

## Classification of spaces of continuous functions on ordinals

LEONID V. GENZE, SERGEI P. GUL'KO, TAT'ANA E. KHYMLEVA

*Abstract.* We conclude the classification of spaces of continuous functions on ordinals carried out by Górak [Górak R., *Function spaces on ordinals*, Comment. Math. Univ. Carolin. **46** (2005), no. 1, 93–103]. This gives a complete topological classification of the spaces  $C_p([0, \alpha])$  of all continuous real-valued functions on compact segments of ordinals endowed with the topology of pointwise convergence. Moreover, this topological classification of the spaces  $C_p([0, \alpha])$  completely coincides with their uniform classification.

*Keywords:* space of continuous functions; pointwise topology; homeomorphism of function spaces; uniform homeomorphism; ordinal number

*Classification:* 54C35

### 1. Introduction

Our terminology basically follows [4]. In particular, we understand cardinals as initial ordinals, compare [4, page 6]. A segment of the ordinals  $[0, \alpha]$  is endowed with a standard order topology. The symbol  $C_p([0, \alpha])$  denotes the set of all continuous real-valued functions defined on  $[0, \alpha]$  and endowed with the topology of pointwise convergence.

A complete linear topological classification of Banach spaces  $C([0, \alpha])$  was carried out in [7] and independently in [8] (for the initial part of this classification, see also [3] and [9]). Similar complete linear topological classification for  $C_p([0, \alpha])$  can be found in [6], [2].

The topological classification of the spaces  $C_p([0, \alpha])$  is carried out in the R. Górak's paper [5], in which the question whether the spaces  $C_p([0, \alpha])$  and  $C_p([0, \beta])$  are homeomorphic is solved for all ordinals  $\alpha$  and  $\beta$  with except for the case  $\alpha = k^+ \cdot k$ ,  $\beta = k^+ \cdot k^+$ , where  $k$  is the initial ordinal, and  $k^+$  is the smallest initial ordinal greater than  $k$ . We note that an ordinal of the form  $k^+$  is always regular ordinal. In this paper we prove the following theorem.

**Theorem 1.** *Let  $\tau$  be an arbitrary initial regular ordinal,  $\sigma$  and  $\lambda$  be initial ordinals satisfying the inequality  $\omega \leq \sigma < \lambda \leq \tau$ . Then the space  $C_p([0, \tau \cdot \sigma])$  is not homeomorphic to the space  $C_p([0, \tau \cdot \lambda])$ .*

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If we combine this result with the results of [5], we get a complete topological classification of the spaces  $C_p([0, \alpha])$  (which coincides with the uniform classification). We can write it in the form of the following theorem.

**Theorem 2.** *Let  $\alpha$  and  $\beta$  be ordinals and  $\alpha \leq \beta$ .*

- (a) *If  $|\alpha| \neq |\beta|$ , then  $C_p([0, \alpha])$  and  $C_p([0, \beta])$  are not homeomorphic.*
- (b) *If  $\tau$  is an initial ordinal,  $|\alpha| = |\beta| = \tau$  and either  $\tau = \omega$  or  $\tau$  is a singular ordinal or  $\beta \geq \alpha \geq \tau^2$ , then the spaces  $C_p([0, \alpha])$  and  $C_p([0, \beta])$  are (uniformly) homeomorphic.*
- (c) *If  $\tau$  is a regular uncountable ordinal and  $\alpha, \beta \in [\tau, \tau^2]$ , then the space  $C_p([0, \alpha])$  is (uniformly) homeomorphic to the space  $C_p([0, \beta])$  if and only if  $\tau \cdot \sigma \leq \alpha \leq \beta < \tau \cdot \sigma^+$ , where  $\sigma$  is the initial ordinal,  $\sigma < \tau$ , and  $\sigma^+$  is the smallest initial ordinal, exceeding  $\sigma$ .*

**2. Proof of Theorem 1**

We need some notation and auxiliary statements. For an arbitrary ordinal  $\alpha$  and the initial ordinal  $\lambda \leq \alpha$  we set

$$A_{\lambda, \alpha} = \{t \in [0, \alpha] : \chi(t) = |\lambda|\},$$

where  $\chi(t)$  is the character of the point  $t \in [0, \alpha]$ . In particular,  $A_{\omega, \alpha}$  is the set of all limit points of  $t \in [0, \alpha]$ , having a countable base of neighborhoods.

Let  $\alpha$  be a limit ordinal. The smallest order type of sets  $A \subset [0, \alpha]$  cofinal in  $[0, \alpha)$ , is called *cofinality* of the ordinal  $\alpha$  and denoted by  $cf(\alpha)$ .

It is easy to see that  $|cf(\alpha)| = \chi(\alpha)$  for the limit ordinal  $\alpha$ . The initial ordinal  $\alpha$  is called *regular* if  $cf(\alpha) = \alpha$ . Otherwise, the initial ordinal is called *singular*.

The symbol  $D(x)$  denotes the set of points of discontinuity of the function  $x$ .

The proof of the following two lemmas is standard (see Example 3.1.27 in [4]).

**Lemma 1.** *Let  $\alpha$  be an arbitrary ordinal and let  $\tau$  be an initial ordinal such that  $\omega < \tau \leq \alpha$ ,  $t_0 \in A_{\tau, \alpha}$  and a function  $x : [0, \alpha] \rightarrow \mathbb{R}$  is continuous at all points of the set  $A_{\omega, \alpha}$ . Then there is an ordinal  $\gamma < t_0$  such that  $x|_{(\gamma, t_0)} = \text{const}$ .*

**Lemma 2.** *If a function  $x : [0, \alpha] \rightarrow \mathbb{R}$  is continuous at all points of the set  $A_{\omega, \alpha}$ , then the set  $D(x)$  is at most countable.*

For the function  $x \in \mathbb{R}^{[0, \alpha]}$  and the initial ordinal  $\lambda \leq \alpha$  the symbol  $G_\lambda(x)$  denotes the family

$$G_\lambda(x) = \left\{ \bigcap_{s \in S} V_s : V_s \text{ is standard neighborhoods of } x \text{ in } \mathbb{R}^{[0, \alpha]} \text{ and } |S| = |\lambda| \right\}.$$

The elements of the family  $G_\lambda(x)$  will be called  $\lambda$ -neighborhoods of the function  $x$ .

For a regular ordinal  $\tau \geq \omega_1$  and an initial ordinal  $\sigma \leq \tau$  we put

$$M_{\tau\sigma} = \{x \in \mathbb{R}^{[0, \tau \cdot \sigma]} : x \text{ is continuous at those points } t \in [0, \tau \cdot \sigma] \text{ for which } \text{cf}(t) < \tau\}.$$

It is clear that  $C([0, \tau \cdot \sigma]) \subset M_{\tau\sigma}$ .

**Lemma 3.** *Let  $\tau \geq \omega_1$  be an initial regular ordinal and let  $\sigma$  be an initial ordinal such that  $\sigma \leq \tau$ . Then*

$$M_{\tau\sigma} = \{x \in \mathbb{R}^{[0, \tau \cdot \sigma]} : V \cap C_p([0, \tau \cdot \sigma]) \neq \emptyset \text{ for every } V \in G_\lambda(x) \text{ and each } \lambda < \tau\}.$$

PROOF: We denote by  $L_{\tau\sigma}$  the right-hand side of the equality and assume that  $x \notin M_{\tau\sigma}$ , that is,  $x$  is discontinuous at some point  $t_0$  for which  $\text{cf}(t_0) < \tau$ . Since  $|\text{cf}(t_0)| = \chi(t_0)$ , there exists a base  $\{U_j(t_0)\}_{j \in J}$  of neighborhoods of the point  $t_0$  such that  $|J| < \tau$ . Since  $x$  is discontinuous at  $t_0$ , there exists a number  $\varepsilon_0 > 0$  such that for each  $j \in J$  there is a point  $t_j \in U_j(t_0)$  such that  $|x(t_j) - x(t_0)| \geq \varepsilon_0$ . Let  $V = \bigcap \{V(x, t_j, t_0, 1/n) : j \in J, n \in \mathbb{N}\}$ , where  $V(x, t_j, t_0, 1/n)$  is the standard neighborhood of the function  $x$  in the space  $\mathbb{R}^{[0, \tau \cdot \sigma]}$ . If  $y \in V$ , then  $y(t_j) = x(t_j)$  and  $y(t_0) = x(t_0)$ . Hence, the function  $y$  is discontinuous at the point  $t_0$  and then  $y \notin C_p([0, \tau \cdot \sigma])$ . Thus,  $V \cap C_p([0, \tau \cdot \sigma]) = \emptyset$ , that is,  $x \notin L_{\tau\sigma}$ .

Now let  $x \in M_{\tau\sigma}$ , i.e. the function  $x$  can be discontinuous only at the points of the set  $A_{\tau, \tau \cdot \sigma}$ . It is easy to see that the set  $A_{\tau, \tau \cdot \sigma}$  has the form

$$A_{\tau, \tau \cdot \sigma} = \{\tau \cdot (\xi + 1) : 0 \leq \xi < \sigma\}, \text{ or} \\ A_{\tau, \tau \cdot \sigma} = \{\tau \cdot (\xi + 1) : 0 \leq \xi < \tau\} \cup \{\tau \cdot \tau\} \quad \text{if } \sigma = \tau.$$

By Lemma 2, the set  $D(x)$  is at most countable and therefore

$$A_{\tau, \tau \cdot \sigma} \cap D(x) = \{\tau \cdot (\xi_n + 1) : \xi_n < \sigma, n \in \mathbb{N}\}, \text{ or} \\ A_{\tau, \tau \cdot \sigma} \cap D(x) = \{\tau \cdot (\xi_n + 1) : \xi_n < \tau, n \in \mathbb{N}\} \cup \{\tau \cdot \tau\} \quad \text{if } \sigma = \tau.$$

Let  $\lambda < \tau$  and  $V(x) = \bigcap \{U(x, \eta, 1/n) : \eta \in S, n \in \mathbb{N}\}$  be a  $\lambda$ -neighbourhood of the point  $x$ . Then  $|S| < |\tau|$ .

Since the countable set  $A_{\tau, \tau \cdot \sigma} \cap D(x)$  is not cofinal in the regular ordinal  $\tau \geq \omega_1$ , for each  $n \in \mathbb{N}$  there is an ordinal  $\gamma_n$  such that  $\tau\xi_n < \gamma_n < \tau(\xi_n + 1)$  and  $(\gamma_n, \tau(\xi_n + 1)) \cap S = \emptyset$ . In the case  $\sigma = \tau$  there is also an ordinal  $\gamma_0 < \tau^2$ , such that  $(\gamma_0, \tau^2) \cap S = \emptyset$  and  $(\gamma_0, \tau^2) \cap \{\tau(\xi_n + 1)\}_{n=1}^\infty = \emptyset$ .

Consider the function

$$\tilde{x}(t) = \begin{cases} x(\tau(\xi_n + 1)) & \text{if } t \in (\gamma_n, \tau(\xi_n + 1)); \\ x(\tau^2) & \text{if } t \in (\gamma_0, \tau^2); \\ x(t) & \text{otherwise.} \end{cases}$$

It is not difficult to see that the function  $\tilde{x}$  is continuous at all points  $t \in [0, \tau \cdot \sigma]$ , and since  $\tilde{x}|_S = x|_S$ ,  $\tilde{x} \in V(x)$ , that is,  $V(x) \cap C_p([0, \tau \cdot \sigma]) \neq \emptyset$  and therefore  $x \in L_{\tau\sigma}$ . □

If  $X$  is a Tychonoff space, then the symbol  $\nu X$  denotes the Hewitt completion of the space  $X$ . The proof of the following lemma can be found in [4, page 218].

**Lemma 4.** *If  $\varphi: X \rightarrow Y$  is a homeomorphism of Tychonoff spaces, then there exists a homeomorphism  $\tilde{\varphi}: \nu X \rightarrow \nu Y$  such that  $\tilde{\varphi}(x) = \varphi(x)$  for each  $x \in X$ .*

**Lemma 5.** *Let  $\alpha$  be an arbitrary ordinal. Then*

$$\nu(C_p([0, \alpha])) = \{x \in \mathbb{R}^{[0, \alpha]} : x \text{ is continuous at all points of the set } A_{\omega, \alpha}\}.$$

PROOF: It is known (see [10, page 382]) that for an arbitrary Tychonoff space  $X$  the space  $\nu(C_p(X))$  coincides with the set of all strictly  $\aleph_0$ -continuous functions from  $X$  to  $\mathbb{R}$ . In this case, the function  $f \in \mathbb{R}^X$  is called strictly  $\aleph_0$ -continuous (see [1]) if for any countable set  $A \subset X$  there is a continuous function  $g \in \mathbb{R}^X$  such that  $f|_A = g|_A$ .

Since for each countable set  $A \subset [0, \alpha]$ , its closure  $\bar{A}$  is also countable, by the Tietze-Urysohn theorem we obtain that the set of all strictly  $\aleph_0$ -continuous functions in  $[0, \alpha]$  in  $\mathbb{R}$  coincides with the set of all those functions that are continuous on each countable subset  $A \subset [0, \alpha]$ . It is easy to see that these are precisely all those functions that are continuous at all points of the set  $A_{\omega, \alpha}$ .  $\square$

**Corollary 6.** *If  $\tau \geq \omega_1$  is the initial regular ordinal and  $\sigma \leq \tau$  is the initial ordinal, then  $M_{\tau\sigma} \subset \nu(C_p([0, \tau \cdot \sigma]))$ .*

For the initial ordinal  $\sigma$  we denote by  $\Gamma_\sigma$  the discrete space of cardinality  $|\sigma|$  and consider the space

$$c_0(\Gamma_\sigma) = \{x \in \mathbb{R}^{\Gamma_\sigma} : \{t \in \Gamma_\sigma : |x(t)| \geq \varepsilon\} \text{ is finite for any } \varepsilon > 0\}.$$

**Lemma 7.** *Let  $\tau \geq \omega_1$  be an initial regular ordinal,  $\sigma \leq \tau$  be an initial ordinal. Then there exists a homeomorphic embedding  $f: c_0(\Gamma_\sigma) \rightarrow M_{\tau\sigma}$  such that  $f(0) = 0$  and  $f(x) \in M_{\tau\sigma} \setminus C_p([0, \tau \cdot \sigma])$  if  $x \neq 0$ .*

PROOF: We enumerate the points of the set  $\Gamma_\sigma$  by the ordinals  $\xi \in [0, \sigma)$ . Then  $\Gamma_\sigma = \{t_\xi\}_{\xi \in [0, \sigma)}$ . For each characteristic function  $\chi_{\{t_\xi\}} \in c_0(\Gamma_\sigma)$  we put  $f(\chi_{\{t_\xi\}}) = \chi_{\{\tau(\xi+1)\}}$ . It is obvious that  $\chi_{\{\tau(\xi+1)\}} \in M_{\tau\sigma} \setminus C_p([0, \tau \cdot \sigma])$ . It remains to extend the map  $f$  in the standard way to the space  $c_0(\Gamma_\sigma)$ .  $\square$

**Lemma 8.** *Let  $\tau \geq \omega_1$  be an initial regular ordinal,  $\sigma, \lambda$  be an initial ordinals and  $\omega \leq \lambda < \sigma \leq \tau$ . If  $f: c_0(\Gamma_\sigma) \rightarrow M_{\tau\lambda}$  is an injective mapping such that  $f(0) = 0$  and  $f(x) \in M_{\tau\lambda} \setminus C_p([0, \tau \cdot \lambda])$  for  $x \neq 0$ , then the map  $f$  is not continuous.*

PROOF: Suppose that there exists a continuous map  $f: c_0(\Gamma_\sigma) \rightarrow M_{\tau\lambda}$  with the above-mentioned properties. As in Lemma 7, let  $\Gamma_\sigma = \{t_\xi\}_{\xi \in [1, \sigma)}$ . Since the space  $c_0(\Gamma_\sigma)$  is considered in the topology of pointwise convergence, any sequence of the form  $\chi_{\{t_{\xi_n}\}}$  converges to zero in this space. Consequently, at each point  $\gamma \in [0, \tau \cdot \lambda]$  only a countable number of functions  $f(\chi_{\{t_\xi\}})$  is nonzero. Since by the condition  $f(\chi_{\{t_\xi\}}) \in M_{\tau\lambda} \setminus C_p([0, \tau \cdot \lambda])$ , each function  $f(\chi_{\{t_\xi\}})$  is discontinuous at some point of the set  $A_{\tau, \tau\lambda} \subset [0, \tau \cdot \lambda]$ .

Let us take

$$B_\gamma = \{f(\chi_{\{t_\xi\}}) : f(\chi_{\{t_\xi\}}) \text{ is discontinuous at a point } \tau(\gamma + 1) \in A_{\tau, \tau\lambda}\}.$$

Since  $\bigcup_{\gamma < \lambda} B_\gamma = f(\{\chi_{\{t_\xi\}} : \xi < \sigma\})$  and  $|\lambda| = |A_{\tau, \tau\lambda}| < |\sigma|$ , there is a point  $\gamma_0 < \lambda$ , such that  $|B_{\gamma_0}| = |\sigma|$ . Since at the point  $\tau(\gamma_0 + 1)$  only a countable number of functions from  $B_{\gamma_0}$  are nonzero, without loss of generality we can assume that all functions from  $B_{\gamma_0}$  at the point  $\tau(\gamma_0 + 1)$  are equal to zero. By Lemma 1, for each function  $f(\chi_{\{t_\xi\}}) \in B_{\gamma_0}$  there exists an ordinal  $\gamma_\xi < \tau(\gamma_0 + 1)$  such that  $f(\chi_{\{t_\xi\}})|_{[\gamma_\xi, \tau(\gamma_\xi + 1))} = \text{const} = C_\xi$ . Since  $|B_{\gamma_0}| = |\sigma| > \omega$ , in  $B_{\gamma_0}$  there is an uncountable family of functions for which  $|C_\xi| \geq \varepsilon_0$ . Consider the sequence  $\{f(\chi_{\{t_{\xi_n}\}})\}_{n=1}^\infty$  of such functions and put  $\gamma_0 = \sup\{\gamma_{\xi_n} : n = 1, 2, \dots\}$ . Since  $\text{cf}(\tau(\gamma_0 + 1)) > \omega$ ,  $\gamma_0 < \tau(\gamma_0 + 1)$  and therefore  $|f(\chi_{\{t_{\xi_n}\}})(t)| \geq \varepsilon_0$  for each  $t \in (\gamma_0, \tau(\gamma_0 + 1))$ . But this contradicts the fact that the sequence  $\{f(\chi_{\{t_{\xi_n}\}})\}_{n=1}^\infty$  converges pointwise to zero.  $\square$

**PROOF OF THEOREM 1:** Suppose that there exists a homeomorphism  $\varphi : C_p([0, \tau \cdot \sigma]) \rightarrow C_p([0, \tau \cdot \lambda])$ . We can assume that  $\varphi(0) = 0$ . By Lemma 4, there exists a homeomorphism  $\tilde{\varphi} : \nu(C_p([0, \tau \cdot \sigma])) \rightarrow \nu(C_p([0, \tau \cdot \lambda]))$  such that  $\tilde{\varphi}(C_p([0, \tau \cdot \sigma])) = C_p([0, \tau \cdot \lambda])$ . By Corollary 6,  $M_{\tau\sigma} \subset \nu(C_p([0, \tau \cdot \sigma]))$ , and by Lemma 3  $\tilde{\varphi}(M_{\tau\sigma}) = M_{\tau\lambda}$ . By Lemma 7 the mapping  $\tilde{\varphi} \cdot f : c_0(\Gamma_\sigma) \rightarrow M_{\tau\lambda}$  is continuous,  $(\tilde{\varphi} \cdot f)(0) = 0$  and  $(\tilde{\varphi} \cdot f)(M_{\tau\sigma}) \subset M_{\tau\lambda} \setminus C_p([0, \tau \cdot \lambda])$  for  $x \neq 0$ . In this case, the map  $\tilde{\varphi}|_{c_0(\Gamma_\sigma)}$  is a homeomorphism of the space  $c_0(\Gamma_\sigma) \subset M_{\tau\sigma}$  onto the subspace  $M_{\tau\lambda}$  such that  $\tilde{\varphi}(0) = 0$  and  $\tilde{\varphi}(x) \subset M_{\tau\lambda} \setminus C_p([0, \tau \cdot \lambda])$  for  $x \neq 0$ . But this is impossible by Lemma 8.  $\square$

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L. V. Genze, S. P. Gul'ko, T. E. Khmyleva:  
FACULTY OF MATHEMATICS AND MEKCHANIKS, TOMSK STATE UNIVERSITY,  
PR. LENINA 36, 634050 TOMSK, RUSSIA

*E-mail:* genze@math.tsu.ru  
gulko@math.tsu.ru  
tex2150@yandex.ru

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