An existence and approximation theorem for solutions of degenerate quasilinear elliptic equations

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Abstract. The main result establishes that a weak solution of degenerate quasilinear elliptic equations can be approximated by a sequence of solutions for non-degenerate quasilinear elliptic equations.

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1. Introduction

Let L be a degenerate elliptic operator in divergence form

$$(1) \qquad Lu(x) = -\sum_{i,j=1}^{n} D_j(a_{ij}(x)\, D_i u(x)), \quad D_j = \frac{\partial}{\partial x_j},$$

where the coefficients a_{ij} are measurable, real-valued functions whose coefficient matrix $A = (a_{ij})$ is symmetric and satisfies the degenerate ellipticity condition

$$(2) \qquad \lambda|\xi|^2\omega(x) \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \;\leq\; \Lambda|\xi|^2\omega(x)$$

for all $\xi \in \mathbb{R}^n$ and almost every x of a bounded open set $\Omega \subset \mathbb{R}^n$, ω is a weight function, λ and Λ are positive constants.

The main purpose of this paper (see Theorem 1) is to establish that a weak solution $u \in W_0^{1,2}(\Omega,\omega)$ for the quasilinear Dirichlet problem

$$(\text{P}) \quad \begin{cases} Lu + g(x,u,\nabla u)\,\omega = f_0 - \displaystyle\sum_{j=1}^{n} D_j f_j & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

can be approximated by a sequence of solutions of non-degenerate quasilinear elliptic equations.

By a *weight*, we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $\omega(x) > 0$ for a.e. $x \in \mathbb{R}^n$. Every weight ω gives rise to a measure on the measurable

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subsets on $\mathbb{R}^n$ through integration. This measure will be denoted by μ. Thus, $\mu(E) = \int_E \omega(x)\,\mathrm{d}x$ for measurable sets $E \subset \mathbb{R}^n$.

In general, the Sobolev spaces $\mathrm{W}^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [2], [3], [5], [7], [8] and [12]). In various applications, we can meet boundary value problems for elliptic equations whose ellipticity is disturbed in the sense that some degeneration or singularity appears. There are several very concrete problems from practice which lead to such differential equations, e.g. from glaceology, non-Newtonian fluid mechanics, flows through porous media, differential geometry, celestial mechanics, climatology, petroleum extraction and reaction-diffusion problems (see some examples of applications of degenerate elliptic equations in [1], [4] and [13]).

A class of weights, which is particularly well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [10]). These classes have found many useful applications in harmonic analysis (see [11]). Another reason for studying A_p-weights is the fact that powers of the distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [9]). There are, in fact, many interesting examples of weights (see [8] for p-admissible weights).

The following lemma can be proved in exactly the same way as Lemma 2.1 in [6] (see also, Lemma 3.1 and Lemma 4.13 in [2]). Our lemma provides a general approximation theorem for A_p weights, $1 \le p < \infty$, by means of weights which are bounded away from 0 and infinity and whose A_p-constants depend only on the A_p-constant of ω. Lemma 1 is the key point for Theorem 2, and the crucial point consists of showing that a weak limit of a sequence of solutions of approximate problems is in fact a solution of the original problem.

In [2] the author studied the existence and uniqueness of solution and demonstrated an approximation theorem in the linear case (i.e., when $g \equiv 0$), and in [3] the author studied the Dirichlet problem with the operator

$$Lu(x) = -\mathrm{div}[\omega(x)\,\mathcal{A}(x, u, \nabla u)] + \sum_{j=1}^{n} b_j(x) D_j u(x) + \alpha\, g(x),$$

where the function $\mathcal{A}\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the Carathéodory conditions, growth condition, monotonicity condition and ellipticity condition.

Lemma 1. *Let $\alpha, \beta > 1$ be given and let $\omega \in A_p$, $1 \le p < \infty$, with A_p-constant $C(\omega, p)$ and let $a_{ij} = a_{ji}$ be measurable, real-valued functions satisfying*

$$\lambda\,\omega(x)|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \Lambda\,\omega(x)\,|\xi|^2, \tag{3}$$

for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$. Then there exist weights $\omega_{\alpha\beta} \ge 0$ a.e. and measurable real-valued functions $a_{ij}^{\alpha\beta}$ such that the following conditions are met.

(i) $c_1(1/\beta) \leq \omega_{\alpha\beta} \leq c_2\alpha$ *in* Ω, *where* c_1 *and* c_2 *depend only on* ω *and* Ω.
(ii) *There exist weights* $\tilde{\omega}_1$ *and* $\tilde{\omega}_2$ *such that* $\tilde{\omega}_1 \leq \omega_{\alpha\beta} \leq \tilde{\omega}_2$, *where* $\tilde{\omega}_i \in A_p$ *and* $C(\tilde{\omega}_i, p)$ *depends only on* $C(\omega, p)$, $i = 1, 2$.
(iii) $\omega_{\alpha\beta} \in A_p$, *with constant* $C(\omega_{\alpha\beta}, p)$ *depending only on* $C(\omega, p)$ *uniformly on* α *and* β.
(iv) *There exists a closed set* $F_{\alpha\beta}$ *such that* $\omega_{\alpha\beta} \equiv \omega$ *in* $F_{\alpha\beta}$ *and* $\omega_{\alpha\beta} \sim \tilde{\omega}_1 \sim \tilde{\omega}_2$ *in* $F_{\alpha\beta}$ *with equivalence constants depending on* α *and* β *(i.e., there are positive constants* $c_{\alpha\beta}$ *and* $C_{\alpha\beta}$ *such that* $c_{\alpha\beta}\,\tilde{\omega}_i \leq \omega_{\alpha\beta} \leq C_{\alpha\beta}\,\tilde{\omega}_i$, $i = 1, 2$). *Moreover,* $F_{\alpha\beta} \subset F_{\alpha'\beta'}$ *if* $\alpha \leq \alpha'$, $\beta \leq \beta'$, *and the complement of* $\bigcup_{\alpha,\beta\geq 1} F_{\alpha\beta}$ *has zero measure.*
(v) $\omega_{\alpha\beta} \to \omega$ *a.e. in* $\mathbb{R}^n$ *as* $\alpha, \beta \to \infty$.
(vi) $\lambda\omega_{\alpha\beta}(x)|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}^{\alpha\beta}(x)\,\xi_i\xi_j \leq \Lambda\omega_{\alpha\beta}(x)|\xi|^2$ *for every* $\xi \in \mathbb{R}$ *and a.e.* $x \in \Omega$, *and* $a_{ij}^{\alpha\beta}(x) = a_{ji}^{\alpha\beta}(x)$.

Proof: See [2], Lemma 3.1 or Lemma 4.13. □

The following theorem will be proved in Section 3.

Theorem 1. *Suppose that*

(H1) *the function* $g\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *satisfies:*
  (i) $x \mapsto g(x, s, \xi)$ *is measurable on* Ω *for all* $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$,
  (ii) *there exists a constant* $C_g > 0$ *such that*

$$|g(x, s_1, \xi_1) - g(x, s_2, \xi_2)| \leq C_g(|s_1 - s_2| + |\xi_1 - \xi_2|)$$

  for all $s_1, s_2 \in \mathbb{R}$, $\xi_1, \xi_2 \in \mathbb{R}^n$ *and almost all* $x \in \Omega$,
  (iii) $g(x, 0, 0) = 0$ *for almost all* $x \in \Omega$;
(H2) $\omega \in A_2$;
(H3) $f_j/\omega \in L^2(\Omega, \omega)$, $j = 0, 1, \ldots, n$;
(H4) *the constant* $\gamma = \lambda - 2C_g\,(C_\Omega^2 + 1) > 0$ *(with* C_Ω *as in Theorem 2).*

Then the problem (P) *has a unique solution* $u \in W_0^{1,2}(\Omega, \omega)$ *and there exists a constant* $C > 0$ *such that*

$$\|u\|_{W_0^{1,2}(\Omega,\omega)} \leq C\left(\sum_{j=0}^{n}\left\|\frac{f_j}{\omega}\right\|_{L^2(\Omega,\omega)}\right). \tag{4}$$

Moreover, u *is the weak limit in* $W_0^{1,2}(\Omega, \tilde{\omega}_1)$ *of a sequence of solutions* $u_m \in W_0^{1,2}(\Omega, \omega_m)$ *of the problems*

$$(\mathrm{P_m})\quad \begin{cases} L_m u_m + g(x, u_m, \nabla u_m)\,\omega_m = f_{0m} + \displaystyle\sum_{j=1}^{n} D_j f_{jm} & \text{in } \Omega, \\ u_m = 0 \quad \text{on } \partial\Omega, \end{cases}$$

with $L_m u_m = -\sum_{i,j=1}^n D_j(a_{ij}^{mm} D_i u_m)$, $f_{jm} = f_j(\omega_m/\omega)^{1/2}$ *and* $\omega_m = \omega_{mm}$ *(where* ω_{mm}, a_{ij}^{mm} *and* $\tilde{\omega}_1$ *are as Lemma 1).*

2. Definitions and basic results

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega(x) < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a constant $C = C(p, \omega)$ such that

$$\left(\frac{1}{|B|}\int_B \omega(x)\,\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)\,\mathrm{d}x\right)^{p-1} \le C$$

for all balls $B \subset \mathbb{R}^n$, where $|\cdot|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \le p$, then $A_q \subset A_p$ (see [7], [8] or [12] for more information about A_p-weights). The weight ω satisfies the doubling condition if there exists a positive constant C such that $\mu(B(x; 2\,r)) \le C\mu(B(x; r))$ for every ball $B = B(x; r) \subset \mathbb{R}^n$, where $\mu(B) = \int_B \omega(x)\,\mathrm{d}x$. If $\omega \in A_p$, then μ is doubling (see Corollary 15.7 in [8]).

As an example of A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is in A_p if and only if $-n < \alpha < n(p-1)$ (see Corollary 4.4, Chapter IX in [11]).

If $\omega \in A_p$, then $(|E|/|B|)^p \le C\mu(E)/\mu(B)$ whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B (see 15.5 *strong doubling property* in [8]). Therefore, $\mu(E) = 0$ if and only if $|E| = 0$; so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e.

Definition 1. Let ω be a weight, and let $\Omega \subset \mathbb{R}^n$ be open. For $0 < p < \infty$ we define $L^p(\Omega, \omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x)\,\mathrm{d}x\right)^{1/p} < \infty.$$

If $\omega \in A_p$, $1 < p < \infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega, \omega) \subset L^1_{\mathrm{loc}}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [12]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega, \omega)$.

Definition 2. Let $\Omega \subset \mathbb{R}^n$ be open, and $\omega \in A_2$. We define the weighted Sobolev space $W^{1,2}(\Omega, \omega)$ as the set of functions $u \in L^2(\Omega, \omega)$ with weak derivatives $D_j u \in L^2(\Omega, \omega)$ for $j = 1, 2, \ldots, n$. The norm of u in $W^{1,2}(\Omega, \omega)$ is defined by

$$\|u\|_{W^{1,2}(\Omega,\omega)} = \left(\int_\Omega |u(x)|^2\,\omega(x)\,\mathrm{d}x + \int_\Omega |\nabla u(x)|^2\,\omega(x)\,\mathrm{d}x\right)^{1/2}. \tag{5}$$

We also define $W^{1,2}_0(\Omega, \omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (5).

If $\omega \in A_2$, then $W^{1,2}(\Omega, \omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm (5) (see Theorem 2.1.4 in [12]). The spaces $W^{1,2}(\Omega, \omega)$ and $W^{1,2}_0(\Omega, \omega)$ are Banach spaces.

It is evident that the weight function ω which satisfies $0 < c_1 \le \omega(x) \le c_2$ for $x \in \Omega$ (c_1 and c_2 positive constants), gives nothing new (the space $\mathrm{W}^{1,2}_0(\Omega, \omega)$

is then identical with the classical Sobolev space $\mathrm{W}_0^{1,2}(\Omega)$). Consequently, we shall be interested above in all such weight functions ω which either vanish in somewhere $\Omega \cup \partial\Omega$ or increase to infinity (or both).

The dual space of $W_0^{1,2}(\Omega,\omega)$ is the space

$$[W_0^{1,2}(\Omega,\omega)]^* = W^{-1,2}(\Omega,\omega) = \Big\{T = f_0 - \operatorname{div} F\colon F = (f_1,\dots,f_n),\ \frac{f_j}{\omega} \in L^2(\Omega,\omega),\ j = 0,\dots,n\Big\},$$

and $\|\cdot\|_*$ denotes the norm in $[W_0^{1,2}(\Omega,\omega)]^*$.

Definition 3. We say that an element $u \in W_0^{1,2}(\Omega,\omega)$ is weak solution of problem (P) if

$$\sum_{i,j=1}^n \int_\Omega a_{ij} D_i u\, D_j\varphi \,\mathrm{d}x + \int_\Omega g(x,u,\nabla u)\,\varphi\,\omega\,\mathrm{d}x = \int_\Omega f_0\,\varphi\,\mathrm{d}x + \sum_{j=1}^n \int_\Omega f_j\, D_j\varphi\,\mathrm{d}x$$

for every $\varphi \in W_0^{1,2}(\Omega,\omega)$.

Remark 1. (a) If $A = (a_{ij})$, we will use the notation

$$\sum_{i,j=1}^n a_{ij} D_i u\, D_j\varphi = (A\nabla u).\nabla\varphi,$$

where the dot denotes here the Euclidian scalar product in $\mathbb{R}^n$.

(b) Since the matrix $A = (a_{ij})$ is symmetric, we have

$$|(A\nabla u).\nabla\varphi| \le [(A\nabla u).\nabla u]^{1/2}\,[(A\nabla\varphi).\nabla\varphi]^{1/2}.$$

Theorem 2 (The weighted Sobolev inequality)**.** *Let Ω be an open bounded set in $\mathbb{R}^n$ and $\omega \in A_2$. There exist positive constants C_Ω and δ such that for all $u \in W_0^{1,2}(\Omega,\omega)$ and all θ satisfying $1 \le \theta \le n/(n-1) + \delta$,*

$$\|u\|_{L^{2\theta}(\Omega,\omega)} \le C_\Omega \|\nabla u\|_{L^2(\Omega,\omega)}, \tag{6}$$

where C_Ω depends only on n, p, the A_p-constant $C(p,\omega)$ of ω and the diameter of Ω.

Proof: Its suffices to prove the inequality for functions $u \in C_0^\infty(\Omega)$ (see Theorem 1.3 in [5]). To extend the estimates (6) to arbitrary $u \in W_0^{1,2}(\Omega,\omega)$, we let $\{u_m\}$ be a sequence of $C_0^\infty(\Omega)$ functions tending to u in $W_0^{1,2}(\Omega,\omega)$. Applying the estimates (6) to differences $u_{m_1} - u_{m_2}$, we see that $\{u_m\}$ will be a Cauchy sequence in $L^2(\Omega,\omega)$. Consequently the limit function u will lie in the desired spaces and satisfy (6). □

Remark 2. By Theorem 2 (with $\theta = 1$) we have

$$\begin{aligned} \|u\|_{W_0^{1,2}(\Omega,\omega)} &= \left(\int_\Omega |u|^2\, \omega \,\mathrm{d}x + \int_\Omega |\nabla u|^2\, \omega \,\mathrm{d}x \right)^{1/2} \\ &\leq \left((C_\Omega^2 + 1) \int_\Omega |\nabla u|^2 \omega \,\mathrm{d}x \right)^{1/2} \\ &= C_1 \, \|\nabla u\|_{L^2(\Omega,\omega)}, \end{aligned} \tag{7}$$

where $C_1 = \sqrt{C_\Omega^2 + 1}$.

3. Proof of Theorem 1

Part 1. Existence and uniqueness of solution.

The basic idea is to reduce the problem (P) to an operator equation $\mathcal{A}u = T$ and apply the theorem below.

Theorem 3. *Let $\mathcal{A}\colon X \to X^*$ be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space X. Then the following assertions hold:*

(a) *for each $T \in X^*$ the equation $\mathcal{A}u = T$ has a solution $u \in X$;*
(b) *if the operator $\mathcal{A}$ is strictly monotone, then equation $\mathcal{A}u = T$ is uniquely solvable in X.*

Proof: See Theorem 26.A in [14]. □

To prove Theorem 1, we define $B\colon W_0^{1,2}(\Omega,\omega) \times W_0^{1,2}(\Omega,\omega) \to \mathbb{R}$ and $T\colon W_0^{1,2}(\Omega,\omega) \to \mathbb{R}$ by

$$\begin{aligned} B(u,\varphi) &= \sum_{i,j=1}^n \int_\Omega a_{ij} D_i u \, D_j \varphi \,\mathrm{d}x + \int_\Omega g(x,u,\nabla u)\, \varphi \, \omega \,\mathrm{d}x \\ &= \int_\Omega (A\nabla u).\nabla\varphi \,\mathrm{d}x + \int_\Omega g(x,u,\nabla u)\, \varphi\, \omega \,\mathrm{d}x; \\ T(\varphi) &= \int_\Omega f_0 \, \varphi \,\mathrm{d}x + \sum_{j=1}^n \int_\Omega f_j \, D_j \varphi \,\mathrm{d}x. \end{aligned}$$

Step 1. By (H1) (ii) and (iii) we have $|g(x,s,\xi)| \leq C_g\,(|s| + |\xi|)$. Using (2) and Remark 1 (b), we obtain

$$\begin{aligned} |B(u,\varphi)| &\leq \int_\Omega |(A\nabla u).\nabla\varphi| \,\mathrm{d}x + \int_\Omega |g(x,u,\nabla u)|\, |\varphi|\, \omega \,\mathrm{d}x \\ &\leq \int_\Omega ((A\nabla u).\nabla u)^{1/2} ((A\nabla\varphi).\nabla\varphi)^{1/2} \,\mathrm{d}x + C_g \int_\Omega (|u| + |\nabla u|)\, |\varphi|\, \omega \,\mathrm{d}x \\ &\leq \left(\int_\Omega (A\nabla u).\nabla u \,\mathrm{d}x \right)^{1/2} \left(\int_\Omega (A\nabla\varphi).\nabla\varphi \,\mathrm{d}x \right)^{1/2} \mathrm{d}x \end{aligned} \tag{8}$$

$$
\begin{aligned}
&+ C_g\left[\left(\int_\Omega |u|^2\,\omega\,\mathrm{d}x\right)^{1/2} + \left(\int_\Omega |\nabla u|^2\,\omega\,\mathrm{d}x\right)^{1/2}\right]\left(\int_\Omega |\varphi|^2\,\omega\,\mathrm{d}x\right)^{1/2}\\
&\le \left(\Lambda\int_\Omega |\nabla u|^2\omega\,\mathrm{d}x\right)^{1/2}\left(\Lambda\int_\Omega |\nabla \varphi|^2\omega\,\mathrm{d}x\right)^{1/2}\\
&\quad + 2C_g\|u\|_{W_0^{1,2}(\Omega,\omega)}\|\varphi\|_{W_0^{1,2}(\Omega,\omega)}\\
&\le (\Lambda + 2C_g)\|u\|_{W_0^{1,2}(\Omega,\omega)}\|\varphi\|_{W_0^{1,2}(\Omega,\omega)},
\end{aligned}
$$

and by (H3)

$$
\begin{aligned}
|T(\varphi)| &\le \int_\Omega \frac{|f_0|}{\omega}|\varphi|\,\omega\,\mathrm{d}x + \sum_{j=1}^n \int_\Omega \frac{|f_j|}{\omega}\,|D_j\varphi|\,\omega\,\mathrm{d}x\\
&\le \left(\sum_{j=0}^n \|f_j/\omega\|_{L^2(\Omega,\omega)}\right)\|\varphi\|_{W_0^{1,2}(\Omega,\omega)}.
\end{aligned}
\tag{9}
$$

Since $B(u,.)$ is linear for each $u \in W_0^{1,2}(\Omega,\omega)$, there is a linear continuous functional on $W_0^{1,2}(\Omega,\omega)$ denoted by $\mathcal{A}u$ such that $\langle \mathcal{A}u,\varphi\rangle = B(u,\varphi)$ for all $\varphi \in W_0^{1,2}(\Omega,\omega)$ (where $\langle f,x\rangle$ denotes the value of the functional f at the point x). Moreover, by (8) we have

$$\|\mathcal{A}u\|_* \le (\Lambda + 2C_g)\|u\|_{W_0^{1,2}(\Omega,\omega)}.$$

Hence, we obtain the operator

$$
\begin{aligned}
\mathcal{A}\colon W_0^{1,2}(\Omega,\omega) &\to [W_0^{1,2}(\Omega,\omega)]^*\\
u &\mapsto \mathcal{A}u.
\end{aligned}
$$

Consequently, problem (P) is equivalent to the operator equation

$$u \in W_0^{1,2}(\Omega,\omega)\colon\ \mathcal{A}u = T.$$

Step 2. The operator $\mathcal{A}$ is strictly monotone and coercive.

In fact, if $u_1, u_2 \in W_0^{1,2}(\Omega,\omega)$ we have, by (2) and Remark 2,

$$
\begin{aligned}
\langle \mathcal{A}u_1 - \mathcal{A}u_2, u_1 - u_2\rangle &= B(u_1, u_1-u_2) - B(u_2, u_1-u_2)\\
&= \int_\Omega (A\nabla(u_1-u_2)).\nabla(u_1-u_2)\,\mathrm{d}x\\
&\quad + \int_\Omega (g(x,u_1,\nabla u_1) - g(x,u_2,\nabla u_2))(u_1-u_2)\,\omega\,\mathrm{d}x\\
&\ge \lambda\int_\Omega |\nabla(u_1-u_2)|^2\,\omega\,\mathrm{d}x - C_g\int_\Omega |u_1-u_2|^2\,\omega\,\mathrm{d}x\\
&\quad - C_g\int_\Omega |u_1-u_2||\nabla u_1 - \nabla u_2|\,\omega\,\mathrm{d}x
\end{aligned}
$$

$$\begin{aligned}
&\geq \frac{\lambda}{C_1^2}\|u_1-u_2\|^2_{W_0^{1,2}(\Omega,\omega)} - 2C_g\|u_1-u_2\|^2_{W_0^{1,2}(\Omega,\omega)}\\
&= \beta\|u_1-u_2\|^2_{W_0^{1,2}(\Omega,\omega)},
\end{aligned}$$

where $\beta = \lambda/C_1^2 - 2C_g > 0$ (by (H4)). Therefore, the operator $\mathcal{A}$ is strongly monotone, and this implies that $\mathcal{A}$ is strictly monotone. Moreover, if $u \in W_0^{1,2}(\Omega,\omega)$ we have

$$\begin{aligned}
\langle \mathcal{A}u, u\rangle = B(u,u) &= \int_\Omega (A\nabla u).\nabla u\,\mathrm{d}x + \int_\Omega g(x,u,\nabla u)\,u\,\omega\,\mathrm{d}x\\
&\geq \lambda\int_\Omega |\nabla u|^2\,\omega\,\mathrm{d}x - C_g\int_\Omega |u|^2\,\omega\,\mathrm{d}x - C_g\int_\Omega |u|\,|\nabla u|\,\omega\,\mathrm{d}x\\
&\geq \frac{\lambda}{C_1^2}\|u\|^2_{W_0^{1,2}(\Omega,\omega)} - 2C_g\|u\|^2_{W_0^{1,2}(\Omega,\omega)}\\
&= \beta\,\|u\|^2_{W_0^{1,2}(\Omega,\omega)}.
\end{aligned}$$

Hence, $\langle \mathcal{A}u, u\rangle/\|u\|_{W_0^{1,2}(\Omega,\omega)} \to \infty$, as $\|u\|_{W_0^{1,2}(\Omega,\omega)} \to \infty$, that is, $\mathcal{A}$ is coercive.

Step 3. We need to show that the operator $\mathcal{A}$ is continuous. Let $u_m \to u$ in $W_0^{1,2}(\Omega,\omega)$. Then, by Remark 1 (b), (2) and (H1), we obtain

$$\begin{aligned}
&|B(u_m,\varphi) - B(u,\varphi)|\\
&\quad\leq \int_\Omega |(A\nabla(u_m-u)).\nabla\varphi|\,\mathrm{d}x + \int_\Omega |g(x,u_m,\nabla u_m) - g(x,u,\nabla u)|\,|\varphi|\,\omega\,\mathrm{d}x\\
&\quad\leq \Lambda\bigg(\int_\Omega |\nabla(u_m-u)|^2\,\omega\,\mathrm{d}x\bigg)^{1/2}\bigg(\int_\Omega |\nabla\varphi|^2\omega\,\mathrm{d}x\bigg)^{1/2}\\
&\qquad + C_g\bigg(\int_\Omega |u_m-u|\,|\varphi|\,\omega\,\mathrm{d}x + \int_\Omega |\nabla(u_m-u)|\,|\varphi|\,\omega\,\mathrm{d}x\bigg)\\
&\quad\leq (\Lambda + 2C_g)\,\|u_m-u\|_{W_0^{1,2}(\Omega,\omega)}\|\varphi\|_{W_0^{1,2}(\Omega,\omega)}
\end{aligned}$$

for all $\varphi \in W_0^{1,2}(\Omega,\omega)$. Then we obtain

$$\|\mathcal{A}u_m - \mathcal{A}u\|_* \leq (\Lambda + 2C_g)\,\|u_m-u\|_{W_0^{1,2}(\Omega,\omega)}.$$

Therefore, $\|\mathcal{A}u_m - \mathcal{A}u\|_* \to 0$ as $m \to \infty$. Hence, $\mathcal{A}$ is continuous and this implies that $\mathcal{A}$ is hemicontinuous.

By Theorem 3, the operator equation $\mathcal{A}u = T$ has unique solution $u \in W_0^{1,2}(\Omega,\omega)$ and it is the unique solution for problem (P).

Part 2. Estimate for $\|u\|_{W_0^{1,2}(\Omega,\omega)}$.

In particular, for $\varphi = u$ in Definition 3, we have

$$\begin{aligned}\sum_{i,j=1}^{n}\int_\Omega a_{ij}D_iu\,D_ju\,\mathrm{d}x &+ \int_\Omega g(x,u,\nabla u)\,u\,\omega\,\mathrm{d}x \\ &= \int_\Omega f_0\,u\,\mathrm{d}x + \sum_{j=1}^{n}\int_\Omega f_j\,D_ju\,\mathrm{d}x.\end{aligned} \tag{10}$$

(i) By (2) and Remark 2, we have

$$\sum_{i,j=1}^{n}\int_\Omega a_{ij}D_iu\,D_ju\,\mathrm{d}x \ge \lambda\int_\Omega |\nabla u|^2\omega\,\mathrm{d}x \ge \frac{\lambda}{C_1^2}\|u\|^2_{W_0^{1,2}(\Omega,\omega)},$$

and by (H3)

$$\begin{aligned}\left|\int_\Omega f_0\,u\,\mathrm{d}x\right| &\le \int_\Omega \frac{|f_0|}{\omega}|u|\,\omega\,\mathrm{d}x \\ &\le \|f_0/\omega\|_{L^2(\Omega,\omega)}\|u\|_{L^2(\Omega,\omega)} \\ &\le \|f_0/\omega\|_{L^2(\Omega,\omega)}\|u\|_{W_0^{1,2}(\Omega,\omega)},\end{aligned}$$

and analogously, for $j = 1, 2\ldots, n$,

$$\left|\int_\Omega f_jD_ju\,\mathrm{d}x\right| \le \|f_j/\omega\|_{L^2(\Omega,\omega)}\|u\|_{W_0^{1,2}(\Omega,\omega)}.$$

(ii) By (H1) (ii) and (iii) we have $|g(x,s,\xi)| \le C_g(|s| + |\xi|)$ for all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^n$. Then we obtain

$$\begin{aligned}\left|\int_\Omega g(x,u,\nabla u)\,u\,\omega\,\mathrm{d}x\right| &\le \int_\Omega |g(x,u,\nabla u)||u|\,\omega\,\mathrm{d}x \\ &\le C_g\left(\int_\Omega |u|^2\omega\,\mathrm{d}x + \int_\Omega |u|\,|\nabla u|\,\omega\,\mathrm{d}x\right) \\ &\le 2C_g\,\|u\|^2_{W_0^{1,2}(\Omega,\omega)}.\end{aligned}$$

Hence, in (10), we obtain

$$\frac{\lambda}{C_1^2}\|u\|^2_{W_0^{1,2}(\Omega,\omega)} - 2C_g\,\|u\|^2_{W_0^{1,2}(\Omega,\omega)} \le \left(\sum_{j=0}^{n}\|f_j/\omega\|_{L^2(\Omega,\omega)}\right)\|u\|_{W_0^{1,2}(\Omega,\omega)}.$$

Therefore, we have

$$\|u\|_{W_0^{1,2}(\Omega,\omega)} \le C\bigg(\sum_{j=0}^{n} \|f_j/\omega\|_{L^2(\Omega,\omega)}\bigg), \tag{11}$$

where $C = C_1^2/(\lambda - 2C_g\, C_1^2) > 0$.

Part 3. Approximation of solution.

Step 1. First, if $f_{jm} = f(\omega/\omega_m)^{-1/2}$, $j = 0, 1, \dots, n$, we note that

$$\left\|\frac{f_{jm}}{\omega_m}\right\|_{L^2(\Omega,\omega_m)} = \left\|\frac{f_j}{\omega}\right\|_{L^2(\Omega,\omega)}.$$

Then, if $u_m \in W_0^{1,2}(\Omega,\omega_m)$ is a solution of problem $(\mathrm{P_m})$ we have (by (11))

$$\begin{aligned}\|u_m\|_{W_0^{1,2}(\Omega,\omega_m)} &\le C\bigg(\sum_{j=0}^{n} \|f_{jm}/\omega_m\|_{L^2(\Omega,\omega_m)}\bigg)\\ &= C\bigg(\sum_{j=0}^{n} \|f_j/\omega\|_{L^2(\Omega,\omega)}\bigg)\\ &= C_3,\end{aligned}$$

where C_3 is independent of m. Using Lemma 1, $\tilde{\omega}_1 \le \omega_m$, we obtain

$$\|u_m\|_{W_0^{1,2}(\Omega,\tilde{\omega}_1)} \le \|u_m\|_{W_0^{1,2}(\Omega,\omega_m)} \le C_3. \tag{12}$$

Consequently, $\{u_m\}$ is a bounded sequence in $W_0^{1,2}(\Omega,\tilde{\omega}_1)$. Therefore, there is a subsequence, again denoted by $\{u_m\}$, and $\tilde{u} \in W_0^{1,2}(\Omega,\tilde{\omega}_1)$ such that

$$u_m \rightharpoonup \tilde{u} \quad \text{in } L^2(\Omega,\tilde{\omega}_1), \tag{13}$$

$$|\nabla u_m| \rightharpoonup |\nabla \tilde{u}| \quad \text{in } L^2(\Omega,\tilde{\omega}_1), \tag{14}$$

$$u_m \to \tilde{u} \quad \text{a.e. in } \Omega, \tag{15}$$

where the symbol "$\rightharpoonup$" denotes weak convergence (see Theorem 1.31 in [8]).

Step 2. We have that $\tilde{u} \in W_0^{1,2}(\Omega,\omega)$. In fact, for F_k fixed, by (13) and (14), for all $\varphi \in W_0^{1,2}(\Omega,\tilde{\omega}_1)$, we obtain

$$\int_\Omega u_m \varphi\, \tilde{\omega}_1\, \mathrm{d}x \;\to\; \int_\Omega \tilde{u}\, \varphi\, \tilde{\omega}_1\, \mathrm{d}x,$$

$$\int_\Omega D_i u_m D_i \varphi\, \tilde{\omega}_1\, \mathrm{d}x \;\to\; \int_\Omega D_i \tilde{u}\, D_i \varphi\, \tilde{\omega}_1\, \mathrm{d}x.$$

If $\psi \in W_0^{1,2}(\Omega,\omega)$, then $\varphi = \psi\, \chi_{F_k} \in W_0^{1,2}(\Omega,\tilde{\omega}_1)$ (since $\omega \sim \tilde{\omega}_1$ in F_k, i.e., there is a constant $c > 0$ such that $\tilde{\omega}_1 \le c\,\omega$ in F_k, and χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{R}^n$) and

$$\int_\Omega \varphi^2\,\tilde\omega_1\,\mathrm{d}x = \int_{F_k}\psi^2\,\tilde\omega_1\,\mathrm{d}x \le c\int_{F_k}\psi^2\omega\,\mathrm{d}x \le c\int_\Omega \psi^2\,\omega\,\mathrm{d}x < \infty,$$

$$\begin{aligned}\int_\Omega (D_i\varphi)^2\,\tilde\omega_1\,\mathrm{d}x = \int_{F_k}(D_i\psi)^2\tilde\omega_1\,\mathrm{d}x &\le c\int_{F_k}(D_i\psi)^2\omega\,\mathrm{d}x\\ &\le c\int_\Omega (D_i\psi)^2\omega\,\mathrm{d}x < \infty.\end{aligned}$$

Consequently, we get

$$\begin{aligned}\int_\Omega u_m\psi\chi_{F_k}\,\tilde\omega_1\,\mathrm{d}x &\to \int_\Omega \tilde u\,\psi\,\chi_{F_k}\,\tilde\omega_1\,\mathrm{d}x,\\ \int_\Omega D_iu_mD_i\psi\,\chi_{F_k}\,\tilde\omega_1\,\mathrm{d}x &\to \int_\Omega D_i\tilde u\,D_i\psi\,\chi_{F_k}\,\tilde\omega_1\,\mathrm{d}x\end{aligned}$$

for all $\psi \in W_0^{1,2}(\Omega,\omega)$, that is, the sequence $\{u_m\,\chi_{F_k}\}$ is weakly convergent in $W_0^{1,2}(\Omega,\omega)$. Therefore, we have

$$\begin{aligned}\|\nabla\tilde u\|^2_{L^2(F_k,\omega)} &= \int_{F_k}|\nabla\tilde u|^2\omega\,\mathrm{d}x\\ &\le \limsup_{m\to\infty}\int_{F_k}|\nabla u_m|^2\,\omega\,\mathrm{d}x,\end{aligned}$$

and for $m \ge k$ we have $\omega = \omega_m$ in F_k. Hence, by (12), we obtain

$$\begin{aligned}\|\nabla\tilde u\|^2_{L^2(F_k,\omega)} &\le \limsup_{m\to\infty}\int_{F_k}|\nabla u_m|^2\,\omega\,\mathrm{d}x\\ &= \limsup_{m\to\infty}\int_{F_k}|\nabla u_m|^2\,\omega_m\,\mathrm{d}x\\ &\le \limsup_{m\to\infty}\int_\Omega |\nabla u_m|^2\omega_m\,\mathrm{d}x \le C_3^2.\end{aligned}$$

By the monotone convergence theorem we obtain $\|\nabla\tilde u\|_{L^2(\Omega,\omega)} \le C_3$. Therefore, we have $\tilde u \in W_0^{1,2}(\Omega,\omega)$.

Step 3. We need to show that $\tilde u$ is a solution of problem (P), i.e, for every $\varphi \in W_0^{1,2}(\Omega,\omega)$ we have

$$\begin{aligned}\sum_{i,j=1}^n\int_\Omega a_{ij}D_i\tilde u\,D_j\varphi\,\mathrm{d}x &+ \int_\Omega g(x,\tilde u,\nabla\tilde u)\,\varphi\,\omega\,\mathrm{d}x\\ &= \int_\Omega f_0\,\varphi\,\mathrm{d}x + \sum_{j=1}^n\int_\Omega f_j\,D_j\varphi\,\mathrm{d}x.\end{aligned}$$

Using the fact that u_m is a solution of ($\mathrm{P_m}$), we have

$$\sum_{i,j=1}^{n}\int_\Omega a_{ij}^{mm}\,D_iu_m\,D_j\varphi\,\mathrm{d}x+\int_\Omega g(x,u_m,\nabla u_m)\,\varphi\,\omega_m\,\mathrm{d}x$$
$$=\int_\Omega f_m\,\varphi\,\mathrm{d}x+\sum_{j=1}^{n}\int_\Omega f_{jm}D_j\varphi\,\mathrm{d}x$$

for every $\varphi\in W_0^{1,2}(\Omega,\omega_m)$. Moreover, over F_k (for $m\geq k$) we have the following properties:

(i) $\omega=\omega_m$;
(ii) $f_{jm}=f_j$, $j=0,1,2\ldots,n$;
(iii) $a_{ij}^{mm}(x)=a_{ij}(x)$.

For $\varphi\in W_0^{1,2}(\Omega,\omega)$ and $k>0$ (fixed), we define $G_1,G_2\colon W_0^{1,2}(\Omega,\tilde\omega_1)\to\mathbb{R}$ by

$$\begin{aligned}G_1(u)&=\sum_{i,j=1}^{n}\int_\Omega a_{ij}\,D_iu\,D_j\varphi\,\chi_{F_k}(x)\,\mathrm{d}x,\\ G_2(u)&=\int_\Omega g(x,u,\nabla u)\,\varphi\,\omega\,\chi_{F_k}(x)\,\mathrm{d}x.\end{aligned}$$

(a) We have that G_1 is linear and continuous functional. In fact, we have (by Lemma 1(iv)) $\omega\sim\tilde\omega_1$ in F_k ($\omega\leq\tilde c_1\,\tilde\omega_1$). By (2) we obtain

$$\begin{aligned}|G_1(u)|&\leq\int_{F_k}|(A\nabla u).\nabla\varphi|\,\mathrm{d}x\\
&\leq\int_{F_k}((A\nabla u).\nabla u)^{1/2}\,((A\nabla\varphi).\nabla\varphi)^{1/2}\,\mathrm{d}x\\
&\leq\bigg(\int_{F_k}(A\nabla u).\nabla u\,\mathrm{d}x\bigg)^{1/2}\bigg(\int_{F_k}((A\nabla\varphi).\nabla\varphi)^{1/2}\,\mathrm{d}x\bigg)^{1/2}\\
&\leq\Lambda\bigg(\int_{F_k}|\nabla u|^2\,\omega\,\mathrm{d}x\bigg)^{1/2}\bigg(\int_{F_k}|\nabla\varphi|^2\,\omega\,\mathrm{d}x\bigg)^{1/2}\\
&\leq\Lambda\bigg(\int_{F_k}\tilde c_1\,|\nabla u|^2\tilde\omega_1\,\mathrm{d}x\bigg)^{1/2}\bigg(\int_\Omega|\nabla\varphi|^2\,\omega\,\mathrm{d}x\bigg)^{1/2}\\
&\leq\Lambda\,\tilde c_1^{1/2}\|\varphi\|_{W_0^{1,2}(\Omega,\omega)}\,\|u\|_{W_0^{1,2}(\Omega,\tilde\omega_1)}.\end{aligned}$$

(b) We have that G_2 is continuous functional. In fact, if $u_1,u_2\in W_0^{1,2}(\Omega,\tilde\omega_1)$, we obtain by (H1)

$$|G_2(u_2)-G_2(u_1)|\leq\int_{F_k}|g(x,u_2,\nabla u_2)-g(x,u_1,\nabla u_1)|\,|\varphi|\,\omega\,\mathrm{d}x$$

$$\begin{aligned}
&\le C_g\Big[\int_{F_k} |u_1-u_2|\,|\varphi|\,\omega\,\mathrm{d}x+\int_{F_k} |\nabla u_1-\nabla u_2|\,|\varphi|\,\omega\,\mathrm{d}x\Big]\\
&\le C_g\Big(\int_{F_k} |\varphi|^2\omega\,\mathrm{d}x\Big)^{1/2}\Big[\Big(\int_{F_k} |u_1-u_2|^2\omega\,\mathrm{d}x\Big)^{1/2}\\
&\quad+\Big(\int_{F_k} |\nabla(u_1-u_2)|^2\,\omega\,\mathrm{d}x\Big)^{1/2}\Big]\\
&\le C_g\ \Big(\int_{\Omega} |\varphi|^2\omega\,\mathrm{d}x\Big)^{1/2}\Big[\Big(\int_{F_k} \tilde{c}_1\,|u_1-u_2|^2\,\tilde{\omega}_1\,\mathrm{d}x\Big)^{1/2}\\
&\quad+\ \Big(\int_{F_k} \tilde{c}_1\,|\nabla(u_2-u_1)|^2\,\tilde{\omega}_1\,\mathrm{d}x\Big)^{1/2}\Big]\\
&\le 2\,\tilde{c}_1^{1/2}\,C_g\,\|\varphi\|_{W_0^{1,2}(\Omega,\omega)}\|u_1-u_2\|_{W_0^{1,2}(\Omega,\tilde{\omega}_1)}.
\end{aligned}$$

Using (a), (b), properties (i), (ii) and (iii), and that u_m is solution of $(\mathrm{P_m})$, we obtain

$$\begin{aligned}
&\sum_{i,j=1}^{n}\int_{F_k} a_{ij}\,D_i\tilde{u}\,D_j\varphi\,\mathrm{d}x+\int_{F_k} g(x,\tilde{u},\nabla\tilde{u})\,\varphi\,\omega\,\mathrm{d}x\\
&\quad=\lim_{m\to\infty}[G_1(u_m)+G_2(u_m)]\\
&\quad=\lim_{m\to\infty}\Big(\sum_{i,j=1}^{n}\int_{F_k} a_{ij}^{mm}\,D_iu_m\,D_j\varphi\,\mathrm{d}x+\int_{F_k} g(x,u_m,\nabla u_m)\,\varphi\,\omega_m\,dx\Big)\\
&\quad=\lim_{m\to\infty}\Big(\sum_{i,j=1}^{n}\int_{\Omega} a_{ij}^{mm}\,D_iu_m\,D_j\varphi\,\mathrm{d}x+\int_{\Omega} g(x,u_m,\nabla u_m)\,\varphi\,\omega_m\,\mathrm{d}x\\
&\qquad-\sum_{i,j=1}^{n}\int_{\Omega\cap F_k^c} a_{ij}^{mm}\,D_iu_m\,D_j\varphi\,dx-\int_{\Omega\cap F_k^c} g(x,u_m,\nabla u_m)\,\varphi\,\omega_m\,dx\Big)\\
&\quad=\lim_{m\to\infty}\Big(\int_{\Omega} f_{0m}\,\varphi\,\mathrm{d}x+\sum_{j=1}^{n}\int_{\Omega} f_{jm}\,D_j\varphi\,\mathrm{d}x\\
&\qquad-\sum_{i,j=1}^{n}\int_{\Omega\cap F_k^c} a_{ij}^{mm}\,D_iu_m\,D_j\varphi\,\mathrm{d}x-\int_{\Omega\cap F_k^c} g(x,u_m,\nabla u_m)\,\varphi\,\omega_m\,\mathrm{d}x\Big),
\end{aligned}\tag{16}$$

where E^c denotes the complement of a set $E\subset\mathbb{R}^n$.

(I) By the Lebesgue Dominated Convergence Theorem and $\tilde{\omega}_2\in A_2$, we obtain (as $m\to\infty$)

$$\int_\Omega f_m\,\varphi\,\mathrm{d}x \;\to\; \int_\Omega f\,\varphi\,\mathrm{d}x,$$
$$\int_\Omega f_{jm} D_j\varphi\,\mathrm{d}x \;\to\; \int_\Omega f_j D_j\varphi\,\mathrm{d}x, \quad j=1,\dots,n.$$

(II) Since the matrix $A^m = (a_{ij}^{mm})$ is symmetric, we have

$$|(A^m\nabla u_m).\nabla\varphi| \le [(A^m\nabla u_m).\nabla u_m]^{1/2}[(A^m\nabla\varphi).\nabla\varphi]^{1/2}.$$

Then, by Lemma 1(vi) and (12), we obtain

$$\begin{aligned}
\Big|\sum_{i,j=1}^n \int_{\Omega\cap F_k^c} a_{ij}^{mm} D_i u_m D_j\varphi\,\mathrm{d}x\Big| &\le \int_{\Omega\cap F_k^c} |(A^m\nabla u_m).\nabla\varphi|\,\mathrm{d}x\\
&\le \Lambda\Big(\int_{\Omega\cap F_k^c}|\nabla u_m|^2\omega_m\,\mathrm{d}x\Big)^{1/2}\Big(\int_{\Omega\cap F_k^c}|\nabla\varphi|^2\omega_m\,\mathrm{d}x\Big)^{1/2}\\
&\le \Lambda\|u_m\|_{W_0^{1,2}(\Omega,\omega_m)}\Big(\int_{\Omega\cap F_k^c}|\nabla\varphi|^2\,w_m\,\mathrm{d}x\Big)^{1/2}\\
&\le \Lambda\,C_3\,\Big(\int_{\Omega\cap F_k^c}|\nabla\varphi|^2\,w_m\,\mathrm{d}x\Big)^{1/2}.
\end{aligned}\tag{17}$$

(III) By (H1), $|g(x,s,\xi)| \le C_g(|s|+|\xi|)$, and (12) we have

$$\begin{aligned}
\Big|\int_{\Omega\cap F_k^c} g(x,u_m,\nabla u_m)\,\varphi\,\omega_m\,\mathrm{d}x\Big| &\le \int_{\Omega\cap F_k^c}|g(x,u_m,\nabla u_m)|\,|\varphi|\,\omega_m\,\mathrm{d}x\\
&\le C_g\int_{\Omega\cap F_k^c}(|u_m|+|\nabla u_m|)\,|\varphi|\,\omega_m\,\mathrm{d}x\\
&\le C_g\Big[\Big(\int_{\Omega\cap F_k^c}|u_m|^2\omega_m\,\mathrm{d}x\Big)^{1/2} + \Big(\int_{\Omega\cap F_k^c}|\nabla u_m|^2\,\omega_m\,\mathrm{d}x\Big)\Big]\\
&\quad\times\Big(\int_{\Omega\cap F_k^c}|\varphi|^2\omega_m\,\mathrm{d}x\Big)^{1/2}\\
&\le 2C_g\|u_m\|_{W_0^{1,2}(\Omega,\omega_m)}\Big(\int_{\Omega\cap F_k^c}|\varphi|^2\omega_m\,\mathrm{d}x\Big)^{1/2}\\
&\le 2C_g\,C_3\Big(\int_{\Omega\cap F_k^c}|\varphi|^2\omega_m\,\mathrm{d}x\Big)^{1/2}.
\end{aligned}\tag{18}$$

Using Lemma 1, we know that $|\Omega \cap F_k^c| \to 0$ when $k \to \infty$. Then

$$\lim_{k\to\infty} \left(\int_{\Omega\cap F_k^c} |\varphi|^2 \omega_m \,\mathrm{d}x \right)^{1/2} = \lim_{k\to\infty} \left(\int_{\Omega\cap F_k^c} |\nabla\varphi|^2 \omega_m \,\mathrm{d}x \right)^{1/2} = 0,$$

and we obtain in (17) and (18)

$$\lim_{k\to\infty} \sum_{i,j=1}^{n} \int_{\Omega\cap F_k^c} a_{ij}^{mm}\, D_i u\, D_j\varphi \,\mathrm{d}x = 0, \tag{19}$$

$$\lim_{k\to\infty} \int_{\Omega\cap F_k^c} g(x, u_m, \nabla u_m)\, \varphi\, \omega_m \,\mathrm{d}x = 0. \tag{20}$$

Therefore, by (16), (19) and (20) we conclude, when $k \to \infty$ (and $m \geq k$),

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij} D_i\tilde{u} D_j\varphi \,\mathrm{d}x + \int_\Omega g(x, \tilde{u}, \nabla\tilde{u})\, \varphi\, \omega \,\mathrm{d}x = \int_\Omega f_0\, \varphi \,\mathrm{d}x + \sum_{j=1}^{n} \int_\Omega f_j D_j\varphi \,\mathrm{d}x$$

for all $\varphi \in W_0^{1,2}(\Omega, \omega)$, that is, $\tilde{u}$ is a solution of problem (P). Therefore, $u = \tilde{u}$ (by the uniqueness).

Example 1. Let $\Omega = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 < 1\}$ and $0 < 2(C_\Omega^2 + 1) < \lambda < \Lambda$. By Theorem 1, with $g\colon \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, $g((x,y), s, \xi) = \cos(xy)\sin(s) + \cos(1/(x^2 + y^2))\sin(|\xi|)$ (with $C_g = 1$), $f_0(x,y) = x|y|$, $f_1(x,y) = |x|y\cos(xy)$, $f_2(x,y) = |x|y\sin(xy)$, $\omega(x,y) = (x^2+y^2)^{-1/2}$ and

$$A(x,y) = \begin{pmatrix} \lambda(x^2+y^2)^{-1/2} & 0 \\ 0 & \Lambda(x^2+y^2)^{-1/2} \end{pmatrix},$$

the problem

$$\begin{cases} Lu + g((x,y), u, \nabla u)\, \omega = f_0 - \dfrac{\partial f_1}{\partial x} - \dfrac{\partial f_2}{\partial y} & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where

$$Lu = -\frac{\partial}{\partial x}\left(\lambda(x^2+y^2)^{-1/2} \frac{\partial u}{\partial x} \right) - \frac{\partial y}{\partial x}\left(\Lambda(x^2+y^2)^{-1/2} \frac{\partial u}{\partial y} \right),$$

has a unique solution $u \in W_0^{1,2}(\Omega, \omega)$ and u can be approximated by a sequence of solutions for non-degenerate quasilinear elliptic equations.

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