Norm inequalities in weighted amalgam

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Abstract. Hardy inequalities for the Hardy-type operators are characterized in the amalgam space which involves Banach function space and sequence space.

Keywords: Hardy's inequality; amalgam space; weight function

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1. Introduction

Amalgam space had been introduced by N. Wiener in [11]. For the study of the amalgam space, we refer to [4]. For the study of the Hardy inequality in the amalgam space, we refer to [3], [5], [6], [10]. Banach function space (BFS) had been introduced by W. A. J. Luxemburg in [9]. For the study of the BFS we refer to [1].

In [3], pair of weights have been characterized for the boundedness of the Hardy operator between two suitable weighted amalgam spaces, both involve weighted Lebesgue space and sequence space. Motivated by this, in this paper, we characterize the boundedness of the Hardy operator between two suitable weighted amalgam spaces, both involve weighted BFS and sequence space. We denote the amalgam space which involves weighted BFS and sequence space as $l^q(X_w)$. Norm of $l^q(X_w)$ is defined as

$$\|f\|_{l^q(X_w)} = \left(\sum_{n\in\mathbb{Z}} \|f\chi_n w\|_X^q \right)^{1/q}.$$

In Section 2 of this paper, we give the necessary and sufficient conditions for the boundedness of the Hardy operator $(Hf)(x) = \int_{-\infty}^x f(t)\,\mathrm{d}t$ and its adjoint operator $(H^*f)(x) = \int_x^\infty f(t)\,\mathrm{d}t$ between amalgam spaces $l^q(X_u)$ and $l^r(Y_v)$ for the cases $1 < r \le q < \infty$ and $1 < q < r < \infty$. Boundedness of the operators H and H^* between two weighted amalgam spaces, one weighted amalgam space made of weighted Lebesgue space and sequence space and the other weighted amalgam space made of weighted BFS and sequence space has been considered in [6] for certain ranges of indices. Precisely, in this paper, we have given answer to the problem mentioned in [6, Remark 8]. In Section 3, we give the corresponding results, as in the Section 2 for the sum of the two Hardy-type operators.

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We say that BFS X and Y satisfy l-condition, if X is l-concave and Y is l-convex simultaneously for a Banach sequence space (BSS) l, see [2]. If a BFS X is l-concave (l-convex) and w is a weight function, then the weighted BFS X_w is also l-concave (l-convex). The boundedness of the Hardy-type operators between two BFS X and Y is shown in [8].

Throughout the paper, u and v are weight functions, that is, a measurable function positive almost everywhere in the appropriate interval, $\chi_n = \chi_{[n,n+1]}$ is the characteristic function defined on $[n, n+1]$, f is a measurable function, $1 < p, q, \bar{p}, \bar{q} < \infty$, $p' = p/(p-1)$ is the conjugate to p and the same is true for the other indices. The set of integers is denoted by $\mathbb{Z}$. For a BFS X, X' is its adjoint space. Throughout the paper, we assume that X_u and Y_v satisfy the l-condition.

2. Boundedness of the Hardy operator

The following result yields the necessary and sufficient condition for the boundedness of $H\colon l^r(Y_v) \to l^q(X_u)$ for the case $1 < r \le q < \infty$:

Theorem 2.1. *Suppose u, v are weight functions, X_u and Y_v are weighted* BFS *and $1 < r \le q < \infty$. There exists a constant $C > 0$ such that the inequality*

$$\|Hf\|_{l^q(X_u)} \le C\|f\|_{l^r(Y_v)} \tag{2.1}$$

holds for all $f \in l^r(Y_v)$ if and only if $\max(C_1, C_2) < \infty$, where

$$C_1 = \sup_{m\in\mathbb{Z}} \Big(\sum_{n=m}^{\infty} \|\chi_n u\|_X^q \Big)^{1/q} \Big(\sum_{n=-\infty}^{m-1} \|\chi_n v^{-1}\|_{Y'}^{r'} \Big)^{1/r'},$$

$$C_2 = \sup_{m\in\mathbb{Z}} \sup_{m<t<m+1} \|\chi_{[t,m+1]} u\|_X \|\chi_{[m,t]} v^{-1}\|_{Y'}.$$

Proof: *Sufficiency.* Since $|Hf| \le H(|f|)$, we assume without the loss of generality that $f \ge 0$. Suppose $\max(C_1, C_2) < \infty$. We have

$$\begin{aligned} \|(Hf)\chi_n\|_{X_u} &= \Big\| \Big(\int_{-\infty}^{n} f + \int_{n}^{x} f \Big) \chi_n \Big\|_{X_u} \\ &\le \Big\| \Big(\sum_{k=-\infty}^{n} \int_{k-1}^{k} f \Big) \chi_n \Big\|_{X_u} + \Big\| \Big(\int_{n}^{x} f \Big) \chi_n \Big\|_{X_u}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|Hf\|_{l^q(X_u)} &\le \Big(\sum_{n\in\mathbb{Z}} \Big(\sum_{k=-\infty}^{n} \int_{k-1}^{k} f \Big)^q \|\chi_n\|_{X_u}^q \Big)^{1/q} + \Big(\sum_{n\in\mathbb{Z}} \Big\| \Big(\int_{n}^{x} f \Big) \chi_n \Big\|_{X_u}^q \Big)^{1/q} \\ &= J_1 + J_2. \end{aligned} \tag{2.2}$$

For $U_n = \|\chi_n\|_{X_u}$, $a_k = \int_{k-1}^{k} f$ and using [3, Corollary 1.3 (i)], we find $C_1 < \infty$ such that

$$J_1 = \left(\sum_{n\in\mathbb{Z}} \left(\sum_{k=-\infty}^{n} a_k\right)^q U_n^q\right)^{1/q} \leq C\left(\sum_{n\in\mathbb{Z}} a_n^r \|\chi_{n-1}v^{-1}\|_{Y'}^{-r}\right)^{1/r}. \tag{2.3}$$

Using the Hölder's inequality, we find

$$a_n^r \leq \|fv\chi_{n-1}\|_Y^r \|v^{-1}\chi_{n-1}\|_{Y'}^r.$$

Substituting the above estimate of a_n^r in the right hand side of the inequality (2.3), we find

$$J_1 \leq C\left(\sum_{n\in\mathbb{Z}} \|fv\chi_{n-1}\|_Y^r\right)^{1/r} = C\|f\|_{l^r(Y_v)}. \tag{2.4}$$

Since $C_2 < \infty$, we find, using an application of [8, Theorem 4], that

$$\left\|\left(\int_n^x f\right)\chi_n\right\|_{X_u} \leq C\|f\chi_n\|_{Y_v}.$$

Therefore, we have

$$J_2 \leq C\|f\|_{l^q(Y_v)} \leq C\|f\|_{l^r(Y_v)} \tag{2.5}$$

because for $r \leq q$, $l^r(Y_v) \subset l^q(Y_v)$ which yields $\|f\|_{l^q(Y_v)} \leq \|f\|_{l^r(Y_v)}$, see [4]. Sufficiency assertions now follow from (2.2), (2.4) and (2.5).
Necessity. For any non-negative sequence $\{a_k\} \in l^r(Y_v)$, we define

$$f = \sum_{k\in\mathbb{Z}} a_k v^{-1}\chi_{[k,k+1]}. \tag{2.6}$$

For $A_k = a_{k-1}\int_{k-1}^{k} v^{-1}$, and $n \leq x < n+1$, we have

$$|(Hf)(x)| = \left|\int_{-\infty}^{n} f + \int_n^x f\right| \geq \left(\sum_{k=-\infty}^{n} A_k\right)$$

which yields

$$\|(Hf)\|_{l^q(X_u)} \geq \left(\sum_{n\in\mathbb{Z}} \left(\sum_{k=-\infty}^{n} A_k\right)^q \|\chi_n\|_{X_u}^q\right)^{1/q}.$$

Also, for f defined as (2.6), we have

$$\|f\|_{l^r(Y_v)} \leq \left(\sum_{n\in\mathbb{Z}} A_n^r \|v^{-1}\chi_{n-1}\|_{Y'}^{-r}\right)^{1/r}.$$

Consequently, the inequality (2.1) implies

$$\left(\sum_{n\in\mathbb{Z}}\left(\sum_{k=-\infty}^{n}A_k\right)^q\|\chi_n\|_{X_u}^q\right)^{1/q}\le C\left(\sum_{n\in\mathbb{Z}}A_n^r\|v^{-1}\chi_{n-1}\|_{Y'}^{-r}\right)^{1/r}$$

for $\{A_k\}\in l^r$. Therefore an application of [3, Corollary 1.3 (i)] yields $C_1<\infty$.

For $g\ge 0$ and $m\in\mathbb{Z}$ fixed, we define

$$f=g\chi_{[m,m+1]}. \tag{2.7}$$

For f defined as (2.7),

$$\|Hf\|_{l^q(X_u)}=\left\|\left(\int_{-\infty}^{x}f\right)\chi_n\right\|_{X_u}\ge\left\|\left(\int_{m}^{x}g\right)\chi_m\right\|_{X_u}$$

and

$$C\|f\|_{l^r(Y_v)}=C\|g\|_{Y_v}.$$

Therefore the inequality (2.1) implies

$$\left\|\left(\int_{m}^{x}g\right)\chi_m\right\|_{X_u}\le C\|g\|_{Y_v}$$

for all $m\in\mathbb{Z}$ with C independent of m. Therefore an application of [8, Theorem 4] implies $C_2<\infty$. This completes the necessity. □

Next, we will consider the boundedness of $H\colon l^r(Y_v)\to l^q(X_u)$ for the case $1<q<r<\infty$.

Theorem 2.2. *Suppose u,v are weight functions, X_u,Y_v are weights* BFS, $1<q<r<\infty$ *and* $1/s=1/q-1/r$. *Define*

$$C_3=\left[\sum_{k\in\mathbb{Z}}\left\{\sum_{n=k}^{\infty}\|\chi_n u\|_X^q\right\}^{s/q}\left\{\sum_{n=-\infty}^{k}\|v^{-1}\chi_{n-1}\|_Y^{r'}\right\}^{s/q'}\|v^{-1}\chi_{n-1}\|_Y^{r'}\right]^{1/s}$$

$$D_n=\sup_{n<t<n+1}\|\chi_{[t,n+1]}u\|_X\|\chi_{[n,t]}v^{-1}\|_{Y'}.$$

There exists a constant $C>0$ such that the inequality (2.1) *holds if $C_3<\infty$ and $\{D_n\}\in l^s$.*

Conversely $C_3<\infty$ and $\sup_{n\in\mathbb{Z}}D_n<\infty$ are necessary for the inequality (2.1).

Proof: *Sufficiency.* We assume $f\ge 0$ without the loss of generality. Making argument similar to the proof of Theorem 2.1, we obtain

$$\|Hf\|_{l^q(X_u)}\le J_1+J_2.$$

Since $q < r$, [3, Corollary 1.3 (ii)] yields

$$J_1 = C\Big(\sum_{n\in\mathbb{Z}} a_n^r \|v^{-1}\chi_n\|_{Y'}^{-r}\Big)^{1/r} \tag{2.8}$$

if and only if $C_3 < \infty$. For the same reason given in the proof of Theorem 2.1, the right hand side of (2.8) is dominated by $C\|f\|_{l^r(Y_v)}$. The sufficiency of $C_3 < \infty$ follows.

By an application of [8, Theorem 4]

$$\Big\|\Big(\int_n^x f\Big)\chi_n\Big\|_{X_u} \le K_n\|f\chi_n\|_{Y_v}$$

if $K_n \sim D_n$. Using Hölder's inequality with the index $\alpha = r/q$, we find

$$J_2 \le \Big(\sum_{n\in\mathbb{Z}} D_n^q \|f\chi_n\|_{Y_v}^q\Big)^{1/q} \le A\Big(\sum_{n\in\mathbb{Z}} D_n^{q\alpha'}\Big)^{1/q\alpha'} \|f\|_{l^q(Y_v)}.$$

Since $q\alpha' = s$ and $\{D_n\} \in l^s$ the sufficiency assertions follow.
Necessity. The necessity of $C_3 < \infty$ can be established by making argument similar to the proof of the necessity of $C_1 < \infty$ as given in Theorem 2.1.

To prove the necessity of $\sup_{n\in\mathbb{Z}} D_n < \infty$, we note that if the inequality (2.1) holds for all $f \in l^r(Y_v)$, then the inequality holds in particular for all f in the subspace $l^q(Y_v)$ of $l^r(Y_v)$ for $q < r$.

Theorem 2.1 is therefore applicable with $r = q$ and the necessity is established. □

Now, we will state the results related to the boundedness of the adjoint operator of H, that is $H^*\colon l^r(Y_v) \to l^q(X_u)$ for the cases $1 < r \le q < \infty$ as well as $1 < q < r < \infty$ as follows:

Theorem 2.3. *Suppose u, v are weight functions, X_u and Y_v are weighted* BFS *and $1 < r \le q < \infty$. There exists a constant $C > 0$ such that the inequality*

$$\|H^*f\|_{l^q(X_u)} \le C\|f\|_{l^r(Y_v)} \tag{2.9}$$

holds for all $f \in l^r(Y_v)$ if and only if $\max(C_1^*, C_2^*) < \infty$, where

$$C_1^* = \sup_{m\in\mathbb{Z}} \Big(\sum_{n=-\infty}^{m} \|\chi_n u\|_X^q\Big)^{1/q} \Big(\sum_{n=m}^{\infty} \|\chi_{n-1} v^{-1}\|_Y^{r'}\Big)^{1/r'},$$

$$C_2^* = \sup_{m\in\mathbb{Z}} \sup_{m<t<m+1} \|\chi_{[m,t]} u\|_X \|\chi_{[t,m+1]} v^{-1}\|_{Y'}.$$

Theorem 2.4. *Suppose u, v are weight functions. The spaces X_u, Y_v are weighted* BFS, $1 < q < r < \infty$, $1/s = 1/q - 1/r$. *Define*

$$C_3^* = \left[\sum_{n\in\mathbb{Z}} \left\{ \sum_{n=-\infty}^{k} \|\chi_n u\|_X^q \right\}^{s/q} \left\{ \sum_{n=k}^{\infty} \|\chi_{n-1} v^{-1}\|_Y^{r'} \right\}^{s/q'} \|\chi_{n-1} v^{-1}\|_Y^{r'} \right]^{1/s},$$

$$D_n^* = \sup_{n<t<n+1} \|\chi_{[n,t]} u\|_X \|\chi_{[t,n+1]} v^{-1}\|_{Y'}.$$

There exists a constant $C > 0$ such that the inequality (2.9) *holds if $C_3^* < \infty$ and $\{D_n^*\} \in l^s$.*

Conversely, $C_3^* < \infty$ and $\sup_{n\in\mathbb{Z}} D_n^* < \infty$ are necessary for the inequality (2.9).

3. Boundedness of the generalized Hardy operator

Consider the operator

$$(Sf)(x) = \varphi_1(x) \int_{-\infty}^{x} \psi_1(t) f(t)\,\mathrm{d}t + \varphi_2(x) \int_{x}^{\infty} \psi_2(t) f(t)\,\mathrm{d}t$$

where φ_i, ψ_i, $i = 1, 2$, are nonzero measurable functions not necessarily nonnegative and f is a measurable function. In this section we give the corresponding results, as in the Section 2, for the boundedness of the operator S between two weighted amalgam spaces, both involve weighted BFS and sequence space. Boundedness of the operators S and S^* between two weighted amalgam spaces, both involve weighted Lebesgue space and sequence space, is available in [5].

Precisely the following are the results:

Theorem 3.1. *Suppose u, v are weight functions, X_u and Y_v are weighted* BFS *and $1 < r \le q < \infty$. There exists a constant $C > 0$ such that the inequality* (2.1) *holds for $H = S$ and $f \in l^r(Y_v)$ if and only if $\max(C_1, C_2, C_3, C_4) < \infty$, where*

$$C_1 = \sup_{m\in\mathbb{Z}} \left(\sum_{n=m}^{\infty} \|\chi_n u|\varphi_1|\|_X^q \right)^{1/q} \left(\sum_{n=-\infty}^{m-1} \|\chi_n v^{-1}|\psi_1|\|_Y^{r'} \right)^{1/r'},$$

$$C_2 = \sup_{m\in\mathbb{Z}} \sup_{m<t<m+1} \|\chi_{[t,m+1]} u|\varphi_1|\|_X \|\chi_{[m,t]} v^{-1}|\psi_1|\|_{Y'},$$

$$C_3 = \sup_{m\in\mathbb{Z}} \left(\sum_{n=-\infty}^{m} \|\chi_n u|\varphi_2|\|_X^q \right)^{1/q} \left(\sum_{n=m}^{\infty} \|\chi_{n-1} v^{-1}|\psi_2|\|_Y^{r'} \right)^{1/r'},$$

$$C_4 = \sup_{m\in\mathbb{Z}} \sup_{m<t<m+1} \|\chi_{[m,t]} u|\varphi_2|\|_X \|\chi_{[t,m+1]} v^{-1}|\psi_2|\|_{Y'}.$$

Proof: *Sufficiency.* Define

$$(S_1 f)(x) = \varphi_1(x) \int_{-\infty}^{x} \psi_1(t) f(t)\,\mathrm{d}t; \qquad (S_2 f)(x) = \varphi_2(x) \int_{x}^{\infty} \psi_2(t) f(t)\,\mathrm{d}t.$$

Then $S = S_1 + S_2$. Consequently

$$\|Sf\|_{l^q(X_u)} \leq \|S_1 f\|_{l^q(X_u)} + \|S_2 f\|_{l^q(X_u)}. \tag{3.1}$$

Using an application of Theorem 2.1, we find $\max(C_1, C_2) < \infty$, which implies that the inequality (2.1) holds for $H = S_1$. Similarly, by an application of Theorem 2.3, we find $\max(C_3, C_4) < \infty$, which implies that the inequality (2.1) holds for $H = S_2$. Sufficiency assertions now follow from the inequality (3.1).
Necessity. Define $f = g\chi_{[m,m+1]}$, $m \in \mathbb{Z}$, such that $f\psi_1 > 0$. The necessity of $C_2 < \infty$ can be established by using the above defined function f in the inequality (2.1) for $H = S$, using [8, Theorem 4] and making arguments similar to those used in [7, (Chapter 2)] and [12]. Similarly the necessity of $C_4 < \infty$ can be established by using the function $f_1 = h\chi_{[m,m+1]}$, $m \in \mathbb{Z}$, such that function $f\psi_2 > 0$ in the inequality (2.1) for $H = S$ and using the dual result of [8, Theorem 4].

For $\alpha \in \mathbb{Z}$, and $\{a_k\} \in l^r(Y_v)$, we define

$$f_2(x) = \begin{cases} \displaystyle\sum_{k\in\mathbb{N}} a_k v^{-1}|\psi_1|^{-1}(\operatorname{sgn}\psi_1)\chi_{[k,k+1]}(x), & x \leq \alpha, \\ 0, & x > \alpha. \end{cases}$$

The necessity of $C_1 < \infty$, now, can be established by using the above defined function f_2 in the inequality (2.1) for $H = S$ and using [3, Corollary 3.1 (i)].

Similarly, for $\alpha \in \mathbb{Z}$, and $\{a_k\} \in l^r(Y_v)$, we define

$$f_3(x) = \begin{cases} \displaystyle\sum_{k\in\mathbb{N}} a_k v^{-1}|\psi_2|^{-1}(\operatorname{sgn}\psi_2)\chi_{[k,k+1]}(x), & x \leq \alpha, \\ 0, & x > \alpha. \end{cases}$$

The necessity of $C_3 < \infty$, now, can be established by using the above defined function f_3 in the inequality (2.1) for $H = S$ and using the dual result of [3, Corollary 3.1 (i)]. □

Theorem 3.2. *Suppose u, v are weight functions, X_u and Y_v are weighted* BFS *and $1 < q < r < \infty$, $1/s = 1/q - 1/r$. Define*

$$C_5 = \left(\sum_{k\in\mathbb{Z}} \left(\sum_{n=k}^{\infty} \|\chi_n u|\varphi_1|\|_X^q \right)^{s/q} \times \left(\sum_{n=-\infty}^{k} \|\chi_{n-1} v^{-1}|\psi_1|\|_Y^{r'} \right)^{s/q'} \|\chi_{n-1} v^{-1}|\psi_1|\|_Y^{r'} \right)^{1/s},$$

$$C_6 = \Big(\sum_{k\in\mathbb{Z}} \Big(\sum_{n=-\infty}^{k} \|\chi_n u|\varphi_2|\|_X^q \Big)^{s/q} \times \Big(\sum_{n=k}^{\infty} \|\chi_{n-1} v^{-1}|\psi_2|\|_Y^{r'} \Big)^{s/q'} \|\chi_{n-1} v^{-1}|\psi_2|\|_Y^{r'} \Big)^{1/s},$$

$$D_n = \sup_{n<t<n+1} \|\chi_{[t,n+1]} u|\varphi_1|\|_X \|\chi_{[n,t]} v^{-1}|\psi_1|\|_{Y'},$$

$$E_n = \sup_{n<t<n+1} \|\chi_{[n,t]} u|\varphi_2|\|_X \|\chi_{[t,n+1]} v^{-1}|\psi_2|\|_{Y'}.$$

There exists a constant $C > 0$ *such that the inequality* (2.1) *exists for* $H = S$ *if* $\max(C_5, C_6) < \infty$, $\{D_n\} \in l^s$, $\{E_n\} \in l^s$.

Conversely, $\max(C_5, C_6) < \infty$, $\sup_n D_n < \infty$ *and* $\sup_n E_n < \infty$ *are necessary for the inequality* (2.1) *for* $H = S$.

Proof: *Sufficiency.* Sufficiency can be established by extending Theorems 2.1 and 2.3 for S_1 and S_3, respectively, and using the inequality (3.1).
Necessity. Necessity can be established in the same way as done in the proof of Theorem 3.1 and using arguments similar to the proof of the Theorem 2.1 and its dual result as given in the Theorem 2.3 and also using the inequality (3.1). We omit the details. □

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