the ring R are the vertex-set of this graph and two distinct elements x and y are adjacent if and only if $xy = 0$. Beck conjectured that $\chi(R) = \omega(R)$ for every ring R. In [2], Anderson and Livingston introduced the zero-divisor graph with vertices $Z(R)^* = Z(R) \setminus \{0\}$, the set of non-zero zero-divisors of R. Some other investigations into properties of zero-divisor graph over commutative semiring may be found in [5], [6]. In [11], Sharma and Bhatwadekar defined another graph on a ring R with vertices as elements of R and there is an edge between two distinct vertices x and y in R if and only if $Rx + Ry = R$. Further, in [10], Maimani et al. studied the graph defined by Sharma and Bhatwadekar and called it *comaximal graph*. Also, in [1], Akbari et al. studied the comaximal graph over non-commutative ring.

Note that throughout this paper all semirings are considered to be commutative semirings with non-zero identity. First, we introduce the concept of *product* of co-ideals in the semiring R. Next, we define an undirected graph over commutative semiring in which vertices are all elements of R and two distinct vertices x and y are adjacent if and only if the product of the co-ideals generated by x and y is R (i.e. $F(x)F(y) = R$). We denote this graph by $\Omega(R)$. In Section 2, we recall some notions of semirings which will be used in this paper. In other sections, we study some graph-theoretic properties of $\Omega(R)$ and its subgraphs such as diameter, radius, girth, clique number and chromatic number.

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In a graph G, we denote the vertex-set of G by $V(G)$ and the edge-set by $E(G)$. A graph G is said to be *connected*, if there is a path between every two distinct vertices and we say that G is *totally disconnected*, if no two vertices of G are adjacent. For a given vertex x, the number of all vertices adjacent to it, is called *degree* of the vertex x, denoted by $deg(x)$. For distinct vertices x and y of G, let $d(x, y)$ be the length of the shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The *diameter* of G is $\text{diam}(G) = \sup\{d(x, y) : x$ and y are distinct vertices of $G\}$. The *girth* of G, denoted by $gr(G)$, is defined as the length of the shortest cycle in G. If G has no cycles, then $gr(G) = \infty$ and G is called a *forest*. Also, G is called a *tree* if G is connected and has no cycles. A *clique* in a graph G is a complete subgraph of G. The *clique number* of G, denoted by $\omega(G)$, is the number of vertices in a largest clique of G. An *independent set* in a graph G is a set of pairwise non-adjacent vertices. A graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. We denote the complete graph on n vertices by K_n. For a positive integer k, a *k-partite* graph is one whose vertex-set can be partitioned into k independent sets. A k-partite graph G is said to be a *complete k-partite* graph, if each vertex is joined to every vertex that is not in the same partition. The *complete bipartite* graph (2-partite graph) with parts of sizes m and n is denoted by $K_{m,n}$. We will sometimes call a $K_{1,n}$ a *star graph*. We write $G \setminus \{x\}$ or $G \setminus S$ for the subgraph of G obtained by deleting a vertex x or set of vertices S. An *induced subgraph* is a subgraph obtained by deleting a set of vertices. Also, a *spanning subgraph* of G is a subgraph with vertex-set $V(G)$. A general reference for graph theory is [12].

2. Preliminaries

In this section, we recall various notions about semirings which will be used throughout the paper. A *semiring* R is an algebraic system $(R, +, \cdot)$ such that $(R, +)$ is a commutative monoid with identity element 0 and $(R, \cdot)$ is a semigroup. In addition, operations $+$ and $\cdot$ are connected by distributivity and 0 annihilates R (i.e. $x0 = 0x = 0$ for each $x \in R$). A semiring R is said to be *commutative* if $(R, \cdot)$ is a commutative semigroup and R is said to have an *identity* if there exists $1 \in R$ such that $1x = x1 = x$.

Recall that, throughout this paper, all semirings are commutative with non-zero identity. The following definitions are given in [7], [9].

2.1 Definition. Let R be a semiring.

(1) A non-empty subset I of R is called a *co-ideal* of R if and only if it is closed under multiplication and satisfies the condition that $a + r \in I$ for all $a \in I$ and $r \in R$. According to this definition, $0 \in I$ if and only if $I = R$. Also, a co-ideal I of R is called *strong*, if $1 \in I$.

(2) A co-ideal I of semiring R is called *subtractive* if $x \in I$ and $xy \in I$, implies $y \in I$ for all $x, y \in R$. So every subtractive co-ideal is a strong co-ideal.

(3) A proper co-ideal P of R is called *prime* if $a + b \in P$, implies $a \in P$ or $b \in P$ for all $a, b \in R$.

(4) A proper co-ideal I of R is called *maximal* if there is no co-ideal J such that $I \subset J \subset R$.

(5) An element a of a semiring R is *multiplicatively idempotent* if and only if $a^2 = a$ and a is called *additively idempotent* if and only if $a + a = a$. A semiring R is said to be idempotent if it is both additively and multiplicatively idempotent.

(6) An element x of a semiring R is called a *zero-sum* of R, if there exists an element $y \in R$ such that $x + y = 0$. It is clear that, y is the unique element which satisfies $x + y = 0$. We will denote the set of all zero-sums of R by $ZS(R)$. It is easy to see that $ZS(R)$ is an ideal of R. Also, a semiring R is a ring if and only if $ZS(R) = R$ and R is called *zero-sumfree* if and only if $ZS(R) = 0$.

(7) If A is a non-empty subset of a semiring R, then the set $F(A)$ of all elements of R of the form $a_1 a_2 \dots a_n + r$, where $a_i \in A$ for all $1 \leq i \leq n$ and $r \in R$, is a co-ideal of R containing A. In fact, $F(A)$ is the unique smallest co-ideal of R containing A.

By the above definition, we can consider the co-ideal generated by a single element $x \in R$ as follows: $F(x) = \{x^n + r : \ r \in R \text{ and } n \in \mathbf{N}\}$. It is obvious that, if $x \in I$ for some co-ideal I, then $F(x) \subseteq I$.

By definition of co-ideal, if R is a ring, then R has no proper co-ideals and so throughout this paper we consider semirings which are not rings. For a semiring R, we denote the set of maximal co-ideals, the union of all the maximal co-ideals and the intersection of all the maximal co-ideals of R by $Co - Max(R)$, $UM(R)$ and $IM(R)$, respectively. Also, if the semiring R has exactly one maximal co-ideal, then we say that the semiring R is *c-local* and R is said to be a *c-semilocal* semiring, if R has only a finite number of maximal co-ideals.

2.2 Lemma ([7]). *Let $I_1, \dots, I_n$ be co-ideals of a semiring R and P be a prime co-ideal containing $\bigcap_{i=1}^n I_i$. Then $I_i \subseteq P$ for some $i = 1, \dots, n$. Moreover, if $P = \bigcap_{i=1}^n I_i$, then $P = I_i$ for some i.*

2.3 Lemma. *Let R be a semiring. Then $x \in \sqrt{ZS(R)}$ if and only if $F(x) = R$.*

PROOF: Let $x \in \sqrt{ZS(R)}$. Thus $x^n \in ZS(R)$ for some positive integer n. This implies $x^n + r = 0$ for some $r \in R$. Hence $0 \in F(x)$, since $x^n + r \in F(x)$ and so $F(x) = R$.

The converse follows, since all conclusions are reversible. □

2.4 Proposition. *Let R be a semiring. Then $R \setminus \sqrt{ZS(R)} = UM(R)$.*

PROOF: Assume that $x \in R \setminus \sqrt{ZS(R)}$. Thus $F(x) \neq R$ and by [7, Proposition 2.1], there exists $m \in Co - Max(R)$ such that $x \in F(x) \subseteq m$. Hence $R \setminus \sqrt{ZS(R)} \subseteq UM(R)$.

Conversely, suppose that $x \in UM(R)$. Thus there is a maximal co-ideal m such that $x \in m$. Now, if $x \in \sqrt{ZS(R)}$, then $F(x) = R$ by Lemma 2.3 and so $R = F(x) \subseteq m$, that is impossible. Hence $UM(R) \subseteq R \setminus \sqrt{ZS(R)}$. This implies $R \setminus \sqrt{ZS(R)} = UM(R)$. □

2.5 Remark. Note that the Prime Avoidance Theorem is explained for subtractive prime co-ideals of a commutative semiring R in [4, Theorem 3.8]. Also, by [8, Proposition 2.5] and [7, Theorem 3.10], every maximal co-ideal is a subtractive and prime co-ideal, so we can conclude that the Prime Avoidance Theorem and Lemma 2.2 also hold for the case where co-ideals are maximal.

In the following, we define the product of co-ideals of a semiring R. It is straightforward to verify that the product of co-ideals with this definition is a co-ideal.

2.6 Definition. Let I and J be two co-ideals of a semiring R. We define the product of I and J as follows:

$$IJ = \{xy + r : \ x \in I, \ y \in J \ \text{ and } \ r \in R\}.$$

Similarly, we define the product of any finite family of co-ideals. Moreover, I^n is defined for any co-ideal I and $I^n = \{a_1 \dots a_n + r : \ a_i \in I \text{ and } r \in R\}$.

Let I and J be co-ideals of R such that $x \in I$ and $y \in J$. Note that with this definition, if I and J are strong co-ideals, then $x, y \in IJ$ because $x = x1 + 0$ and $y = 1y + 0$ but this may not be true in general.

3. Some basic properties of $\Omega(R)$

As mentioned in the introduction, the graph $\Omega(R)$ is a graph with all the elements of R as its vertex-set and two distinct vertices x and y are adjacent if and only if $F(x)F(y) = R$. Let $\Omega_1(R)$ be the subgraph of $\Omega(R)$ with vertex-set $\sqrt{ZS(R)}$ and $\Omega_2(R)$ be the subgraph of $\Omega(R)$ with vertex-set $UM(R)$. If $x \in \sqrt{ZS(R)}$, then by Lemma 2.3, $F(x) = R$ and this implies x is adjacent to any other vertex of R. With this comment, we can say that $\Omega_1(R)$ is a complete graph. Also, if $x, y \in m$ for some maximal co-ideal m of R, then x and y cannot be adjacent because $F(x)F(y) \subseteq m$. Hence, if the semiring R has one maximal co-ideal, then $\Omega_2(R)$ is a totally disconnected graph.

3.1 Lemma. *Let m be a maximal co-ideal of a semiring R and $x \in R$. If $x \notin m$, then $mF(x) = R$.*

PROOF: Suppose that $x \notin m$. Thus $F(m \cup \{x\}) = R$ since $m \subsetneq F(m \cup \{x\})$ and m is a maximal co-ideal. Now, since $0 \in R$, we split the proof into three cases for $F(m \cup \{x\})$:

Case 1: There exist $a_1, \dots, a_k \in m$ and $r \in R$ for some positive integer k such that $a_1 \dots a_k + r = 0$. This implies $0 \in m$ since m is co-ideal. This is a contradiction because m is a maximal co-ideal.

Case 2: $x^t + r = 0$ for some $r \in R$ and a positive integer t. In this case, $F(x) = R$ because $0 = x^t + r \in F(x)$ and so $mF(x) = R$.

Case 3: $yx^t + r = 0$ for some $y \in m$, $r \in R$ and a positive integer t. Hence $mF(x) = R$ since $0 = yx^t + r \in mF(x)$. □

As an immediate consequence of Lemma 3.1, we have the next proposition:

3.2 Proposition. *Let m be a maximal co-ideal of a semiring R and $x \in R$. If $x \notin m$, then there is an element $y \in m$ such that x is adjacent to y in $\Omega(R)$.*

PROOF: Suppose that m is a maximal co-ideal and $x \notin m$. By Lemma 3.1, we have $mF(x) = R$. This implies $y(x^t + r) + k = 0$ for some $r, k \in R$, $y \in m$ and a positive integer t. Hence $yx^t + s = 0$ for some $s \in R$ and so $F(x)F(y) = R$ since $0 = yx^t + s \in F(x)F(y)$. Therefore, x and y are adjacent in $\Omega(R)$. □

3.3 Proposition. *Let R be a semiring and $x \in R$. Then $x \in IM(R)$ if and only if x is adjacent to no vertex of $\Omega_2(R)$.*

PROOF: Let $x \in IM(R)$. Assume contrary that $y \in UM(R)$ is adjacent to x in $\Omega_2(R)$. Thus there exists $m \in Co-Max(R)$ such that $y \in m$ and $F(x)F(y) = R$. On the other hand, $x \in IM(R)$ gives $x \in m$. Hence $F(x)F(y) \subseteq m$, that is a contradiction.

Conversely, assume that x is not adjacent to any vertex of $\Omega_2(R)$. If $x \notin IM(R)$, there exists $m \in Co - Max(R)$ such that $x \notin m$. By Proposition 3.2, there is an element $y \in m$ such that x is adjacent to y, which is contrary to our assumption. □

By Proposition 3.3, for each $x \in IM(R)$, $deg_{\Omega_2(R)}(x) = 0$. So it will be interesting to study the properties of the graph $\Omega_2(R) \setminus IM(R)$ with vertex-set $UM(R) \setminus IM(R)$. Note that if R is a c-local semiring, then $\Omega_2(R) \setminus IM(R)$ is an empty graph.

3.4 Theorem. *Let R be a semiring which is not c-local. Then $\Omega_2(R) \setminus IM(R)$ is a complete bipartite graph if and only if R has exactly two maximal co-ideals.*

PROOF: First, assume that $\Omega_2(R) \setminus IM(R)$ is a complete bipartite graph with vertex-sets V_1 and V_2. Clearly, m is contained in one of the partitions for any maximal co-ideal m. Thus, suppose that $m_i \setminus IM(R) \subseteq V_i$ for $i = 1, 2$. If R has another maximal co-ideal such as m_3, then $m_3 \setminus IM(R) \subseteq V_i$ for some $i = 1, 2$, which is impossible, since $m_1m_3 = m_2m_3 = R$. Hence R can have only two maximal co-ideals.

Conversely, suppose that $Co - Max(R) = \{m_1, m_2\}$. Then the vertex-set of $\Omega_2(R) \setminus IM(R)$ is $(m_1 \setminus m_2) \cup (m_2 \setminus m_1)$. Clearly, the subgraphs $m_1 \setminus m_2$ and $m_2 \setminus m_1$ are totally disconnected. Let $x \in m_1 \setminus m_2$ and $y \in m_2 \setminus m_1$. Now to complete the proof, it suffices to show that $F(x)F(y) \nsubseteq m_1$ and $F(x)F(y) \nsubseteq m_2$. If $F(x)F(y) \subseteq m_1$, then $xy \in m_1$. This implies that $y \in m_1$, since m_1 is subtractive, a contradiction. Similarly, it can be shown that $F(x)F(y) \nsubseteq m_2$. Therefore we have $F(x)F(y) = R$. Hence $\Omega_2(R) \setminus IM(R)$ is complete bipartite graph with vertex-set $m_1 \setminus m_2$ and $m_2 \setminus m_1$. □

In the following, we give an example of semiring R in which R has two maximal co-ideals and show that $\Omega_2(R) \setminus IM(R)$ is complete bipartite graph.

3.5 Example. Let $S = \{0, 1, a\}$ be an idempotent semiring in which $a + 1 = 1 + a = a$ and let $R = S \times S$. The maximal co-ideals of R are as follows:

$$m_1 = \{(0,1),(0,a),(1,a),(a,1),(1,1),(a,a)\},$$
$$m_2 = \{(1,0),(a,0),(1,a),(a,1),(1,1),(a,a)\}.$$

It can be shown that $\Omega_2(R) \setminus IM(R)$ is complete bipartite with vertex-sets $\{(0,1),(0,a)\}$ and $\{(1,0),(a,0)\}$.

In the next theorem, we study the clique number of the graph $\Omega_2(R) \setminus IM(R)$ for a c-semilocal semiring. Also, with this theorem, we give a result about the girth of $\Omega_2(R) \setminus IM(R)$.

3.6 Theorem. *Let R be a c-semilocal semiring and $|Co - Max(R)| \geq n$ with $n \geq 2$. Then $\Omega_2(R) \setminus IM(R)$ has a clique of order n. In particular, if $|Co - Max(R)| = n$, then $\omega(\Omega_2(R) \setminus IM(R)) = n$.*

Proof: Let $\{m_1, \dots, m_n\}$ be a subset of $Co - Max(R)$. We claim that for any $x_1 \in m_1 \setminus \bigcup_{j=2}^{n} m_j$, there exists a clique with vertex-set $\{x_1, \dots, x_n\}$ in $\Omega_2(R) \setminus IM(R)$, where $x_i \in m_i \setminus \bigcup_{\substack{j=1 \\ j \neq i}}^{n} m_j$ for $i = 1, \dots, n$. We prove this claim by induction on n. For $n = 2$, the proof is similar to the proof of Theorem 3.4. Now, suppose that $n \geq 3$. By Remark 2.5, $m_1 \cap m_n \nsubseteq \bigcup_{j=2}^{n-1} m_j$. Thus there exists $y \in (m_1 \cap m_n) \setminus \bigcup_{j=2}^{n-1} m_j$ and so $x_1 + y \in (m_1 \cap m_n) \setminus \bigcup_{j=2}^{n-1} m_j$. By induction hypothesis, there is a clique with vertex-set $\{x_1 + y, x_2, \dots, x_{n-1}\}$, where $x_i \in m_i \setminus \bigcup_{\substack{j=1 \\ j \neq i}}^{n-1} m_j$ for $2 \leq i \leq n-1$. Indeed, $x_2, \dots, x_{n-1} \notin m_n$ since $x_1 + y \in m_n$. On the other hand, since $x_1 + y$ is adjacent to $x_2, \dots, x_{n-1}$, hence x_1 is adjacent to $x_2, \dots, x_{n-1}$ because $F(x_1 + y) \subseteq F(x_1)$. Now, since $x_1 + \cdots + x_{n-1} \notin m_n$ (m_n is prime), so by Proposition 3.2, there exists $x_n \in m_n$ which is adjacent to $x_1 + \cdots + x_{n-1}$. This implies that x_n is adjacent to $x_1, \dots, x_{n-1}$ and we can conclude $\{x_1, \dots, x_n\}$ is a clique of order n in $\Omega_2(R) \setminus IM(R)$.

Now, suppose that $|Co - Max(R)| = n$. Thus we have $\omega(\Omega_2(R) \setminus IM(R)) \geq n$. If $\Omega_2(R) \setminus IM(R)$ has a clique of order k in which $k \geq n$, then by the Pigeon Hole Principal, two elements of the clique should belong to one maximal co-ideal, which is a contradiction. Hence $\omega(\Omega_2(R) \setminus IM(R)) = n$. □

Theorem 3.6 leads to the following corollary:

3.7 Corollary. *Let R be a c-semilocal semiring with $|Co - Max(R)| \geq 3$. Then $gr(\Omega_2(R) \setminus IM(R)) = 3$.*

Proof: Let $|Co - Max(R)| \geq 3$. By Theorem 3.6, $\Omega_2(R) \setminus IM(R)$ has a clique of order 3, so $gr(\Omega_2(R) \setminus IM(R)) = 3$. □

In the next theorem, we will compute the girth of $\Omega_2(R) \setminus IM(R)$ when R is a c-semilocal semiring.

3.8 Theorem. *Let R be a c-semilocal semiring with $|Co - Max(R)| \geq 2$. If $\Omega_2(R) \setminus IM(R)$ contains a cycle, then $gr(\Omega_2(R) \setminus IM(R)) \leq 4$.*

Proof: Assume that $\Omega_2(R) \setminus IM(R)$ contains a cycle and $gr(\Omega_2(R) \setminus IM(R)) \neq 3$. So Corollary 3.7 implies that $|Co - Max(R)| = 2$. Hence by Theorem 3.4, $\Omega_2(R) \setminus IM(R)$ is complete bipartite graph and so $gr(\Omega_2(R) \setminus IM(R)) = 4$. □

3.9 Example. Let $X = \{a, b, c\}$ and $R = (P(X), \cup, \cap)$ be a semiring, where $P(X)$ is the power set of X. For this semiring we have $1_R = X$ and $0_R = \emptyset$. In this case, the maximal co-ideals of semiring R are as follows:

$$m_1 = \{\{a\}, \{a, b\}, \{a, c\}, X\},$$
$$m_2 = \{\{b\}, \{a, b\}, \{b, c\}, X\},$$
$$m_3 = \{\{c\}, \{a, c\}, \{b, c\}, X\}.$$

For the graph $\Omega_2(R) \setminus IM(R)$ the vertex-set is $P(X) \setminus \{\emptyset, X\}$ and {{a},{b},{c}} is a maximal clique. This implies that $\omega(\Omega_2(R) \setminus IM(R)) = 3$ and so $gr(\Omega_2(R) \setminus IM(R)) = 3$.

3.10 Proposition. *Let R be a c-semilocal semiring with $|Co - Max(R)| \geq 2$. Then $\Omega_2(R) \setminus IM(R)$ is star graph if and only if there is a vertex of $\Omega_2(R) \setminus IM(R)$ which is adjacent to every other vertex.*

Proof: The necessity is obvious by definition, thus we need to prove the sufficiency. Assume that there exists $x \in \Omega_2(R) \setminus IM(R)$ that is adjacent to every other vertex. Let $x \in m$ for some $m \in Co - Max(R)$. We must have $|m \setminus IM(R)| = 1$, because if x and y are distinct vertices of $m \setminus IM(R)$, then by assumption x and y are adjacent, which is impossible. Now, if $|Co - Max(R)| \geq 3$, then $|m \setminus IM(R)| \geq 3$ for any maximal co-ideal m of R. Hence R cannot contain more than two maximal co-ideals. It is straightforward to verify that $\Omega_2(R) \setminus IM(R)$ is a star graph by Theorem 3.4. □

3.11 Theorem. *Let R be a c-semilocal semiring with $|Co - Max(R)| \geq 2$. Then the following statements are equivalent:*

(1) *$\Omega_2(R) \setminus IM(R)$ is a tree;*
(2) *$\Omega_2(R) \setminus IM(R)$ is a forest;*
(3) *$|Co - Max(R)| = 2$ and $|m \setminus IM(R)| = 1$ for some $m \in Co - Max(R)$;*
(4) *$\Omega_2(R) \setminus IM(R)$ is a star graph.*

Proof: (1) ⇒ (2), (3) ⇒ (4) and (4) ⇒ (1) are clear.

(2) ⇒ (3) Let $\Omega_2(R) \setminus IM(R)$ be a forest. Thus by Corollary 3.7, we have $|Co - Max(R)| = 2$. Now, if $|m \setminus IM(R)| \geq 2$ for each maximal co-ideal m, then $\Omega_2(R) \setminus IM(R)$ contains a cycle of order 4, because by Theorem 3.4, $\Omega_2(R) \setminus IM(R)$ is a complete bipartite graph, a contradiction. Hence $|m \setminus IM(R)| = 1$ for some $m \in Co - Max(R)$. □

3.12 Proposition. *Let R be a c-semilocal semiring. Then $\Omega_2(R) \setminus IM(R)$ is a complete graph if and only if it is in the form $K_{1,1}$.*

Proof: Let $\Omega_2(R) \setminus IM(R)$ be a complete graph. So we can say that there is a vertex of $\Omega_2(R) \setminus IM(R)$ that is adjacent to every other vertex. Hence by

Proposition 3.10, $\Omega_2(R)\setminus IM(R)$ is a star graph and Theorem 3.11 implies that R has exactly two maximal co-ideals m_1 and m_2 so that $|m_i\setminus IM(R)| = 1$ for some i. Now, since for each maximal co-ideal m_i, the vertex-set $m_i\setminus IM(R)$ is a partition of $\Omega_2(R)\setminus IM(R)$, we must have $|m_i\setminus IM(R)| = 1$ for any i, because the elements of $m_i\setminus IM(R)$ are not adjacent to each other. In this case, $\Omega_2(R)\setminus IM(R)$ is in the form $K_{1,1}$.

The converse is obvious. □

3.13 Example. Let $X = \{a, b\}$ and $R = (P(X), \cup, \cap)$ be a semiring, where $P(X)$ is power set of X and $1_R = X$ and $0_R = \emptyset$. The maximal co-ideals of semiring R are as follows:

$$m_1 = \{\{a\}, X\},$$
$$m_2 = \{\{b\}, X\}.$$

Thus by Theorem 3.4, $\Omega_2(R)\setminus IM(R)$ is a complete bipartite graph with vertex-sets $V_1 = \{\{a\}\}$ and $V_2 = \{\{b\}\}$. Indeed, $\Omega_2(R)\setminus IM(R)$ forms $K_{1,1}$. Hence $\Omega_2(R)\setminus IM(R)$ is complete graph that is a star graph and a tree. Also, since $\Omega_2(R)\setminus IM(R)$ does not contain any cycle, so it is a forest and $gr(\Omega_2(R)\setminus IM(R)) = \infty$.

3.14 Theorem. *Let R be a c-semilocal semiring which is not a c-local. Then the following hold.*

(i) *If $|Co - Max(R)| = n$, then $\Omega_2(R)\setminus IM(R)$ is n-partite.*
(ii) *If $\Omega_2(R)\setminus IM(R)$ is n-partite, then $|Co - Max(R)| \leq n$. In this case, if $\Omega_2(R)\setminus IM(R)$ is not $(n-1)$-partite, then $|Co - Max(R)| = n$.*

Proof: (i) Suppose that $Co - Max(R) = \{m_1, \ldots, m_n\}$. Let $V_1 = m_1\setminus IM(R)$ and $V_i = m_i \setminus \bigcup_{j=1}^{i-1} m_j$ for $2 \leq i \leq n$. By Remark 2.5, $V_i \neq \emptyset$ for each i. Also, clearly that $\bigcup_{i=1}^{n} V_i = UM(R)\setminus IM(R)$ and for every $x, y \in V_i$, they are not adjacent in $\Omega_2(R)\setminus IM(R)$. Hence $\Omega_2(R)\setminus IM(R)$ is n-partite graph.

(ii) Assume contrary that $|Co - Max(R)| \geq n+1$. By Theorem 3.6, $\Omega_2(R)\setminus IM(R)$ has a clique with cardinality $n+1$. Thus by the Pigeon Hole Principal, two elements of this clique should belong to one part of $\Omega_2(R)\setminus IM(R)$, which is a contradiction.

Now, if $\Omega_2(R)\setminus IM(R)$ is not $(n-1)$-partite and $|Co - Max(R)| = k < n$, then by part (i), $\Omega_2(R)\setminus IM(R)$ can be a k-partite graph, a contradiction. □

3.15 Proposition. *Let R be a semiring with $|Co - Max(R)| \geq 2$. If $\Omega_2(R)\setminus IM(R)$ is complete n-partite graph, then $n = 2$.*

Proof: Let $\{m_1, m_2\} \subseteq Co - Max(R)$. By Proposition 3.2, it is clear that there exists at least one element of $m_1\setminus IM(R)$ which is adjacent to one element of $m_2\setminus IM(R)$. Also, $m_i\setminus IM(R)$ is totally disconnected for any $m_i \in Co - Max(R)$, so $m_1\setminus IM(R)$ and $m_2\setminus IM(R)$ are entirely contained in one of partitions of $\Omega_2(R)\setminus IM(R)$. This implies that $(m_1\setminus IM(R)) \cap (m_2\setminus IM(R)) = \emptyset$ and hence

$m_1 \cap m_2 \subseteq IM(R)$. Therefore we have $m_1 \cap m_2 = IM(R)$. Thus $|Co-Max(R)| = 2$ and by Theorem 3.4, $\Omega_2(R) \setminus IM(R)$ is a complete bipartite graph. □

As mentioned in the introduction, Beck conjectured that $\chi(R) = \omega(R)$ for every ring R. In the following theorem we want to establish Beck's conjecture for the graph $\Omega_2(R) \setminus IM(R)$ of c-semilocal semiring.

We recall that the *chromatic number* of the graph G, denoted by $\chi(G)$, is the minimal number of colors which can be assigned to the vertices of G in such a way that any two adjacent vertices have different colors.

3.16 Theorem. *Let R be a c-semilocal semiring with $|Co-Max(R)| = n$. Then $\chi(\Omega_2(R) \setminus IM(R)) = \omega(\Omega_2(R) \setminus IM(R)) = n$.*

Proof: Let $Co - Max(R) = \{m_1, \dots, m_n\}$. By Theorem 3.6, we know that $\omega(\Omega_2(R) \setminus IM(R)) = n$. Also, it is obvious that $\chi(G) \geq \omega(G)$ for any graph G, so $\chi(\Omega_2(R) \setminus IM(R)) \geq n$. On the other hand, $\Omega_2(R) \setminus IM(R)$ is n-partite by Theorem 3.14, thus the elements of each part can be colored by an identical color because these elements are not adjacent. Hence $\chi(\Omega_2(R) \setminus IM(R)) = n$. □

4. Diameter and radius of $\Omega(R)$

In this section, we show that $\Omega_2(R) \setminus IM(R)$ is a connected graph and $\text{diam}(\Omega_2(R) \setminus IM(R)) \leq 3$. Also, we compute the eccentricity of the vertices of $\Omega_2(R) \setminus IM(R)$.

4.1 Theorem. *Let R be a semiring. The graph $\Omega_2(R) \setminus IM(R)$ is connected with* $\text{diam}(\Omega_2(R) \setminus IM(R)) \leq 3$.

Proof: Let $x, y \in \Omega_2(R) \setminus IM(R)$ that are not adjacent. We consider two cases:

Case 1: Suppose that $x + y \notin IM(R)$. By Proposition 3.3, $F(x+y)F(a) = R$, for some $a \in \Omega_2(R) \setminus IM(R)$. This implies that $F(x)F(a) = F(y)F(a) = R$ since $F(x+y) \subseteq F(x), F(y)$. Hence $x - a - y$ is a path in $\Omega_2(R) \setminus IM(R)$ and $d(x, y) = 2$.

Case 2: Suppose that $x + y \in IM(R)$. Thus for each $m \in Co - Max(R)$, we have $x \in m$ or $y \in m$. Since $x \notin IM(R)$, by Proposition 3.3, there exists $a \in \Omega_2(R) \setminus IM(R)$ such that x is adjacent to a in $\Omega_2(R) \setminus IM(R)$. Hence if $x \in m$ for maximal co-ideal m, then $a \notin m$. Now, there exists $n \in Co - Max(R)$ in which $y \notin n$, since $y \notin IM(R)$. This implies that $x \in n$ and $a \notin n$. As n is prime co-ideal, we have $a + y \notin IM(R)$. So by Case 1, $d(a, y) \leq 2$ and hence $d(x, y) \leq 3$. □

We recall that for a graph G, the *eccentricity* of a vertex x is $e(x) = Max\{d(y, x); y \in V(G)\}$. A vertex x with smallest eccentricity is called a *center* of G and its eccentricity is called the *radius* of G and is denoted by $rad(G)$.

4.2 Proposition. *Let R be a c-semilocal semiring with $|Co - Max(R)| \geq 3$. If $x \in \Omega_2(R) \setminus IM(R)$ belongs to at least two maximal co-ideals, then $e(x) = 3$.*

Proof: Suppose that for $x \in \Omega_2(R)\setminus IM(R)$ there exist at least two maximal co-ideals m_i and m_j so that x is contained in $m_i \cap m_j$. By Theorem 4.1, $d(x,y) \leq 3$ for any $y \in \Omega_2(R)\setminus IM(R)$. Now to complete the proof, it suffices to show that, there is an element y in $\Omega_2(R)\setminus IM(R)$ such that $d(x,y)=3$. Let $y \in \bigcap_{\substack{k=1\\k\neq i}}^{n} m_k \setminus IM(R)$. Clearly that $d(x,y)\neq 1$, since $x,y \in m_j$. If $d(x,y)=2$, then $x-a-y$ is a path for some $a \in \Omega_2(R)\setminus IM(R)$. Now, as $x \in m_i \cap m_j$, thus $a \notin m_i, m_j$. Also, $y \in \bigcap_{\substack{k=1\\k\neq i}}^{n} m_k \setminus IM(R)$ implies that $a \notin m_k$, for $1 \leq k \leq n$ and $k \neq i$. Indeed, this implies that $a \notin m$ for any $m \in Co-Max(R)$, that is impossible. So we can conclude that $d(x,y)=3$ and hence $e(x)=3$. □

4.3 Corollary. *Let R be a c-semilocal semiring with $|Co-Max(R)| \geq 3$. Then* $\operatorname{diam}(\Omega_2(R)\setminus IM(R))=3$.

Proof: We know that $\operatorname{diam}(\Omega_2(R)\setminus IM(R)) \leq 3$, by Theorem 4.1. On the other hand, $|Co-Max(R)| \geq 3$ implies that there is an element x in $\Omega_2(R)\setminus IM(R)$ that belongs to at least two maximal co-ideals. Now, the proof is immediate from Proposition 4.2. □

4.4 Proposition. *Let R be a semiring with $|Co-Max(R)|=2$. If $|m_i \setminus IM(R)| \geq 2$ for some i, then* $\operatorname{diam}(\Omega_2(R)\setminus IM(R))=2$.

Proof: Assume that $|Co-Max(R)|=2$. By Theorem 3.4, $\Omega_2(R)\setminus IM(R)$ is complete bipartite graph and thus $\operatorname{diam}(\Omega_2(R)\setminus IM(R)) \leq 2$. On the other hand, $\operatorname{diam}(\Omega_2(R)\setminus IM(R)) \neq 1$ because $|m_i \setminus IM(R)| \geq 2$ for some i. Hence $\operatorname{diam}(\Omega_2(R)\setminus IM(R))=2$. □

4.5 Theorem. *Let R be a semiring. If* $\operatorname{diam}(\Omega_2(R)\setminus IM(R))=2$, *then R has an infinite number of maximal co-ideals or $|Co-Max(R)|=2$ such that $|m_i \setminus IM(R)| \geq 2$ for some $i=1,2$.*

Proof: Assume that $\operatorname{diam}(\Omega_2(R)\setminus IM(R))=2$ and $|Co-Max(R)|$ is finite. If $n \geq 3$, then by Corollary 4.3, $\operatorname{diam}(\Omega_2(R)\setminus IM(R))=3$, which is a contradiction. Thus we must have $|Co-Max(R)|=2$. Now, if $|m_i \setminus IM(R)|=1$ for each i, then $\operatorname{diam}(\Omega_2(R)\setminus IM(R))=1$ because $\Omega_2(R)\setminus IM(R)$ is a complete bipartite graph, this is a contradiction. Hence $|m_i \setminus IM(R)| \geq 2$ for some i. □

4.6 Theorem. *Let R be a c-semilocal semiring with $|Co-Max(R)|=n \geq 2$. If $\Omega_2(R)\setminus IM(R)$ is not a star graph, then we have:*

$$e(x)=\begin{cases} 2 & \text{if } x \in m_i \setminus \bigcup_{\substack{j=1\\j\neq i}}^{n} m_j \\ 3 & \text{otherwise.}\end{cases}$$

Proof: First, we claim that for any $a \in \Omega_2(R)\setminus IM(R)$, $e(a)\neq 1$. Suppose that there is an element x of $\Omega_2(R)\setminus IM(R)$ such that $e(x)=1$. This means that x is adjacent to any vertex of $\Omega_2(R)\setminus IM(R)$ and so $\Omega_2(R)\setminus IM(R)$ is a star graph by Proposition 3.10, which is a contradiction. Now, suppose that $x \in m_i \setminus \bigcup_{\substack{j=1\\j\neq i}}^{n} m_j$.

For any $y \in \bigcup_{\substack{j=1 \\ j \neq i}}^{n} m_j \setminus m_i$, if $F(x)F(y) \neq R$, then $F(x)F(y) \subseteq m_k$ for some $m_k \in Co - Max(R)$. Hence $x, y \in m_k$, that is a contradiction. Therefore, in this case $d(x, y) = 1$. But, if $y \in m_i \setminus IM(R)$ and $y \neq x$, then by proof of Theorem 4.1, $d(x, y) \leq 2$ since $x + y \notin IM(R)$. Clearly x and y are not adjacent and so $d(x, y) = 2$. According to the assumption, since $\Omega_2(R) \setminus IM(R)$ is not star graph thus by Theorem 3.11 ((4) $\Rightarrow$ (3)) $|Co - Max(R)| \geq 2$ and $|m \setminus IM(R)| \geq 2$ for each $m \in Co - Max(R)$. Hence $e(x) = 2$ for any $x \in m_i \setminus \bigcup_{\substack{j=1 \\ j \neq i}}^{n} m_j$.

Now, suppose that $x \notin m_i \setminus \bigcup_{\substack{j=1 \\ j \neq i}}^{n} m_j$ for any maximal co-ideal m_i. Hence there are at least two maximal co-ideals m_k and m_j so that x is contained in $m_k \cap m_j$. This implies that $|Co - Max(R)| \geq 3$, thus by Proposition 4.2 we have $e(x) = 3$. □

4.7 Corollary. *Let R be a c-semilocal semiring with $|Co - Max(R)| = n \geq 2$. If $\Omega_2(R) \setminus IM(R)$ is not a star graph, then the elements of $m_i \setminus \bigcup_{\substack{j=1 \\ j \neq i}}^{n} m_j$ are center of $\Omega_2(R) \setminus IM(R)$ for each $m_i \in Co - Max(R)$ and $rad(\Omega_2(R) \setminus IM(R)) = 2$.*

PROOF: This is an immediate consequence of Theorem 4.6. □

4.8 Proposition. *Let R be a semiring with $|Co - Max(R)| = 2$. Then $rad(\Omega_2(R) \setminus IM(R)) = 1$ or 2.*

PROOF: We know by Theorem 3.4, $\Omega_2(R) \setminus IM(R)$ is a complete bipartite graph when $|Co - Max(R)| = 2$. Now, if $\Omega_2(R) \setminus IM(R)$ is a star graph, clearly $rad(\Omega_2(R) \setminus IM(R)) = 1$. Otherwise, $rad(\Omega_2(R) \setminus IM(R)) = 2$ and all elements of $\Omega_2(R) \setminus IM(R)$ are center. □

5. The relations between $\Omega(R)$ and $\Gamma(R)$

In this section, we will investigate the relations between the zero-divisor graph $\Gamma(R)$ and $\Omega(R)$. We show that $\Gamma(R)$ is a subgraph of the $\Omega(R)$. Also, we determine a family of commutative semirings whose zero-divisor graph $\Gamma(R)$ and $\Omega_2(R)$ are isomorphic.

We recall that an *isomorphism* from a simple graph G to a simple graph H is a bijection $f : V(G) \to V(H)$ such that x and y are adjacent in G if and only if $f(x)$ and $f(y)$ are adjacent in H. We say G is isomorphic to H, if there is an isomorphism from G to H, denoted by $G \cong H$.

5.1 Theorem. *The zero-divisor graph $\Gamma(R)$ is a subgraph of the graph $\Omega(R)$.*

PROOF: Suppose that x and y are two distinct adjacent vertices in $\Gamma(R)$. Thus $xy = 0$ and this implies $F(x)F(y) = R$, since $0 = xy \in F(x)F(y)$. Hence x and y are adjacent in $\Omega(R)$. Now, since the vertex-set of zero-divisor graph is $Z(R)^*$, thus we can conclude that $\Gamma(R)$ is a subgraph of $\Omega(R)$. □

5.2 Theorem. *Let R be a multiplicatively idempotent and zero-sumfree semiring. Then the zero-divisor graph $\Gamma(R)$ is an induced subgraph of the graph $\Omega(R)$.*

Proof: By Theorem 5.1, $\Gamma(R)$ is a subgraph of $\Omega(R)$. Thus it is enough to show that if $x, y \in Z(R)^*$ and they are adjacent in $\Omega(R)$, then x and y are adjacent in $\Gamma(R)$. Assume that $x, y \in Z(R)^*$ and $F(x)F(y) = R$. So we have $(x^n + r)(y^m + s) + k = 0$ for some positive integers n, m and $r, s, k \in R$. Since R is a multiplicatively idempotent, then we have $xy + a = 0$ for some $a \in R$. Hence $xy = 0$ because R is a zero-sumfree semiring. This implies x and y are adjacent in $\Gamma(R)$. □

Note that if $UM(R) = Z(R)^*$, then $\Gamma(R)$ is a spanning subgraph of $\Omega_2(R)$ by Theorem 5.1. Thus, if R is a multiplicatively idempotent and zero-sumfree semiring, then we have the following result:

5.3 Corollary. *Let R be a multiplicatively idempotent and zero-sumfree semiring. If $Z(R)^* = UM(R)$, then the zero-divisor graph $\Gamma(R)$ and $\Omega_2(R)$ are isomorphic. In particular, if $Z(R)^* = UM(R) \setminus IM(R)$, then $\Gamma(R)$ and $\Omega_2(R) \setminus IM(R)$ are isomorphic.*

Proof: This is an immediate consequence of Theorems 5.1 and 5.2. □

To this end, we give an example that clarifies the previous results:

5.4 Example. Let $S = \{0, 1, a\}$ and $R = (S \times S, +, \cdot)$ be a semiring as defined in Example 3.5. We know that R is a multiplicatively idempotent. For this semiring, the vertex-set of $\Gamma(R)$ is

$$Z(R)^* = \{(0,1), (1,0), (0,a), (a,0)\}$$

and the vertex-set of $\Omega_2(R)$ is $UM(R) = R \setminus \{(0,0)\}$. Clearly $\Gamma(R)$ is an induced subgraph of $\Omega(R)$ and $\Omega_2(R)$. On the other hand, $(0,0)$ is only zero-sum of R, thus R is zero-sumfree semiring. We see that $UM(R) \setminus IM(R) = Z(R)^*$, so we can conclude that $\Gamma(R)$ and $\Omega_2(R) \setminus IM(R)$ are isomorphic by Corollary 5.3.

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