Automorphism liftable modules

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Abstract. We introduce the notion of an automorphism liftable module and give a characterization to it. We prove that category equivalence preserves automorphism liftable. Furthermore, we characterize semisimple rings, perfect rings, hereditary rings and quasi-Frobenius rings by properties of automorphism liftable modules. Also, we study automorphism liftable modules with summand sum property (SSP) and summand intersection property (SIP).

Keywords: dual automorphism invariant module; supplemented module; semisimple ring; perfect ring; summand sum property

Classification: 16L30, 16D40, 16W20

1. Introduction

Throughout this paper, R denotes an associative ring with identity and all modules are assumed to be unital right R-modules unless otherwise stated. The category of right (or left) R-modules denoted by $\mathcal{M}_R$ or mod-R (${}_R\mathcal{M}$ or R-mod, respectively).

A right R-module M is said to be an automorphism-extendable module if for each submodule N in M, every automorphism of the module N can be extended to an endomorphism of M. Such modules were introduced by Tuganbaev in [12].

A right R-module M is called an automorphism-invariant module if every isomorphism between two essential submodules of M extends to an automorphism of M. Equivalently, M is an automorphism-invariant module if for any automorphism σ of $E(M)$, $\sigma(M) \subseteq M$, where $E(M)$ is the injective hull of M. A right R-module M is called quasi-injective (or pseudo-injective) module if M is invariant under any endomorphism (or monomorphism, respectively) of $E(M)$. Clearly, any automorphism-invariant module is automorphism-extendable.

The dual notion of automorphism-invariant modules was introduced by Singh and Srivastava in [11] and they called such modules as dual automorphism invariant modules. Further study on dual automorphism invariant modules was carried out in [8] and [10]. An R-submodule N of an R-module M is said to be *small* in

DOI 10.14712/1213-7243.2015.237

The first author was partially supported by UGC SAP (Grant No. F.510/7/DRS-1/2016 (SAP-1) dated 24-02-2016). The second author was supported by CSIR - Human Resource Development Group, CSIR Complex, Library Avenue, Pusa, New Delhi - 110 012, India, Grant No. 09/810(0020)/2012-EMR-I.

M if $N+K \neq M$ for any proper submodule K of M and it is denoted by $N \ll M$. An epimorphism is said to be *small epimorphism* if its kernel is small. Given two R-modules N and M, N is called *M-projective* (*projective relative to* M) if for every submodule A of M, any R-homomorphism $f\colon N \to M/A$ can be lifted to an R-homomorphism $f'\colon N \to M$.

A right R-module M is called a *dual automorphism invariant module* if whenever K_1 and K_2 are small submodules of M, then any epimorphism $\eta\colon M/K_1 \to M/K_2$ with small kernel lifts to an endomorphism ϕ of M. Then ϕ is an isomorphism by [11].

A submodule K of M is said to be weak supplement of N if $N + K = M$ and $N \cap K \ll M$. A module M is a weakly supplemented module if every submodule of M has a weak supplement. A ring R-module M is said to be ADS*-module if for every decomposition $M = S \oplus T$ of M and every weak supplement T' of S we have $M = S \oplus T'$.

In this paper we introduce the notion of an automorphism liftable module and give a characterization to it. We prove that category equivalence preserves automorphism liftable. Furthermore, we characterize semisimple rings, perfect rings, hereditary rings and quasi-Frobenius rings by properties of automorphism liftable modules. Also, we study automorphism liftable modules with summand sum property (SSP) and summand intersection property (SIP).

2. Automorphism liftable modules

In this section we introduce the notion of an automorphism liftable module, which is the dual notion of automorphism extendable and generalization of quasi-projective. Also, here we discuss some basic properties of automorphism liftable modules.

Definition 2.1. An R-module M is said to be an *automorphism liftable module* if for each R-submodule N of M, every automorphism of the factor module M/N can be lifted to an endomorphism of M.

In the following proposition we will prove the category of automorphism liftable modules closed under direct summand.

Proposition 2.2. *Any direct summand of an automorphism liftable module is automorphism liftable.*

PROOF: Let $M = M_1 \oplus M_2$ be an R-module, N_1 be a submodule of M_1 and $f\colon M_1/N \to M_1/N$ be an automorphism. Then $f' = f \oplus I_{M_2}\colon M_1/N \oplus M_2 \to M_1/N \oplus M_2$, where I_{M_2} is the identity map on M_2, is an isomorphism. Then by hypothesis there exists an endomorphism $g\colon M \to M$, which is a lifting of f'. Let $g = \left\{ \begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{smallmatrix} \right\}$. Clearly, $g_{11}\colon M_1 \to M_1$ is a lifting of f. □

Remark. The direct sum of two automorphism liftable modules need not be automorphism liftable.

For example, $\mathbb{Z}_2$ and $\mathbb{Z}_4$ are automorphism liftable $\mathbb{Z}$-modules. Consider the submodule $N = \{0 \oplus 0, 0 \oplus 2\}$ of $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ and an automorphism $f\colon M/N \to M/N$ defined by $\{\overline{0 \oplus 0} \to \overline{0 \oplus 0}, \overline{0 \oplus 1} \to \overline{1 \oplus 1}, \overline{1 \oplus 0} \to \overline{0 \oplus 1}, \overline{1 \oplus 1} \to \overline{1 \oplus 0}\}$. Clearly, f has no lifting to M. Hence $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ is not an automorphism liftable $\mathbb{Z}$-module.

Lemma 2.3. *Let M be a weakly supplemented R-module, Then M is automorphism liftable module if and only if for any small R-submodule S of M, every automorphism of the factor module M/S can be lifted to an endomorphism of M.*

Proof: Let N_1 be an R-submodule of M and $f\colon M/N_1 \to M/N_1$ be an automorphism. Since M is weakly supplemented, there exists an R-submodule N_2 of M such that $N_1 + N_2 = M$ and $N_1 \cap N_2 \ll M$. Then $N_1/(N_1 \cap N_2) \oplus N_2/(N_1 \cap N_2) = M/(N_1 \cap N_2)$ and $N_2/(N_1 \cap N_2) \cong M/N_1$. Let $g\colon N_2/(N_1 \cap N_2) \to M/N_1$ be an isomorphism defined by $g(n_2 + (N_1 \cap N_2)) = n_2 + N_1$. Then we get an isomorphism $\phi = I_1 \oplus \phi'\colon (N_1/(N_1 \cap N_2) \oplus N_2/(N_1 \cap N_2)) \to (N_1/(N_1 \cap N_2) \oplus N_2/(N_1 \cap N_2))$, where I_1 is the identity map on $N_1/(N_1 \cap N_2)$ and $\phi' = g^{-1} \circ f \circ g$. By hypothesis, there exists an endomorphism $f'\colon M \to M$ which is a lifting of ϕ.

We shall prove that f' is a lifting of f. Let $m \in M$ and $m = m_1 + m_2$, where $m_1 \in N_1$ and $m_2 \in N_2$. Because of $(N_1 \cap N_2) \subseteq N_1$ and $I_1(m_1) \in N_1$, we have

$$\begin{aligned} f'(m) + N_1 &= \phi'(m_2) + I_1(m_1) + N_1 \\ &= \phi'(m_2) + N_1 \\ &= g^{-1} f g(m_2) + N_1 \\ &= f(m_2) + N_1. \end{aligned}$$

Suppose $m \in N_1$, $f'(m) + N_1 = 0 + N_1$ and suppose $m \in N_2$, $f'(m) + N_1 = f(m) + N_1$. Hence f' is a lifting of f. This completes the proof. □

From [11] dual automorphism invariant module is also a generalization of quasi-projective module. This motivates us to rise the question: "Is there any relation between dual automorphism invariant module and automorphism liftable module?". This motivation directs us to the following proposition.

Proposition 2.4. *Let M be a weakly supplemented dual automorphism invariant R-module, then M is an automorphism liftable module.*

Proof: It follows from Lemma 2.3. □

Recall that an R-module M is said to be pseudo-projective if for every submodule N of M every epimorphism $f\colon M \to M/N$ can be lifted to an endomorphism $g\colon M \to M$.

Proposition 2.5. *Any pseudo-projective module is automorphism liftable.*

Proof: Let N be an R-submodule of a pseudo-projective R-module M and $f\colon M/N \to M/N$ be an automorphism. Then $f' = f \circ n\colon M \to M/N$ is an epimorphism, where n is the natural projection from M to M/N. By pseudo-projectivity of M, f' can be lifted to $g\colon M \to M$. Clearly, g is a lifting of f. □

Theorem 2.6. *Let M be a module of finite length. Then M is automorphism liftable if and only if for every submodule N of M, every automorphism of the factor module M/N can be lifted to an automorphism of M.*

Proof: Suppose that M is an automorphism liftable module. Let N be a submodule of M and f_1 be an automorphism of the factor module M/N. We need to prove that f_1 can be lifted to an automorphism of M.

Since M is a module of finite length, M is a supplemented R-module. Then there exists an R-submodule N' of M such that $N + N' = M$ and $N \cap N' \ll M$. Therefore $N/(N \cap N') \oplus N'/(N \cap N') = M/(N \cap N')$ and $N'/(N \cap N') \cong_g M/N$. Thus the isomorphism $I_{N/(N\cap N')} \oplus (g \circ f_1 \circ g^{-1})$ is a lifting of f_1. Hence we can assume that N is a small submodule of M.

Let W be the set of all pairs $(M/S', f')$ such that M/S' is a quotient of M, where S' is a small submodule of M contains in N, f' is an automorphism of M/S' and f' is a lifting of f_1. Define a partial order relation "$\leq$" on W such that $(M/S'_1, f'_1) \leq (M/S'_2, f'_2) \Leftrightarrow S'_2 \subseteq S'_1$ and f'_2 is a lifting of f'_1. Let $(M/S'_1, f'_1) \leq (M/S'_2, f'_2) \leq \dots (M/S'_n, f'_n) \leq \dots$ be a chain in W. Since M is artinian, the descending chain $\dots S'_n \subseteq \dots \subseteq S'_2 \subseteq S'_1$ is stationary. Therefore every chain of W has a maximal element in W. Hence M has a maximal element $(M/S, f)$. Clearly, $S' \supseteq S$ and f' has f as a lifting.

Suppose $S = 0$, then f is a required automorphism of M.

Let $S \neq 0$. Then S has a maximal element and the nonzero Jacobson radical $J(S)$. Let g be a lifting of f. Since $\mathrm{Im}(g) + S = M$, S is small in M. Hence $\mathrm{Im}(g) = M$, i.e., g is an epimorphism. In addition, $J(S)$ is invariant under g. So we get an epimorphism $h\colon M/J(S) \to M/J(S)$. Clearly, h is a lifting of f_1.

We need to prove that h is one-one. It is enough to prove that $h|_{S/J(S)}$ is one-one. Since $S/J(S)$ is semisimple and finitely generated, $h|_{S/J(S)}$ is one-one. Hence h is automorphism, which is a contradiction to $(M/S, f)$ being maximal. Hence f_1 can be lifted to an automorphism of M.

The converse part is trivial. □

Theorem 2.7. *Let M_1, M_2 be right R-modules. If $M = M_1 \oplus M_2$ is automorphism liftable, then M_1 is M_2-projective and M_2 is M_1-projective.*

Proof: Let $f\colon M_1 \to M_2/N$ be an R-homomorphism. It induces an R-homomorphism $\sigma\colon M/N \to M/N$ given by $\sigma(x_1 + x_2 + N) = x_1 + f(x_1) + (x_2 + N)$ for $x_1 \in M_1$, $x_2 \in M_2$. Let $x_1 \in M_1$, $x_2 \in M_2$, $m_1 \in M_1$ and $m_2 \in M_2$, $x_1 + x_2 + N \in M/N$ and $m_1 + m_2 + N \in M/N$. To prove injectivity of σ, we suppose that

$$\sigma(x_1 + x_2 + N) = \sigma(m_1 + m_2 + N), \quad \text{i.e.,}$$
$$x_1 + f(x_1) + (x_2 + N) = m_1 + f(m_1) + (m_2 + N).$$

Then $x_1 = m_1$, $f(x_1) = f(m_1)$. Thus $x_2 + N = m_2 + N$.

To prove ontoness of σ, consider $x_1+x_2+N \in M/N$, then $x_1+x_2-f(x_1)+N \in M/N$ maps to $x_1+x_2+N \in M/N$ under the R-homomorphism σ. Hence σ is an isomorphism. Since M is automorphism liftable, σ lifts to an endomorphism η of M. Let $x_1 \in M_1$ and $\eta(x_1+0) = u_1+u_2$, where $u_1 \in M_1$, $u_2 \in M_2$. Then $u_1+u_2+N = x_1+f(x_1) \in M_1 \oplus M_2/N$, $u_2+N = f(x_1)$.

Let $\phi_2\colon M \to M_2$ be the natural projection. Then $\phi_2 \circ \eta|_{M_1}\colon M_1 \to M_2$ is a lifting of f. Hence M_1 is M_2-projective. Similarly, it can be shown that M_2 is M_1-projective. □

Corollary 2.8. *Let M_1, M_2 be right R-modules and $M = M_1 \oplus M_2$ is a weakly supplemented dual automorphism invariant R-module, then M_1 is M_2-projective and M_2 is M_1-projective.*

Proof: This follows from Proposition 2.4 and Theorem 2.7. □

Corollary 2.9. *Every automorphism liftable module is ADS*-module.*

Proof: It follows from Theorem 2.7 and the proof of Theorem 2.1 in [13]. □

Corollary 2.10. *A sufficient condition for an epimorphism $\lambda\colon N \to M$ to split is that $N \oplus M$ is automorphism liftable.*

Proof: By Theorem 2.7, N and M are mutually projective. Then there exists an epimorphism $\alpha\colon M \to N$ such that $\lambda\alpha = I_M$. Hence λ is a splitting epimorphism. □

Theorem 2.11. *Let R be a ring, M be an automorphism liftable right R-module, and let X and Y be two submodules in M with $X+Y = M$. If $f\colon M/Y \to M/X$ is an R-homomorphism, then there exists an endomorphism g of the module M that is a lifting of f.*

Proof: Let $f\colon M/Y \to M/X$ be an R-homomorphism. Consider the isomorphisms $\alpha_1\colon X/(X\cap Y) \to M/Y$ and $\alpha_2\colon Y/(X\cap Y) \to M/X$ defined by $\alpha_1(x + X\cap Y) = x+Y$ and $\alpha_2(y + X\cap Y) = y+X$, respectively. Then we get an R-homomorphism $f_1 = \alpha_2^{-1} \circ f \circ \alpha_1\colon X/(X\cap Y) \to Y/(X\cap Y)$.

We define an endomorphism α of the module $M/(X\cap Y)$ by the relation $\alpha(x+y+X\cap Y) = x+f_1(x+X\cap Y)+y+X\cap Y$ for all $x \in X$ and $y \in Y$. Then

$$\begin{aligned}
\alpha(x+y+X\cap Y) = 0+X\cap Y \\
&\Rightarrow x+f_1(x+X\cap Y)+y+X\cap Y = 0+X\cap Y \\
&\Rightarrow x+f_1(x+X\cap Y)+y \in X\cap Y \\
&\Rightarrow x \in Y \qquad [f_1(x+X\cap Y)+y \in Y] \\
&\Rightarrow f_1(x+X\cap Y) = 0 \\
&\Rightarrow x+y+X\cap Y = 0+X\cap Y.
\end{aligned}$$

Hence α is a monomorphism.

Let $x+y+X\cap Y \in M/X\cap Y$. Take $x+y-f_1(x+X\cap Y)+X\cap Y \in M/X\cap Y$, then

$$\alpha(x+y-f_1(x+Y)+X\cap Y) = x+y+X\cap Y.$$

Hence α is an isomorphism.

Since M is automorphism liftable, α can be lifted to g. Clearly, g is a lifting of f. □

Let R and S be two rings, a covariant functor $T\colon \mathcal{M}_R \to \mathcal{M}_S$ is said to be category equivalence if there exists a covariant functor $T'\colon \mathcal{M}_S \to \mathcal{M}_R$ such that $T'T$ and TT' are naturally equivalent to the identities $I_{\mathcal{M}_R}\colon \mathcal{M}_R \to \mathcal{M}_R$ and $I_{\mathcal{M}_S}\colon \mathcal{M}_S \to \mathcal{M}_S$, respectively. In the following theorem we prove that category equivalence preserves the automorphism liftable module.

Theorem 2.12. *Let R and S be two rings and $T\colon \mathcal{M}_R \to \mathcal{M}_S$ be a category equivalence. Then M is automorphism liftable if and only if $T(M)$ is automorphism liftable.*

PROOF: The proof is trivial. □

By [6] for any ring R and its matrix ring R_n (the set of all $n\times n$ matrix over R) are category equivalence by the functors $T\colon \mathcal{M}_R \to \mathcal{M}_{R_n}$ defined by

$$T(M) = M_n \quad \text{(the set of all } n\times n \text{ matrix over } M)$$

and $T'\colon \mathcal{M}_{R_n} \to \mathcal{M}_R$ defined by

$$T'(M_n) = (e_{11}U),$$

where $e_{11} \in S$ is a matrix with 1_R in the $(1,1)$th position and zero elsewhere.

Corollary 2.13. *Let R be a ring and $S = R_n$. Then*

(1) *M is automorphism liftable over R if and only if M_n is automorphism liftable over S;*
(2) *U is automorphism liftable over S if and only if $(e_{11}U)$ is automorphism liftable over R.*

By [6], let R be a ring and I be an ideal of R contained in the annihilator of a module M, then $\mathcal{M}_R$ and $\mathcal{M}_{R/I}$ are category equivalent. Hence we have the following.

Corollary 2.14. *Let M be a right R-module and I a two-sided ideal of R contained in the annihilator of M. Then M is automorphism liftable over R if and only if it is automorphism liftable over R/I.*

3. Characterization of rings using automorphism liftable modules

In this section we characterize some special rings using automorphism liftable modules.

Proposition 3.1. *If every 2-generated R-module is automorphism liftable, then R is a semisimple ring.*

Proof: Let M be a simple R-module and let $f\colon R \to M$ be an epimorphism defined by $f(r) = rm$ for all $r \in R$ and for any $m \in M$. Consider the module $N = R \oplus M$. Hence by hypothesis N is an automorphism liftable module. By Theorem 2.7, M and R are relatively projective to each other. Then the map f is split. Therefore M is projective and hence by [9] R is a semisimple ring. □

Corollary 3.2. *For any ring R the following are equivalent:*

(1) *any automorphism liftable module is projective;*
(2) *direct sum of automorphism liftable modules is automorphism liftable;*
(3) *$R \oplus M$ is automorphism liftable module for any simple module M;*
(4) *every finitely generated R-module is automorphism liftable;*
(5) *every 2-generated R-module is automorphism liftable;*
(6) *R is semisimple.*

Proof: (1) ⇒ (2), (2) ⇒ (3) and (6) ⇒ (1) are trivial. By the proof of Proposition 3.1, we have (3) ⇒ (6). (4) ⇒ (5) and (6) ⇒ (4) are trivial. By Proposition 3.1, (5) ⇒ (6). □

Note. In [11, Theorem 5], Singh and Srivastava proved that a ring such that every finitely generated right R-module is dual automorphism invariant is a V-ring. By Proposition 3.1, dual automorphism invariant module need not be an automorphism liftable module.

In the sense of Bass [2], a ring R satisfies the descending chain condition on principal right ideals means that for every sequence $\langle a_i \rangle$ of elements of R there exists an m such that $a_1 \cdots a_m R = a_1 \cdots a_{m+k} R$ for all $k \leq 0$. The proof of the following theorem is based on [5, Theorem 3.1].

Theorem 3.3. *The ring R is left perfect if and only if every flat left R-module is an automorphism liftable left R-module.*

Proof: The ring R is left perfect implies that every flat left R-module is an automorphism liftable left R-module which follows from [5, Theorem 3.1]. By [2, Theorem P], we have to show that R satisfies the descending chain condition on principal right ideals. Let $F = \bigoplus_{i=1}^{\infty} Rx_i$ be a countably-generated free module and define $G_n = \bigoplus_{i=1}^{\infty} R(x_i - a_i x_{i+1})$, $n = 1, 2, \ldots, \infty$. Then F/G_n is free (hence flat) for all $n < \infty$ and so $F/G = \lim_{\to} F/G_n$ is the direct limit of flat left R-modules and so is flat. A module F itself is flat and hence so is $F \oplus F/G$. By hypothesis $F \oplus F/G$ is an automorphism liftable left R-module and hence G is a direct summand of F. By [2, Lemma 1.3], we have done the proof. □

We will adopt the following notations from [7]. Let G be an injective cogenerator over the ring $\mathbb{Z}$ of integers and let $\chi = \mathrm{Hom}_{\mathbb{Z}}(-, G)$ be the G-character functor. Then χ can be considered as a faithful exact contravariant functor $R\text{-}\mathrm{mod} \to$

mod-R (or mod-$R \to R$-mod) which commutes with finite direct sums. Furthermore, for any R-module M we have a canonical embedding $M \to \chi^2(M)$. The module $\chi(M)$ is called the character module of M (with respect to G).

Theorem 3.4. *The following are equivalent for a ring R:*

(1) *R is completely reducible;*
(2) *$\chi(M)$ is automorphism liftable for every right (left) R-module M;*
(3) *$\chi^2(M)$ is automorphism liftable for every left (right) R-module M.*

Proof: Let $\chi(M)$ be an automorphism liftable for all M. Since χ commutes with finite direct sums, $\chi(M) \oplus \chi(M) \cong \chi(M \oplus M)$. Then $\chi(M) \oplus \chi(M)$ is automorphism liftable and hence $\chi(M)$ is quasi-projective. Similarly, $\chi^2(M)$ is automorphism liftable for all M implies $\chi^2(M)$ is quasi-projective for all M. Hence the proof follows from [7, Theorem B]. □

Recall that, a ring R is said to be quasi-Frobenius if every projective module is injective. Equivalently, every injective module is projective. In the following theorem we give some other equivalent conditions for quasi-Frobenius ring.

Theorem 3.5. *The following are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *every injective module is automorphism liftable;*
(3) *every projective module is automorphism extendable.*

Proof: Let M be an injective (projective) module. Then $M \oplus M$ is injective (projective). Therefore $M \oplus M$ is automorphism liftable (extendable) and hence M is quasi-injective (quasi-projective). By [3, Corollary 2.3], R is a quasi-Frobenius ring.

(1) ⇒ (2) and (1) ⇒ (3) are the trivial implication. □

Recall that a ring R is called (semi)hereditary if all (finitely generated) submodules of projective modules are again projective.

Theorem 3.6. *A ring R is (semi)hereditary if and only if every (finitely generated) submodule of a projective module is automorphism liftable.*

Proof: Let M be a (finitely generated) submodule of a projective module P, $M \oplus M$ is a (finitely generated) submodule of a projective module $P \oplus P$. Then $M \oplus M$ is an automorphism liftable module and hence M is quasi-projective. By [6, Theorem 4.3], R is (semi)hereditary if and only if every (finitely generated) submodule of a projective module is automorphism liftable. □

4. Automorphism-liftable module with SSP and SIP

Recall that a module M has the summand sum property (SSP) if the sum of two direct summands is a direct summand of M. Also, a module M has the summand intersection property (SIP) if the intersection of two direct summands is a direct summand of M.

Lemma 4.1 ([1]). *A module M has the summand sum property if and only if for every decomposition $M = A \oplus B$ and every R-homomorphism $f\colon A \to B$, the image of f is a direct summand of B.*

Lemma 4.2 ([1]). *A module M has the summand intersection property if and only if for every decomposition $M = A \oplus B$ and every R-homomorphism $f\colon A \to B$, the kernel of f is a direct summand of A.*

Recall that an R-module M has C_3 condition, if A and B are direct summands of M with $A \cap B = \{0\}$, then $A + B$ is a direct summand of M.

Also an R-module M has D_3 condition, if A and B are direct summands of M with $A + B = M$, then $A \cap B$ is a direct summand of M.

Theorem 4.3. *An automorphism-liftable module M has the summand sum property if and only if M has C_3 condition and summand intersection property.*

Proof: Let $M = A \oplus B$ for some modules A and B. Let $f\colon A \to B$ be an R-homomorphism. Since the module M has the summand sum property, by Lemma 4.1, $\mathrm{Im} f$ is a direct summand of B and $A \oplus \mathrm{Im} f$ is a direct summand of M. Then $A \oplus \mathrm{Im} f$ is automorphism-liftable. By Theorem 2.7 A and $\mathrm{Im}\, p|_A$ are relatively projective to each other. Therefore the epimorphism $f\colon A \to \mathrm{Im} f$ is split. Hence $\ker p|_A$ is a direct summand of A. Then by Lemma 4.2, M has the summand intersection property. If an R-module M has the summand sum property, then M satisfies the C_3 condition.

Conversely, suppose M has C_3 condition and summand intersection property, then by [1, Lemma 19], M has the summand sum property. □

Dually, we can prove the following theorem for an automorphism-extendable module.

Theorem 4.4. *An automorphism-extendable module M has the summand intersection property if and only if M has D_3 condition and summand sum property.*

Proof: The proof is dual to the proof of Theorem 4.3. □

Corollary 4.5. *Let M be an R-module and $S = \mathrm{End}(M)$. Then*

(1) *if M is an automorphism-liftable module, then M has the summand sum property if and only if S has the summand sum property;*
(2) *if M is an automorphism-extendable module, then M has the summand intersection property if and only if S has the summand sum property.*

Proof: By [4, Theorem 2.3], S has the summand sum property. □

In [4], Garcia proved the following theorem for quasi-projective and quasi-injective modules. Here we generalize it to automorphism-liftable and automorphism-extendable modules.

Proposition 4.6. *Let M be an R-module and $S = \mathrm{End}(M)$. Then*

(1) *if M is an automorphism-liftable module, then $M \oplus M$ has the summand sum property if and only if S is regular;*

(2) *if M is an automorphism-extendable module, then $M \oplus M$ has the summand intersection property if and only if S is regular.*

Proof: The proof is similar to that of [4, Theorem 2.8]. Let M be an automorphism-liftable module and assume that $M \oplus M$ has the SSP. Then M has both the SSP and SIP by Theorem 4.3. So that the kernel and the image of each endomorphism of M are direct summands of M by Lemma 4.1 and Lemma 4.2, and so S is regular by [14, Lemma 3.1]. Conversely, if S is regular, then $\mathrm{End}(M \oplus M)$ is also regular and $M \oplus M$ has the SSP by [4, Corollary 2.4]. The automorphism-extendable case is similar. □

Acknowledgment. The authors are very grateful to the referee for several suggestions and comments that greatly improved the paper.

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(*Received* July 17, 2017, *revised* December 12, 2017)