

The universal Banach space with a K -suppression unconditional basis

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Abstract. Using the technique of Fraïssé theory, for every constant $K \geq 1$, we construct a universal object \mathbb{U}_K in the class of Banach spaces possessing a normalized K -suppression unconditional Schauder basis.

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1. Introduction

A Banach space X is *complementably universal* for a given class of Banach spaces if X belongs to this class and every space from the class is isomorphic to a complemented subspace of X .

In 1969 A. Pełczyński in [11] constructed a complementably universal Banach space for the class of Banach spaces with a Schauder basis. In 1971 M. I. Kadec in [7] constructed a complementably universal Banach space for the class of spaces with the *bounded approximation property* (BAP). In the same year A. Pełczyński in [12] showed that every Banach space with BAP is complemented in a space with a basis. A. Pełczyński and P. Wojtaszczyk in [13] constructed a universal Banach space for the class of spaces with a finite-dimensional decomposition. Applying Pełczyński's decomposition argument, see [10], one immediately concludes that all three universal spaces are isomorphic. It is worth mentioning a negative result of W. B. Johnson and A. Szankowski, see [6], saying that no separable Banach space can be complementably universal for the class of all separable Banach spaces. In [4] the second author constructed an isometric version of the Kadec-Pełczyński-Wojtaszczyk space. The universal Banach space from [4] was constructed using the general categorical technique of Fraïssé limits, see [8]. This method was also applied by W. Kubiś and S. Solecki in [9] for constructing the Gurarii space, see [5], which possesses the property of extension of almost isometries, which implies the universality property that is stronger than the standard universality property of the Banach spaces l_∞ or $C[0, 1]$.

In this paper we apply the categorical method of Fraïssé limits for constructing a universal space \mathbb{U}_K in the class of Banach spaces with a normalized K -suppression unconditional Schauder basis. The universal space constructed by this method has a nice property of extension of almost isometries, which is better than just the standard universality, established in the papers of A. Pełczyński, see [11], and G. Schechtman, see [14], (who gave a short alternative construction of universal space for class of Banach spaces with an unconditional bases). We also prove that the universal space \mathbb{U}_K is isomorphic to the complementably universal space \mathbb{U} for Banach spaces with unconditional basis, which was constructed by A. Pełczyński in [11].

2. Preliminaries

All Banach spaces considered in this paper are separable and over the field \mathbb{R} of real numbers.

2.1 Definitions. Let X be a Banach space with a Schauder basis $(e_n)_{n=1}^\infty$ and let $(e_n^*)_{n=1}^\infty$ be the corresponding sequence of coordinate functionals. The basis $(e_n)_{n=1}^\infty$ is called K -suppression for a real constant K if for every finite subset $F \subset \mathbb{N}$ the projection $\text{pr}_F: X \rightarrow X$, $\text{pr}_F: x \mapsto \sum_{n \in F} e_n^*(x) \cdot e_n$, has norm $\|\text{pr}_F\| \leq K$. It is well-known, see [1, 3.1.5], that each K -suppression Schauder basis $(e_n)_{n=1}^\infty$ is unconditional. So for any $x \in X$ and any permutation π of \mathbb{N} the series $\sum_{n=1}^\infty e_{\pi(n)}^*(x) \cdot e_{\pi(n)}$ converges to x . This means that we can forget about the ordering and think of a K -suppression basis of a Banach space as a subset $\beta \subset X$ such that for some bijection $e: \mathbb{N} \rightarrow \beta$ the sequence $(e(n))_{n=1}^\infty$ is a K -suppression Schauder basis for X .

More precisely, by a *normalized K -suppression basis* for a Banach space X we shall understand a subset $\beta \subset X$ for which there exists a family $\{e_b^*\}_{b \in \beta} \subset X$ of continuous functionals such that

- $\|b\| = 1 = e_b^*(b)$ for any $b \in \beta$;
- $e_b^*(b') = 0$ for every $b \in \beta$ and $b' \in \beta \setminus \{b\}$;
- $x = \sum_{b \in \beta} e_b^*(x) \cdot b$ for every $x \in X$;
- for any finite subset $F \subset \beta$ the projection $\text{pr}_F: X \rightarrow X$, $\text{pr}_F: x \mapsto \sum_{b \in F} e_b^*(x) \cdot b$, has norm $\|\text{pr}_F\| \leq K$.

The equality $x = \sum_{b \in \beta} e_b^*(x) \cdot b$ in the third item means that for every $\varepsilon > 0$ there exists a finite subset $F \subset \beta$ such that $\|x - \sum_{b \in F} e_b^*(x) \cdot b\| < \varepsilon$ for every finite subset $E \subset \beta$ containing F .

By a *K -based Banach space* we shall understand a pair (X, β_X) consisting of a Banach space X and a normalized K -suppression basis β_X for X . By a *based Banach space* we understand a K -based Banach space for some $K \geq 1$. We shall say that a based Banach space (X, β_X) is a *subspace* of a based Banach space (Y, β_Y) if $X \subseteq Y$ and $\beta_X = X \cap \beta_Y$.

For a Banach space X by $\|\cdot\|_X$ we denote the norm of X and by $B_X := \{x \in X: \|x\|_X \leq 1\}$ the closed unit ball of X .

A finite dimensional based Banach space (X, β_X) is called *rational* if its unit ball B_X is a convex polyhedron spanned by finitely many vectors with rational coordinates in the basis β_X . A based Banach space X is called *rational* if each finite-dimensional based subspace of X is rational.

2.2 Categories. Let \mathfrak{K} be a category. For two objects A, B of the category \mathfrak{K} , by $\mathfrak{K}(A, B)$ we will denote the set of all \mathfrak{K} -morphisms from A to B . A *subcategory* of \mathfrak{K} is a category \mathfrak{L} such that each object of \mathfrak{L} is an object of \mathfrak{K} and each morphism of \mathfrak{L} is a morphism of \mathfrak{K} . Morphisms and isomorphisms of a category \mathfrak{K} will be called *\mathfrak{K} -morphisms* and *\mathfrak{K} -isomorphisms*, respectively.

A subcategory \mathfrak{L} of a category \mathfrak{K} is *full* if each \mathfrak{K} -morphism between objects of the category \mathfrak{L} is an \mathfrak{L} -morphism. A subcategory \mathfrak{L} of a category \mathfrak{K} is *cofinal* in \mathfrak{K} if for every object A of \mathfrak{K} there exists a \mathfrak{K} -morphism $f: A \rightarrow B$ to an object B of \mathfrak{L} .

A category \mathfrak{K} has the *amalgamation property* if for every objects A, B, C of \mathfrak{K} and \mathfrak{K} -morphisms $f: A \rightarrow B$ and $g: A \rightarrow C$ there exist an object D of \mathfrak{K} and \mathfrak{K} -morphisms $f': B \rightarrow D$ and $g': C \rightarrow D$ such that $f' \circ f = g' \circ g$.

In this paper we shall work in the category \mathfrak{B} , whose objects are based Banach spaces and morphisms are linear continuous operators $T: X \rightarrow Y$ between based Banach spaces (X, β_X) and (Y, β_Y) such that $T(\beta_X) \subseteq \beta_Y$.

A morphism $T: X \rightarrow Y$ of the category \mathfrak{B} is called an *isometry* (or else an *isometry morphism*) if $\|T(x)\|_Y = \|x\|_X$ for any $x \in X$. By \mathfrak{BI} we denote the category whose objects are based Banach spaces and morphisms are isometry morphisms of based Banach spaces. The category \mathfrak{BI} is a subcategory of the category \mathfrak{B} .

For any real number $K \geq 1$ let \mathfrak{B}_K (or \mathfrak{BI}_K) be the category whose objects are K -based Banach spaces and morphisms are (isometry) \mathfrak{B} -morphisms between K -based Banach spaces. So, \mathfrak{B}_K and \mathfrak{BI}_K are full subcategories of the categories \mathfrak{B} and \mathfrak{BI} , respectively.

By \mathfrak{FI}_K we denote the full subcategory of \mathfrak{BI}_K , whose objects are finite-dimensional K -based Banach spaces, and by \mathfrak{RI}_K the full subcategory of \mathfrak{FI}_K whose objects are rational finite-dimensional K -based Banach spaces. So, we have the inclusions $\mathfrak{RI}_K \subset \mathfrak{FI}_K \subset \mathfrak{BI}_K$ of categories.

From now on we assume that $K \geq 1$ is some fixed real number.

2.3 Amalgamation. In this section we prove that the categories \mathfrak{FI}_K and \mathfrak{RI}_K have the amalgamation property.

Lemma 2.1 (Amalgamation lemma). *Let X, Y, Z be finite-dimensional K -based Banach spaces and $j: Z \rightarrow X, i: Z \rightarrow Y$ be \mathfrak{BI} -morphisms. Then there exist a finite-dimensional K -based Banach space W and \mathfrak{BI} -morphisms $j': Y \rightarrow W$ and $i': X \rightarrow W$ such that the diagram*

$$\begin{array}{ccc} Y & \xrightarrow{j'} & W \\ i \uparrow & & \uparrow i' \\ Z & \xrightarrow{j} & X \end{array}$$

is commutative.

Moreover, if the K -based Banach spaces X, Y, Z are rational, then so is the K -based Banach space W .

PROOF: We shall prove this lemma in the special case when the isometries i, j are identity inclusions; the general case is analogous but has more complicated notation. Our assumptions on i, j ensure that $Z = X \cap Y$ and $\beta_Z = \beta_X \cap \beta_Y$, where $\beta_X, \beta_Y, \beta_Z$ are the normalized K -suppression bases of the K -based Banach spaces X, Y, Z . It follows from $\beta_Z = \beta_X \cap \beta_Y$ that the coordinate functionals of the bases β_X and β_Y agree on the intersection $Z = X \cap Y$.

Consider the direct sum $X \oplus Y$ of the Banach spaces X, Y endowed with the norm $\|(x, y)\| = \|x\|_X + \|y\|_Y$. Let $W = (X \oplus Y)/\Delta$ be the quotient space of $X \oplus Y$ by the subspace $\Delta = \{(z, -z) : z \in Z\}$.

We define linear operators $i' : X \rightarrow W$ and $j' : Y \rightarrow W$ by $i'(x) = (x, 0) + \Delta$ and $j'(y) = (0, y) + \Delta$.

Let us show i' and j' are isometries. Indeed, for every $x \in X$

$$\|i'(x)\|_W = \text{dist}((x, 0), \Delta) \leq \|(x, 0)\| = \|x\|_X + \|0\|_Y = \|x\|_X.$$

On the other hand, for every $z \in Z$

$$\begin{aligned} \|(x, 0) - (z, -z)\| &= \|(x - z, z)\| = \|x - z\|_X + \|z\|_Y \\ &= \|x - z\|_X + \|z\|_X \geq \|x - z + z\|_X = \|x\|_X \end{aligned}$$

and hence $\|x\|_X \leq \inf_{z \in Z} \|(x, 0) - (z, -z)\| = \|i'(x)\|_W$. Therefore $\|i'(x)\|_W = \|x\|_X$. Similarly, we can show that j' is an isometry.

We shall identify X and Y with their images $i'(X)$ and $j'(Y)$ in W . In this case we can consider the union $\beta_W := \beta_X \cup \beta_Y$ and can show that β_W is a normalized Schauder basis for the (finite-dimensional) Banach space W . Let $\{e_b^*\}_{b \in \beta_W} \subset W^*$ be the sequence of coordinate functionals of the basis β_W .

Let us show that the basis β_W is K -suppression. Given any subset D of β_W we should prove that the projection $\text{pr}_D : W \rightarrow W$, $\text{pr}_D : w \mapsto \sum_{b \in D} e_b^*(w)b$, has norm $\|\text{pr}_D\| \leq K$.

Write the set D as $D = D_Z \cup D_X \cup D_Y$, where $D_Z = D \cap \beta_Z = D \cap \beta_X \cap \beta_Y$, $D_X = D \setminus \beta_Y$ and $D_Y = D \setminus \beta_X$.

Taking into account that the bases β_X and β_Y are K -suppression, for any $w \in W$ we obtain:

$$\begin{aligned} \|\text{pr}_D(w)\|_W &= \inf\{\|x\|_X + \|y\|_Y : x \in X, y \in Y, x + y = \text{pr}_D(w)\} \\ &= \inf\{\|\text{pr}_{D_X}(w) + z'\|_X + \|z'' + \text{pr}_{D_Y}(w)\|_Y : \\ &\quad z', z'' \in Z, z' + z'' = \text{pr}_{D_Z}(w)\} \\ &\leq \inf\{\|\text{pr}_{D_X}(w) + z'\|_X + \|z'' + \text{pr}_{D_Y}(w)\|_Y : \\ &\quad z', z'' \in \text{pr}_{D_Z}(Z), z' + z'' = \text{pr}_{D_Z}(w)\} \end{aligned}$$

$$\begin{aligned}
 &= \inf \{ \|\text{pr}_{D_X}(w) + \text{pr}_{D_Z}(z')\|_X + \|\text{pr}_{D_Z}(z'') + \text{pr}_{D_Y}(w)\|_Y : \\
 &\quad z', z'' \in Z, z' + z'' = \text{pr}_{B_Z}(w) \} \\
 &\leq K \inf \{ \|\text{pr}_{B_X \setminus B_Z}(w) + z'\|_X + \|z' + \text{pr}_{B_Y \setminus B_X}(w)\|_Y : \\
 &\quad z', z'' \in Z, z' + z'' = \text{pr}_{B_Z}(w) \} \\
 &= K \inf \{ \|x\|_X + \|y\|_Y : x + y = w \} = K \|w\|_W.
 \end{aligned}$$

If the finite-dimensional based Banach spaces X and Y are rational, then so is their sum $X \oplus Y$ and so is the quotient space W of $X \oplus Y$. \square

3. \mathfrak{B} -universal based Banach spaces

Definition 3.1. A based Banach space U is defined to be \mathfrak{B} -universal if each based Banach space X is \mathfrak{B} -isomorphic to a based subspace of U .

Definition 3.1 implies that each \mathfrak{B} -universal based Banach space is complementably universal for the class of Banach spaces with unconditional basis. Reformulating Pełczyński’s uniqueness Theorem 3, see [11], we obtain the following uniqueness result.

Theorem 3.2 (Pełczyński). *Any two \mathfrak{B} -universal based Banach spaces are \mathfrak{B} -isomorphic.*

A \mathfrak{B} -universal based Banach space \mathbb{U} was constructed by A. Pełczyński in [11]. In the following sections we shall apply the technique of Fraïssé limits to construct many \mathfrak{B} -isomorphic copies of the Pełczyński’s \mathfrak{B} -universal space \mathbb{U} .

4. $\mathfrak{R}\mathfrak{J}_K$ -universal based Banach spaces

Definition 4.1. A based Banach space X is called $\mathfrak{R}\mathfrak{J}_K$ -universal if for any rational finite-dimensional K -based Banach space A , any isometry morphism $f: \Lambda \rightarrow X$ defined on a based subspace Λ of A can be extended to an isometry morphism $\bar{f}: A \rightarrow X$.

We recall that $\mathfrak{R}\mathfrak{J}_K$ denotes the full subcategory of $\mathfrak{B}\mathfrak{J}$ whose objects are rational finite-dimensional K -based Banach spaces. Obviously, up to isomorphism the category $\mathfrak{R}\mathfrak{J}_K$ contains countably many objects. By Lemma 2.1, the category $\mathfrak{R}\mathfrak{J}_K$ has the amalgamation property. We now use the concepts from [8] for constructing a “generic” sequence in $\mathfrak{R}\mathfrak{J}_K$.

A sequence $(X_n)_{n \in \omega}$ of objects of the category $\mathfrak{B}\mathfrak{J}_K$ is called a *chain* if each K -based Banach space X_n is a subspace of the K -based Banach space X_{n+1} .

Definition 4.2. A chain of $(U_n)_{n \in \omega}$ of objects of the category $\mathfrak{R}\mathfrak{J}_K$ is *Fraïssé* if for any $n \in \omega$ and $\mathfrak{R}\mathfrak{J}_K$ -morphism $f: U_n \rightarrow Y$ there exist $m > n$ and an $\mathfrak{R}\mathfrak{J}_K$ -morphism $g: Y \rightarrow U_m$ such that $g \circ f: U_n \rightarrow U_m$ is the identity inclusion of U_n to U_m .

Definition 4.2 implies that the Fraïssé sequence $\{U_n\}_{n \in \omega}$ is cofinal in the category $\mathfrak{R}\mathfrak{J}_K$ in the sense that each object A of the category $\mathfrak{F}\mathfrak{J}_K$ admits an $\mathfrak{R}\mathfrak{J}_K$ -morphism $A \rightarrow U_n$ for some $n \in \omega$. This means that the category $\mathfrak{R}\mathfrak{J}_K$ is countably cofinal.

The name “Fraïssé sequence”, as in [8], is motivated by the model-theoretic theory of Fraïssé limits developed by R. Fraïssé in [3]. One of the results in [8] is that every countably cofinal category with amalgamation has a Fraïssé sequence. Applying this general result to our category $\mathfrak{R}\mathfrak{J}_K$ we get:

Theorem 4.3 ([8], Theorem 3.7). *The category $\mathfrak{R}\mathfrak{J}_K$ has a Fraïssé sequence.*

From now on, we fix a Fraïssé sequence $(U_n)_{n \in \omega}$ in $\mathfrak{R}\mathfrak{J}_K$, which can be assumed to be a chain of finite-dimensional rational K -based Banach spaces. Let \mathbb{U}_K be the completion of the union $\bigcup_{n \in \omega} U_n$ and $\mathfrak{B}_{\mathbb{U}_K} = \bigcup_{n \in \omega} \mathfrak{B}_{U_n} \subset \mathbb{U}_K$.

Theorem 4.4. *The pair $(\mathbb{U}_K, \mathfrak{B}_{\mathbb{U}_K})$ is an $\mathfrak{R}\mathfrak{J}_K$ -universal rational K -based Banach space.*

PROOF: First we show that $\mathfrak{B}_{\mathbb{U}_K} = \bigcup_{n \in \omega} \mathfrak{B}_{U_n}$ is a normalized K -suppression basis for \mathbb{U}_K . The fact that $\mathfrak{B}_{\mathbb{U}_K}$ is an unconditional Schauder basis with suppression constant K follows from Lemma 6.2 and Fact 6.3 in [2]. For each n the spaces U_n are K -based Banach spaces, so $\|b\| = 1$ for every $b \in \mathfrak{B}_{U_n}$. This shows that the basis $\mathfrak{B}_{\mathbb{U}_K}$ is normalized.

The based Banach space $(\mathbb{U}_K, \mathfrak{B}_{\mathbb{U}_K})$ is rational, since each finite-dimensional based subspace of $(\mathbb{U}_K, \mathfrak{B}_{\mathbb{U}_K})$ is contained in some rational based Banach space $(U_n, \mathfrak{B}_{U_n})$ and hence is rational.

The $\mathfrak{R}\mathfrak{J}_K$ -universality of the based Banach space $(\mathbb{U}_K, \mathfrak{B}_{\mathbb{U}_K})$ follows from the construction and [8, Proposition 3.1]. □

To shorten notation, let \mathbb{U}_K is the $\mathfrak{R}\mathfrak{J}_K$ -universal rational K -based Banach space $(\mathbb{U}_K, \mathfrak{B}_{\mathbb{U}_K})$. The following theorem shows that such space is unique up to $\mathfrak{B}\mathfrak{J}$ -isomorphism.

Theorem 4.5. *Any $\mathfrak{R}\mathfrak{J}_K$ -universal rational K -based Banach spaces X, Y are $\mathfrak{B}\mathfrak{J}$ -isomorphic, which means that there exists a linear bijective isometry $X \rightarrow Y$ preserving the bases of X and Y .*

PROOF: By definition, the rational K -based Banach spaces X, Y can be written as the completions of unions $\bigcup_{n \in \omega} X_n$ and $\bigcup_{n \in \omega} Y_n$ of chains $(X_n)_{n \in \omega}$ and $(Y_n)_{n \in \omega}$ of rational finite-dimensional K -based Banach spaces such that $X_0 = \{0\}$ and $Y_0 = \{0\}$ are trivial K -based Banach spaces.

We define inductively sequences of $\mathfrak{R}\mathfrak{J}_K$ -morphisms $\{f_k\}_{k \in \omega}, \{g_k\}_{k \in \omega}$ and increasing number sequences $(n_k), (m_k)$ such that the following conditions are satisfied for every $k \in \omega$:

- (1) $f_k: X_{n_{k-1}} \rightarrow Y_{m_k}$ and $g_k: Y_{m_k} \rightarrow X_{n_k}$ are morphisms of category $\mathfrak{R}\mathfrak{J}_K$;
- (2) $f_{k+1} \circ g_k = \text{id} \upharpoonright Y_{m_k}$ and $g_{k+1} \circ f_{k+1} = \text{id} \upharpoonright X_{n_k}$.

We start the inductive construction letting $n_0 = 0 = m_0$ and $f_0: X_0 \rightarrow Y_0$, $g_0: Y_0 \rightarrow X_0$ be the unique isomorphisms of the trivial K -based Banach spaces X_0 and Y_0 . To make an inductive step, assume that for some $k \in \omega$, the numbers n_k, m_k and $\mathfrak{R}\mathfrak{J}_K$ -morphisms $f_k: X_{n_{k-1}} \rightarrow Y_{m_k}$, $g_k: Y_{m_k} \rightarrow X_{n_k}$ have been constructed. By Definition 4.1, the $\mathfrak{B}\mathfrak{J}$ -morphism $g_k^{-1}: g_k(Y_{m_k}) \rightarrow Y$ defined on the based subspace $g_k(Y_{m_k})$ of the rational finite-dimensional K -based Banach space X_{n_k} extends to a $\mathfrak{B}\mathfrak{J}$ -morphism $f_{k+1}: X_{n_k} \rightarrow Y$. So, $f_{k+1} \circ g_k = \text{id} \upharpoonright Y_{m_k}$. Since $f_{k+1}(\beta_{X_{n_k}}) \subset \beta_Y = \bigcup_{i \in \omega} \beta_{Y_i}$, there exists a number m_{k+1} such that $f_{k+1}(\beta_{X_{n_k}}) \subset \beta_{Y_{m_{k+1}}}$ and hence $f_{k+1}(X_{n_k}) \subset Y_{m_{k+1}}$. Since the based space Y is rational, its based subspace $Y_{m_{k+1}}$ is an object of the category $\mathfrak{R}\mathfrak{J}_K$ and the morphism $f_{k+1}: X_{n_k} \rightarrow Y_{m_{k+1}}$ is an $\mathfrak{R}\mathfrak{J}_K$ -morphism.

By analogy we can use the $\mathfrak{R}\mathfrak{J}_K$ -universality of the based Banach space X to find a number $n_{k+1} > n_k$ and an $\mathfrak{R}\mathfrak{J}_K$ -morphism $g_{k+1}: Y_{m_{k+1}} \rightarrow X_{n_{k+1}}$ such that $g_{k+1} \circ f_{k+1}$ is the identity inclusion X_{n_k} in $X_{n_{k+1}}$. This completes the inductive step.

After completing the inductive construction consider two isometries $f: \bigcup_{n \in \omega} X_n \rightarrow \bigcup_{m \in \omega} Y_m$ and $g: \bigcup_{m \in \omega} Y_m \rightarrow \bigcup_{n \in \omega} X_n$ such that $f \upharpoonright X_{n_k} = f_{k+1}$ and $g \upharpoonright Y_{m_k} = g_k$ for every $k \in \omega$.

By the uniform continuity, the isometries f, g extend to isometries $\bar{f}: X \rightarrow Y$ and $\bar{g}: Y \rightarrow X$.

The condition (2) of the inductive construction implies that $\bar{f} \circ \bar{g} = \text{id}_Y$ and $\bar{g} \circ \bar{f} = \text{id}_X$, so f and g are isometric isomorphisms of the Banach spaces X and Y . Since the isometries $g_k: Y_{m_k} \rightarrow X_{n_k}$ are morphisms of based Banach spaces, we get

$$g(\beta_Y) = g\left(\bigcup_{k \in \omega} \beta_{Y_{m_k}}\right) = \bigcup_{k \in \omega} g(\beta_{Y_{m_k}}) = \bigcup_{k \in \omega} g_k(\beta_{Y_{m_k}}) \subset \bigcup_{k \in \omega} \beta_{X_{n_k}} = \beta_X.$$

By analogy we can show that $f(\beta_X) \subset \beta_Y$. So, f and g are $\mathfrak{B}\mathfrak{J}$ -isomorphisms. \square

5. Almost $\mathfrak{F}\mathfrak{J}_K$ -universality

By analogy with the $\mathfrak{R}\mathfrak{J}_K$ -universal based Banach space, one can try to introduce a $\mathfrak{F}\mathfrak{J}_K$ -universal based Banach space. However such notion is vacuous as each based Banach space has only countably many finite-dimensional based subspaces whereas the category $\mathfrak{F}\mathfrak{J}_K$ contains continuum many pairwise non $\mathfrak{B}\mathfrak{J}$ -isomorphic 2-dimensional based Banach spaces. A “right” definition is that of an almost $\mathfrak{F}\mathfrak{J}_K$ -universal based Banach space, introduced with the help of ε -isometries.

For a positive real number ε , a linear operator $f: X \rightarrow Y$ between Banach spaces X and Y is called an ε -isometry if

$$(1 + \varepsilon)^{-1} \|x\|_X < \|f(x)\|_Y < (1 + \varepsilon) \|x\|_X$$

for every $x \in X \setminus \{0\}$. This definition implies that each ε -isometry is an injective linear operator.

A morphism of the category \mathfrak{B} of based Banach spaces is called an ε -isometry \mathfrak{B} -morphism if it is an ε -isometry of the underlying Banach spaces.

Definition 5.1. A based Banach space X called *almost \mathfrak{J}_K -universal* if for any $\varepsilon > 0$ and finite dimensional K -based Banach space A , any ε -isometry \mathfrak{B} -morphism $f: \Lambda \rightarrow X$ defined on a based subspace Λ of A can be extended to an ε -isometry \mathfrak{B} -morphism $\bar{f}: A \rightarrow X$.

Theorem 5.2. Any $\mathfrak{R}\mathfrak{J}_K$ -universal rational K -based Banach space X is almost \mathfrak{J}_K -universal.

PROOF: We shall use the fact that the norm of any finite-dimensional based Banach space can be approximated by a rational norm (which means that its unit ball coincides with the convex hull of finitely many points having rational coordinates in the basis).

To prove that X is almost \mathfrak{J}_K -universal, take any $\varepsilon > 0$, any finite-dimensional K -based Banach space A and an ε -isometry \mathfrak{B} -morphism $f: \Lambda \rightarrow X$ defined on a based subspace Λ of A . We recall that by $\|\cdot\|_A$ and $\|\cdot\|_\Lambda$ we denote the norms of the Banach spaces A and Λ . The morphism f determines a new norm $\|\cdot\|'_\Lambda$ on Λ , defined by $\|a\|'_\Lambda = \|f(a)\|_X$ for $a \in \Lambda$. Since X is rational and K -based, $\|\cdot\|'_\Lambda$ is a rational norm on Λ such that $\|\text{pr}_F(a)\|'_\Lambda \leq K\|a\|'_\Lambda$ for every $a \in \Lambda$ and every subset $F \subset \mathfrak{B}_\Lambda$. Taking into account that f is an ε -isometry, we conclude that $(1 + \varepsilon)^{-1} < \|a\|'_\Lambda < (1 + \varepsilon)$ for every $a \in \Lambda$ with $\|a\|_\Lambda = 1$. By the compactness of the unit sphere in Λ , there exists a positive $\delta < \varepsilon$ such that $(1 + \delta)^{-1} < \|a\|'_\Lambda < (1 + \delta)$ for every $a \in \Lambda$ with $\|a\|_\Lambda = 1$. This inequality implies $(1 + \delta)^{-1}B_\Lambda \subset B'_\Lambda \subset (1 + \delta)B_\Lambda$, where $B_\Lambda = \{a \in \Lambda: \|a\|_\Lambda \leq 1\}$ and $B'_\Lambda = \{a \in \Lambda: \|a\|'_\Lambda \leq 1\}$ are the closed unit balls of Λ in the norms $\|\cdot\|_\Lambda$ and $\|\cdot\|'_\Lambda$. Choose δ' such that $\delta < \delta' < \varepsilon$. Also choose a nonnegative real number $c \leq K - 1$ such that $K - c$ is rational and $K/(K - c) < 1 + \delta$.

Let $B_A = \{a \in A: \|a\|_A \leq 1\}$ be the closed unit ball of the Banach space A . Choose a rational polyhedron P in A such that $P = -P$ and $(1 + \delta')^{-1}B_A \subset P \subset (1 + \delta)^{-1}B_A$. Next consider the convex hull $B'_A := \text{conv}(P')$ of the set $P' = B'_\Lambda \cup P \cup \bigcup_{F \subset \mathfrak{B}_A} (K - c)^{-1} \text{pr}_F(P)$ and observe that B'_A is a rational polyhedron in the based Banach space A . Taking into account that $P \subset (1 + \delta)^{-1}B_A$, $B'_\Lambda \subset (1 + \delta)B_\Lambda \subset (1 + \delta)B_A$, and A is a K -based Banach space, we conclude that

$$\begin{aligned} P' &\subset B'_\Lambda \cup \frac{1}{1 + \delta} \left(B_A \cup \bigcup_{F \subset \mathfrak{B}_A} \frac{1}{K - c} \text{pr}_F(B_A) \right) = B'_\Lambda \cup \frac{1}{1 + \delta} \left(B_A \cup \frac{K}{K - c} B_A \right) \\ &= B'_\Lambda \cup \frac{1}{1 + \delta} \frac{K}{K - c} B_A \subset (1 + \delta)B_\Lambda \cup \frac{1}{1 + \delta} (1 + \delta)B_A \subset (1 + \delta)B_A \end{aligned}$$

and hence

$$\frac{1}{1 + \delta'} B_A \subset P \subset B'_A := \text{conv}(P') \subset (1 + \delta)B_A.$$

The convex symmetric set $B'_A := \text{conv}(P')$ determines a rational norm $\|\cdot\|'_A$ on A whose unit ball coincides with B'_A . We claim that the base β_A of the Banach space $A' := (A, \|\cdot\|'_A)$ is K -suppression. Indeed, for any set $F \subset \beta_A$ we have

$$\begin{aligned} \text{pr}_F(P') &= \text{pr}_F(B'_\Lambda) \cup \text{pr}_F(P) \cup \bigcup_{E \subset \beta_A} \frac{1}{K-c} \text{pr}_F \circ \text{pr}_E(P) \\ &\subset K B'_\Lambda \cup (K-c)P' \cup P' \subset K P' \end{aligned}$$

and hence

$$\begin{aligned} \text{pr}_F(B'_A) &= \text{pr}_F(\text{conv}(P')) = \text{conv}(\text{pr}_F(P')) \\ &\subset \text{conv}(K P') = K \text{conv}(P') = K B'_A, \end{aligned}$$

which means that the projection $\text{pr}_F: A' \rightarrow A'$ has norm less than or equal to K and A' is a K -based Banach space.

It remains to check that $\|a\|'_A = \|a\|'_\Lambda$ for each $a \in \Lambda$, which is equivalent to the equality $B'_A \cap \Lambda = B'_\Lambda$. The inclusion $B'_\Lambda \subset B'_A \cap \Lambda$ is evident. To prove the reverse inclusion $B'_\Lambda \supset B'_A \cap \Lambda$ observe that

$$\begin{aligned} \Lambda \cap B'_A &= \Lambda \cap \text{conv}(P') \subset \Lambda \cap \text{conv}\left(B'_\Lambda \cup \frac{1}{1+\delta} B_A\right) \\ &= \Lambda \cap \left\{ t\lambda + (1-t)a : t \in [0, 1], \lambda \in B'_\Lambda, a \in \frac{1}{1+\delta} B_A \right\} \\ &= \left\{ t\lambda + (1-t)a : t \in [0, 1], \lambda \in B'_\Lambda, a \in \frac{1}{1+\delta} (\Lambda \cap B_A) \right\} \\ &\subset \text{conv}(B'_\Lambda \cup B'_\Lambda) = B'_\Lambda. \end{aligned}$$

The inclusions $(1 + \delta')^{-1} B_A \subset B'_A \subset (1 + \delta) B_A$ imply the strict inequality

$$(1) \quad (1 + \varepsilon)^{-1} \|a\|_A < \|a\|'_A < (1 + \varepsilon) \|a\|_A$$

holding for all $a \in A \setminus \{0\}$.

Let Λ' and A' be the K -based Banach spaces Λ and A endowed with the new rational norms $\|\cdot\|'_\Lambda$ and $\|\cdot\|'_A$, respectively. It is clear that $\Lambda' \subset A'$. The definition of the norm $\|\cdot\|'_\Lambda$ ensures that $f: \Lambda' \rightarrow X$ is a $\mathfrak{B}\mathfrak{J}$ -morphism. Using the $\mathfrak{R}\mathfrak{J}_K$ -universality of X , extend the isometry morphism $f: \Lambda' \rightarrow X$ to an isometry morphism $\bar{f}: A' \rightarrow X$. The inequalities (1) ensure that $\bar{f}: A \rightarrow X$ is an ε -isometry \mathfrak{B} -morphism from A , extending the ε -isometry f . This completes the proof of the almost $\mathfrak{J}\mathfrak{J}_K$ -universality of X . \square

Theorem 5.3. *Let X and Y be almost $\mathfrak{J}\mathfrak{J}_K$ -universal K -based Banach spaces and $\varepsilon > 0$. Each ε -isometry \mathfrak{B} -morphism $f: X_0 \rightarrow Y$ defined on a finite-dimensional based subspace X_0 of the K -based Banach space X can be extended to an ε -isometry \mathfrak{B} -isomorphism $\bar{f}: X \rightarrow Y$.*

PROOF: Fix a positive real number ε . Using the compactness of the unit sphere of the finite dimensional Banach space X_0 , we can find a positive $\delta < \varepsilon$ such that f is a δ -isometry. Write X and Y as the completions of the unions $\bigcup_{n \in \omega} X_n$ and $\bigcup_{n \in \omega} Y_n$ of chains of finite dimensional K -based Banach spaces such that $Y_0 = f(X_0)$. We define inductively sequences of \mathfrak{B} -morphisms $\{f_k\}_{k \in \omega}$, $\{g_k\}_{k \in \omega}$ and increasing number sequences (n_k) , (m_k) such that $n_0 = m_0 = 0$, $f_0 = f$ and the following conditions are satisfied for every $k \in \omega$:

- (1) $f_k: X_{n_{k-1}} \rightarrow Y_{m_k}$ and $g_k: Y_{m_k} \rightarrow X_{n_k}$ are δ -isometry \mathfrak{B} -morphisms;
- (2) $f_{k+1} \circ g_k = \text{id} \upharpoonright Y_{m_k}$ and $g_{k+1} \circ f_{k+1} = \text{id} \upharpoonright X_{n_k}$.

To make the inductive step assume that for some $k \in \omega$, the numbers n_k , m_k and δ -isometries $f_k: X_{n_{k-1}} \rightarrow Y_{m_k}$, $g_k: Y_{m_k} \rightarrow X_{n_k}$ have been constructed. Definition 5.1 of almost \mathfrak{FJ}_K -universality of the based Banach space Y yields a δ -isometry \mathfrak{B} -morphism $f_{k+1}: X_{n_k} \rightarrow Y$ such that $f_{k+1} \upharpoonright g_k(Y_{m_k}) = g_k^{-1} \upharpoonright g_k(Y_{m_k})$ and hence $f_{k+1} \circ g_k = \text{id} \upharpoonright Y_{m_k}$. Since $f_{k+1}(\beta_{X_{n_k}})$ is a finite subset of the basis $\beta_Y = \bigcup_{i \in \omega} \beta_{Y_i}$ of Y , there exists a number $m_{k+1} > m_k$ such that $f_{k+1}(\beta_{X_{n_k}}) \subset \beta_{Y_{m_{k+1}}}$ and hence $f_{k+1}(X_{n_k}) \subset Y_{m_{k+1}}$.

By analogy, we can use the almost \mathfrak{FJ}_K -universality of the based Banach space X and find a number $n_{k+1} > n_k$ and a δ -isometry \mathfrak{B} -morphism $g_{k+1}: Y_{m_{k+1}} \rightarrow X_{n_{k+1}}$ such that $g_{k+1} \circ f_{k+1} = \text{id} \upharpoonright X_{n_k}$. This completes the inductive step.

After completing the inductive construction consider two δ -isometries $\tilde{f}: \bigcup_{n \in \omega} X_n \rightarrow \bigcup_{m \in \omega} Y_m$ and $\tilde{g}: \bigcup_{m \in \omega} Y_m \rightarrow \bigcup_{n \in \omega} X_n$ such that for every $k \in \omega$, $\tilde{f} \upharpoonright X_{n_k} = f_{k+1}$ and $\tilde{g} \upharpoonright Y_{m_k} = g_k$. The condition (2) of the inductive construction implies that $\tilde{f} \circ \tilde{g}$ and $\tilde{g} \circ \tilde{f}$ are the identity maps of $\bigcup_{n \in \omega} X_n$ and $\bigcup_{m \in \omega} Y_m$, respectively.

Using the uniform continuity, the δ -isometries \tilde{f} , \tilde{g} extend to ε -isometries $\bar{f}: X \rightarrow Y$ and $\bar{g}: Y \rightarrow X$ such that $\bar{f} \circ \bar{g} = \text{id}_Y$ and $\bar{g} \circ \bar{f} = \text{id}_X$. Taking into account that f_n and g_n are \mathfrak{B} -morphisms, we can show (repeating the argument from the proof of Theorem 4.5) that the operators \tilde{f} and \tilde{g} preserve the bases of the K -based Banach spaces X and Y and hence are \mathfrak{B} -isomorphisms. \square

Corollary 5.4. *For any almost \mathfrak{FJ}_K -universal K -based Banach spaces X and Y and any $\varepsilon > 0$ there exists an ε -isometry \mathfrak{B} -isomorphism $f: X \rightarrow Y$.*

Theorem 5.5. *Let U be an almost \mathfrak{FJ}_K -universal K -based Banach space. For any $\varepsilon > 0$ and any K -based Banach space X there exists an ε -isometry \mathfrak{B} -morphism $f: X \rightarrow U$.*

PROOF: Write X as the completion of the union $\bigcup_{n \in \omega} X_n$ of a chain of finite dimensional K -based Banach subspaces X_n of X such that $X_0 = \{0\}$. Fix a positive real number ε and choose any $\delta < \varepsilon$. We shall define inductively a sequence of δ -isometry \mathfrak{B} -morphisms $(f_k: X_k \rightarrow U)_{k=0}^\infty$ such that $f_k \upharpoonright X_{k-1} = f_{k-1}$ for every $k > 0$.

We set $f_0 = 0$. Suppose that for some $k \in \omega$ a δ -isometry \mathfrak{B} -morphism $f_k: X_k \rightarrow U$ has already been constructed. Using the definition of the almost \mathfrak{J}_K -universality of the space U , we can find a δ -isometry \mathfrak{B} -morphism $f_{k+1}: X_{k+1} \rightarrow U$ such that $f_{k+1} \upharpoonright X_k = f_k$. This completes the inductive step.

After completing the inductive construction consider the δ -isometry f such that $f \upharpoonright X_k = f_k$ for every $k \in \omega$; $f: \bigcup_{k \in \omega} X_k \rightarrow U$.

By the uniform continuity, the δ -isometry f extends to an ε -isometry $\bar{f}: X \rightarrow U$ such that

$$f(\mathfrak{B}_X) = f\left(\bigcup_{k \in \omega} B_{X_k}\right) = \bigcup_{k \in \omega} f(B_{X_k}) = \bigcup_{k \in \omega} f_k(B_{X_k}) \subset \mathfrak{B}_U,$$

which means that f is a \mathfrak{B}_K -morphism. □

Corollary 5.6. *Each almost \mathfrak{J}_K -universal K -based Banach space U is \mathfrak{B} -universal.*

PROOF: Given a based Banach space X , we need to prove that X is \mathfrak{B} -isomorphic to a based subspace of U . Denote by X_1 the based Banach space X endowed with the equivalent norm

$$\|x\|_1 = \sup_{F \subset \mathfrak{B}_X} \|\text{pr}_F(x)\|.$$

It is easy to check that X_1 is a 1-based Banach space. By Theorem 5.5, for $\varepsilon = 1/2$ there exists an ε -isometry \mathfrak{B} -morphism $f: X_1 \rightarrow U$. Then f is a \mathfrak{B} -isomorphism between X and the based subspace $f(X) = f(X_1)$ of the based Banach space U . □

Corollary 5.6 combined with the Uniqueness Theorem 3.2 of Pełczyński implies

Corollary 5.7. *Each almost \mathfrak{J}_K -universal K -based Banach space U_K is \mathfrak{B} -isomorphic to the \mathfrak{B} -universal space \mathbb{U} of Pełczyński.*

Combining Corollary 5.7 with Theorem 5.2, we get another model of the \mathfrak{B} -universal Pełczyński's space \mathbb{U} .

Corollary 5.8. *Each \mathfrak{R}_K -universal rational K -based Banach space U_K is \mathfrak{B} -isomorphic to the \mathfrak{B} -universal Pełczyński's space \mathbb{U} .*

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