

On the mappings \mathcal{Z}_A and \mathfrak{S}_A in intermediate rings of $C(X)$

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Abstract. In this article, we investigate new topological descriptions for two well-known mappings \mathcal{Z}_A and \mathfrak{S}_A defined on intermediate rings $A(X)$ of $C(X)$. Using this, coincidence of each two classes of z -ideals, \mathcal{Z}_A -ideals and \mathfrak{S}_A -ideals of $A(X)$ is studied. Moreover, we answer five questions concerning the mapping \mathfrak{S}_A raised in [J. Sack, S. Watson, *C and C^* among intermediate rings*, Topology Proc. **43** (2014), 69–82].

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1. Introduction

Throughout this article all topological spaces are assumed to be Tychonoff. For a given topological space X , $C(X)$ denotes the algebra of all real-valued continuous functions on X and $C^*(X)$ denotes the subalgebra of $C(X)$ consisting of all bounded elements. A subring of $C(X)$ containing $C^*(X)$ is called an intermediate ring. An intermediate ring which is isomorphic with $C(Y)$ for some Tychonoff space Y is called an intermediate C -ring. The reader is referred to [5] for undefined terms and notations concerning $C(X)$. For each element f of an intermediate ring $A(X)$, we set $S_A(f) = \{p \in \beta X : (fg)^*(p) = 0, \forall g \in A(X)\}$. As stated in [11], we could easily observe that $S_A(fg) = S_A(f) \cup S_A(g)$, $S_A(f^2 + g^2) = S_A(f) \cap S_A(g)$ and $S_A(f^n) = S_A(f)$ for each $f, g \in A(X)$ and each $n \in \mathbb{N}$. Also, $S_C(f) = \text{cl}_{\beta X} Z(f)$ for each $f \in C(X)$ and $S_{C^*}(f) = Z(f^\beta)$ for each $f \in C^*(X)$. Moreover, $\text{cl}_{\beta X} Z(f) \subseteq S_A(f) \subseteq Z(f^*)$ and thus $S_A(f) \cap X = Z(f)$ for each $f \in A(X)$. For each $p \in \beta X$, we use M_A^p (or O_A^p) to denote the set $\{f \in A(X) : p \in S_A(f)\}$ ($\{f \in A(X) : p \in \text{int}_{\beta X} S_A(f)\}$), respectively. Evidently, $M_C^p = M^p$ (or $O_C^p = O^p$) and $M_{C^*}^p = M^{*p}$ ($O_{C^*}^p = O^{*p}$), respectively. Moreover, we can see that $\text{int}_{\beta X} S_A(f) = \text{int}_{\beta X} \text{cl}_{\beta X} Z(f)$ for each $f \in A(X)$ and thus $O_A^p = O^p \cap A(X)$ for each $p \in \beta X$. An ideal I of a commutative ring R is called a z -ideal if $M_f(R) \subseteq I$ whenever $f \in I$ in which $M_f(R)$ denotes the intersection of all the maximal ideals of R containing f . It is well-known that $M_f(C(X)) = \{g \in C(X) : Z(f) \subseteq Z(g)\}$ for each $f \in C(X)$. Also, from [9, Proposition 2.7], it follows that $M_f(A(X)) = \{g \in A(X) : S_A(f) \subseteq S_A(g)\}$ for each element f of an intermediate ring $A(X)$. Therefore, an ideal I in $C(X)$

(or in an intermediate ring $A(X)$) is a z -ideal if and only if whenever $Z(f) \subseteq Z(g)$ (or $S_A(f) \subseteq S_A(g)$), where $f \in I$ and $g \in C(X)$ ($g \in A(X)$, respectively), then $g \in I$. The aim of this paper is to answer the five basic questions concerning the mapping \mathfrak{S}_A which have been raised in [13]. This is done by investigating new topological descriptions for two well-known mappings \mathcal{Z}_A and \mathfrak{S}_A defined on intermediate rings of $C(X)$. This paper consists of 3 sections. Section 1, which is already noticed, is the introduction. In Section 2, we study the mapping \mathcal{Z}_A which was first introduced in [12] and more studied in [3]. Moreover, coincidence of z -ideals of $A(X)$ with \mathcal{Z}_A -ideals and z_A -ideals is studied. In Section 3, we study the mapping \mathfrak{S}_A which was first introduced in [8] and further studied in [14] and [13]. We establish a topological description for this mapping. Coincidence of \mathfrak{S}_A -ideals with z -ideals and \mathcal{Z}_A -ideals are studied. Moreover, the five questions concerning the mapping \mathfrak{S}_A raised in [13] are answered.

2. The mapping \mathcal{Z}_A

The mapping \mathcal{Z}_A was first introduced in [12] on an intermediate ring $A(X)$ as follows: for each element f of $A(X)$,

$$\mathcal{Z}_A(f) = \{E \in Z(X) : \exists g \in A(X), fg|_{X \setminus E} = 1\}.$$

By the following statement we provide a new topological description for \mathcal{Z}_A .

Theorem 2.1. *For each element f of an intermediate ring $A(X)$, we have*

$$\mathcal{Z}_A(f) = \{E \in Z(X) : S_A(f) \subseteq \text{int}_{\beta X} \text{cl}_{\beta X} E\}.$$

PROOF: If $E \in \mathcal{Z}_A(f)$, then there exists $g \in A(X)$ such that $fg|_{X \setminus E} = 1$. Thus, $(fg)^*|_{\text{cl}_{\beta X}(X \setminus E)} \neq 0$. This means that $S_A(f) \cap \text{cl}_{\beta X}(X \setminus E) = \emptyset$ and thus $S_A(f) \subseteq \beta X \setminus \text{cl}_{\beta X}(X \setminus E) = \beta X \setminus \text{cl}_{\beta X}(\beta X \setminus \text{cl}_{\beta X} E) \subseteq \text{cl}_{\beta X} E$; i.e., $S_A(f) \subseteq \text{int}_{\beta X} \text{cl}_{\beta X} E$. For the reverse inclusion, let $E \in Z(X)$ be such that $S_A(f) \subseteq \text{int}_{\beta X} \text{cl}_{\beta X} E$. Thus, $S_A(f) \cap (\beta X \setminus \text{int}_{\beta X} \text{cl}_{\beta X} E) = \emptyset$. Hence, there exists some $h \in C^*(X)$ such that $h^\beta(S_A(f)) = \{1\}$ and $(\beta X \setminus \text{int}_{\beta X} \text{cl}_{\beta X} E) \subseteq Z(h^\beta)$. These imply that $S_A(f) \cap S_A(h) = \emptyset$ and $X \setminus E \subseteq X \setminus \text{int}_X E \subseteq Z(h)$. Therefore, $S_A(f^2 + h^2) = S_A(f) \cap S_A(h) = \emptyset$ which implies that there exists some $k \in A(X)$ such that $(f^2 + h^2)k = 1$. Thus, clearly $fk \in A(X)$ and $(f^2k)|_{X \setminus E} = ((f^2 + h^2)k)|_{X \setminus E} = 1$; i.e., $E \in \mathcal{Z}_A(f)$. \square

The following properties of the mapping \mathcal{Z}_A follow from Theorem 2.1. Note that for each two z -filters \mathcal{F} and \mathcal{F}' , we denote by $\mathcal{F} \wedge \mathcal{F}'$ and $\mathcal{F} \vee \mathcal{F}'$ the meet and join on the lattice of z -filters, respectively.

Proposition 2.2. *The following statements hold for each two elements f, g of an intermediate ring $A(X)$ and each $n \in \mathbb{N}$.*

- (a) *If $0 \leq f \leq g$, then $\mathcal{Z}_A(f) \subseteq \mathcal{Z}_A(g)$.*
- (b) *$\bigcap \mathcal{Z}_A(f) = Z(f)$.*

- (c) $\mathcal{Z}_A(fg) = \mathcal{Z}_A(f) \wedge \mathcal{Z}_A(g)$.
- (d) $\mathcal{Z}_A(f + g) \subseteq \mathcal{Z}_A(f) \vee \mathcal{Z}_A(g)$.
- (e) $\mathcal{Z}_A(f^2 + g^2) = \mathcal{Z}_A(f) \vee \mathcal{Z}_A(g)$.
- (f) $\mathcal{Z}_A(f^n) = \mathcal{Z}_A(f)$.

PROOF: Parts (c) through (f) easily follow from Theorem 2.1.

(a) It suffices to show that $S_A(g) \subseteq S_A(f)$. Let $p \notin S_A(f)$. Thus, $f \notin M_A^p$ and hence $g \notin M_A^p$, i.e., $p \notin S_A(g)$, since $0 \leq f \leq g$ and from [3, Theorem 2.5] it follows that every maximal ideal in $A(X)$ is a convex ideal. Therefore, $S_A(g) \subseteq S_A(f)$.

(b) It is evident that $Z(f) \subseteq E$ for each $E \in \mathcal{Z}_A(f)$. Thus, $Z(f) \subseteq \bigcap \mathcal{Z}_A(f)$. Let $x \in X$ and $x \notin Z(f)$. Hence, $x \notin S_A(f)$ and thus there exists a zero-set E such that $x \notin \text{cl}_{\beta X} E$ and $S_A(f) \subseteq \text{int}_{\beta X} \text{cl}_{\beta X} E$. It follows that $E \in \mathcal{Z}_A(f)$, however, $x \notin E$. This implies that $\bigcap \mathcal{Z}_A(f) \subseteq Z(f)$ and the equality follows. \square

It is shown in [8, Theorem 1.2] that the mapping \mathcal{Z}_A could characterize $C^*(X)$ among intermediate rings. The following statement shows that the mapping \mathcal{Z}_A could also characterize $C(X)$ among intermediate rings in the case that X is a P -space.

Theorem 2.3. *Let $A(X)$ be an intermediate ring of $C(X)$. Then $\mathcal{Z}_A(f) = (Z(f))$ for each $f \in A(X)$ if and only if X is a P -space and $A(X) = C(X)$.*

PROOF: (\Rightarrow) As $\mathcal{Z}_A(f) = (Z(f))$ for each $f \in A(X)$, we have $Z(f) \in \mathcal{Z}_A(f)$ and thus $S_A(f) \subseteq \text{int}_{\beta X} \text{cl}_{\beta X} Z(f)$. This implies that $S_A(f) = \text{cl}_{\beta X} Z(f) = \text{int}_{\beta X} \text{cl}_{\beta X} Z(f)$ for each $f \in A(X)$. Hence, X is a P -space and, by [10, Theorem 2.3], $A(X) = C(X)$

(\Leftarrow) It is clear that $\mathcal{Z}_C(f) \subseteq (Z(f))$. If $E \in (Z(f))$, then, $Z(f) \subseteq E$ and as X is a P -space, we have $S_C(f) = \text{cl}_{\beta X} Z(f) \subseteq \text{int}_{\beta X} \text{cl}_{\beta X} E$. Thus, $(Z(f)) \subseteq \mathcal{Z}_C(f)$. \square

For an ideal I of an intermediate ring $A(X)$, we denote by $\mathcal{Z}_A(I)$ the set $\bigcup_{f \in I} \mathcal{Z}_A(f)$. Also, for a z -filter \mathcal{F} on X we denote by $\mathcal{Z}_A^{-1}(\mathcal{F})$ the set $\{f \in A(X) : \mathcal{Z}_A(f) \subseteq \mathcal{F}\}$. It is clear that $I \subseteq \mathcal{Z}_A^{-1} \mathcal{Z}_A(I)$ and $\mathcal{Z}_A \mathcal{Z}_A^{-1}(\mathcal{F}) \subseteq \mathcal{F}$. We call an ideal I in $A(X)$ a \mathcal{Z}_A -ideal, if $\mathcal{Z}_A^{-1} \mathcal{Z}_A(I) = I$. Also, a z -filter \mathcal{F} on X is called a \mathcal{Z}_A -filter, if $\mathcal{Z}_A \mathcal{Z}_A^{-1}(\mathcal{F}) = \mathcal{F}$. Evidently, $\mathcal{Z}_A^{-1} \mathcal{Z}_A(M_A^p) = M_A^p$. Hence, every maximal ideal in $A(X)$ is \mathcal{Z}_A -ideal.

Remark 2.4. It easily follows from Theorem 2.1 that $\mathcal{Z}_A(M_A^p) = \mathcal{Z}_A(O_A^p)$ for each $p \in \beta X$. Moreover, $\mathcal{Z}_A^{-1}(\mathcal{U}^p) = M_A^p$ for each $p \in \beta X$. Also, one can easily prove that $\mathcal{Z}_A(M)$ is contained in a unique z -ultrafilter for each maximal ideal M in $A(X)$. These provide a new approach to [3, Theorem 3.2].

It easily follows from Theorem 2.1 that every \mathcal{Z}_A -ideal of an intermediate ring $A(X)$ is a z -ideal. However, the converse of this fact does not hold, in general. For example, let $X = \mathbb{R}$ and $A(X) = C^*(X) = C^*(\mathbb{R})$ and $p \in X$; then $O_A^p \subsetneq M_A^p$, easily verifiable. It follows that O_A^p is not a \mathcal{Z}_A -ideal in $A(X)$, however, it is clearly a z -ideal. In [7, Theorem 2.14] it is stated that whenever every ideal of

an intermediate ring $A(X)$ is a \mathcal{Z}_A -ideal, then X is a P -space. The next theorem shows that even when every z -ideal is a \mathcal{Z}_A -ideal, then we have $A(X) = C(X)$. Note that we call an ideal I in $A(X)$ a z_A -ideal, if whenever $Z(f) \subseteq Z(g)$ where $f \in I$ and $g \in A(X)$, then $g \in I$. It is easy to see that the ideals O_A^p and $M^p \cap A(X)$, for each $p \in \beta X$, are z_A -ideals in $A(X)$. Also, the ideal M_A^p , for each $p \in \beta X \setminus v_A X$, is a maximal ideal in $A(X)$ which is not a z_A -ideal, refer to [2], [1], and [10] for more details.

Theorem 2.5. *The following statements are equivalent for an intermediate ring $A(X)$.*

- (a) Every z -ideal in $A(X)$ is a \mathcal{Z}_A -ideal.
- (b) Every z_A -ideal in $A(X)$ is a \mathcal{Z}_A -ideal.
- (c) X is a P -space and $A(X) = C(X)$.

PROOF: (a) \Rightarrow (c) The proof is straightforward by using [7, Theorem 2.5 and Theorem 3.10].

(c) \Rightarrow (a) By our hypothesis and Theorem 2.3, $\mathcal{Z}_C(f) = (Z(f))$ for each $f \in C(X)$. This clearly implies that every z -ideal in $C(X)$ is a \mathcal{Z}_C -ideal.

(b) \Rightarrow (c) As O_A^p , for each $p \in \beta X$, is a z_A -ideal, by our hypothesis, O_A^p would be a \mathcal{Z}_A -ideal. Hence, $O_A^p = \mathcal{Z}_A^{-1} \mathcal{Z}_A(O_A^p) = \mathcal{Z}_A^{-1} \mathcal{Z}_A(M_A^p) = M_A^p$. This means that $A(X)$ is a regular ring. Thus, by [7, Theorem 2.5 and Theorem 3.3], X is a P -space and $A(X) = C(X)$.

(c) \Rightarrow (b) It is evident that whenever $A(X) = C(X)$, then z_A -ideals coincide with z -ideals in $A(X)$. Moreover, as X is a P -space, by [4, Theorem 3.13], every z_A -ideal is a \mathcal{Z}_A -ideal. □

3. The mapping \mathfrak{S}_A

The mapping \mathfrak{S}_A on an intermediate ring $A(X)$ was first introduced in [8] as follows: for each $f \in A(X)$

$$\mathfrak{S}_A(f) = \{E \in Z(X) : \text{for all zero-sets } H \subseteq X \setminus E, \exists g \in A(X), fg|_H = 1\}.$$

The following statement provides a new topological description to the mapping \mathfrak{S}_A which could be proved similar to Theorem 2.1.

Theorem 3.1. *Let $A(X)$ be an intermediate ring of $C(X)$. For each $f \in A(X)$ we have*

$$\mathfrak{S}_A(f) = \{E \in Z(X) : \forall H \in Z(X) \text{ and } H \subseteq X \setminus E, S_A(f) \subseteq \text{int}_{\beta X} \text{cl}_{\beta X}(X \setminus H)\}.$$

The following question concerning the basic properties of the mapping \mathfrak{S}_A has been raised in [13].

Question 1. Let $f, g \in A(X)$. Which properties analogous to those of Proposition 2.2 hold with \mathfrak{S}_A in place of \mathcal{Z}_A ?

The answer to Question 1 will be given by the next statement which is an easy consequence of Theorem 3.1.

Proposition 3.2. *For each two elements f, g of an intermediate ring $A(X)$, the following statements hold.*

- (a) *If $0 \leq f \leq g$, then $\mathfrak{S}_A(f) \subseteq \mathfrak{S}_A(g)$.*
- (b) *$\bigcap \mathfrak{S}_A(f) = Z(f)$.*
- (c) *$\mathfrak{S}_A(fg) = \mathfrak{S}_A(f) \wedge \mathfrak{S}_A(g)$.*
- (d) *$\mathfrak{S}_A(f + g) \subseteq \mathfrak{S}_A(f) \vee \mathfrak{S}_A(g)$.*
- (e) *$\mathfrak{S}_A(f^2 + g^2) = \mathfrak{S}_A(f) \vee \mathfrak{S}_A(g)$.*
- (f) *$\mathfrak{S}_A(f^n) = \mathfrak{S}_A(f)$ for each $n \in \mathbb{N}$.*

It is stated in [3, Lemma 1.3] that for a z -filter \mathcal{F} and $f \in A(X)$, we have $\lim_{\mathcal{F}} fg = 0$ for each $g \in A(X)$ if and only if $\mathcal{F} \subseteq \mathcal{Z}_A(f)$. Note that for a z -filter \mathcal{F} on X and $f \in C(X)$, we use $\lim_{\mathcal{F}} f$ to denote the limit of the filter base $f(\mathcal{F})$. The following question has been raised in [13].

Question 2. Let $f \in A(X)$ and \mathcal{F} be a z -filter on X . Is it the case that $\mathfrak{S}_A(f) \subseteq \mathcal{F}$ if and only if $\lim_{\mathcal{F}} fg = 0$ for every $g \in A(X)$?

ANSWER TO QUESTION 2: As stated in [13], if $\mathfrak{S}_A(f) \subseteq \mathcal{F}$, then $\lim_{\mathcal{F}} fg = 0$ for every $g \in A(X)$. We show that the converse of this statement does not hold, in general. Take $X = \mathbb{R}$, $f(x) = x$ for each $x \in \mathbb{R}$ and \mathcal{F} be the z -filter of all zero set neighbourhood of 0 in \mathbb{R} . Then for all $h \in C(\mathbb{R})$, $\lim_{\mathcal{F}} fh = 0$ but $\mathfrak{S}_C(f) = \{Z \in Z[X] : 0 \in Z\}$ which contains \mathcal{F} properly.

In [7, Theorem 2.8] it is stated that a topological space X is a P -space if and only if, for every intermediate ring $A(X)$, we have $\mathfrak{S}_A(M_A^p) = \mathfrak{S}_A(O_A^p)$ for each $p \in X$. The next theorem extends this fact for each $p \in \beta X$. We use the following lemma which could be proved by a little modification of the arguments of [7, Proposition 2.7] and exploiting the complete regularity of βX .

Lemma 3.3. *Let $A(X)$ be an intermediate ring of $C(X)$. Then $\mathfrak{S}_A(O_A^p) = \mathcal{Z}_A(M_A^p)$ for each $p \in \beta X$.*

Theorem 3.4. *A topological space X is a P -space if and only if for every intermediate ring $A(X)$ we have $\mathfrak{S}_A(M_A^p) = \mathfrak{S}_A(O_A^p)$ for each $p \in \beta X$.*

PROOF: (\Rightarrow) This easily follows from [7, Theorem 2.8].

(\Leftarrow) By our hypothesis, $\mathfrak{S}_C(M^p) = \mathfrak{S}_C(O^p)$. Hence, $Z(M_A^p) = \mathfrak{S}_C(O^p) = Z(O^p)$ for each $p \in \beta X$ which clearly implies that X is a P -space. \square

For an ideal I of an intermediate ring $A(X)$, we denote by $\mathfrak{S}_A(I)$ the set $\bigcup_{f \in I} \mathfrak{S}_A(f)$. Moreover, for a z -filter \mathcal{F} on X , we use $\mathfrak{S}_A^{-1}(\mathcal{F})$ to denote the set $\{f \in A(X) : \mathfrak{S}_A(f) \subseteq \mathcal{F}\}$. An ideal I in $A(X)$ is called a \mathfrak{S}_A -ideal, if $\mathfrak{S}_A^{-1}\mathfrak{S}_A(I) = I$. Also, a z -filter \mathcal{F} on X is called a \mathfrak{S}_A -filter, if $\mathfrak{S}_A\mathfrak{S}_A^{-1}(\mathcal{F}) = \mathcal{F}$. It easily follows from Theorem 2.1 and Theorem 3.1 that every \mathcal{Z}_A -ideal in $A(X)$ is a \mathfrak{S}_A -ideal and every \mathfrak{S}_A -ideal in $A(X)$ is a z -ideal. However, the converse of these facts does not hold, in general, see the next example.

Example 3.5. (a) Whenever X is not a P -space, then there exists $p \in \beta X$ such that $O^p \neq M^p$. Hence, O^p is clearly a z -ideal and thus is a \mathfrak{S}_C -ideal in $C(X)$, however, it is not a \mathcal{Z}_C -ideal, since, $\mathcal{Z}_C^{-1}\mathcal{Z}_C(O^p) = M^p$.

(b) Let $X = \mathbb{N}$ and $A(X) = C^*(\mathbb{N})$. It is clear that there exists $p \in \beta\mathbb{N}$ such that $M_A^p \neq O_A^p$. By Theorem 3.4, $\mathfrak{S}_A(M_A^p) = \mathfrak{S}_A(O_A^p)$ for each $p \in \beta\mathbb{N}$. Thus, $\mathfrak{S}_A^{-1}\mathfrak{S}_A(O_A^p) = \mathfrak{S}_A^{-1}\mathfrak{S}_A(M_A^p) = M_A^p \neq O_A^p$ which means that O_A^p is not a \mathfrak{S}_A -ideal, however, it is evidently a z -ideal.

The following theorem shows that coincidence of z_A -ideals with \mathfrak{S}_A -ideals characterizes $C(X)$ among intermediate rings.

Theorem 3.6. *Let $A(X)$ be an intermediate ring of $C(X)$. Then every \mathfrak{S}_A -ideal in $A(X)$ is a z_A -ideal if and only if $A(X) = C(X)$.*

PROOF: (\Rightarrow) Since $A(X) \neq C(X)$, there exists $f \in C(X) \setminus A(X)$. Take $g = 1/(1 + |f|)$, then g is a non-unit of $A(X)$. Consequently, $g \in M_A^p$ for some $p \in \beta X$. It is clear that, since $Z(g) = \emptyset$, M_A^p is not a z_A -ideal in $A(X)$, but M_A^p being a maximal ideal is a \mathfrak{S}_A -ideal in $A(X)$.

(\Leftarrow) This is evident, since, \mathfrak{S}_C -ideals of $C(X)$ are nothing other than z -ideals. □

Remark 3.7. It is stated in [7, Theorem 2.14] that whenever every ideal of an intermediate ring $A(X)$ is a \mathfrak{S}_A -ideal, then X is a P -space. We show that this condition implies $A(X) = C(X)$. It is evident that whenever every ideal in $A(X)$ is a \mathfrak{S}_A -ideal, then every ideal in $A(X)$ is a z -ideal and thus, by [6, Theorem 1.2], $A(X)$ is a regular ring. Hence, we have $A(X) = C(X)$, since, otherwise, there exists $f \in A(X)$ such that $Z(f) = \emptyset$ and $S_A(f) \neq \emptyset$. Hence, if we choose some $p \in S_A(f)$, then $M^p \cap A(X) \neq M_A^p$ which means that $M^p \cap A(X)$ is a prime ideal in $A(X)$ which is not a maximal ideal and thus $A(X)$ is not a regular ring. Other than this, in the proof of this theorem in [7] it is stated that $\mathfrak{S}_A^{-1}\mathcal{Z}_A(M_A^p) = M_A^p$ for each $p \in X$. We claim that this equality does not hold, in general. For example, if we let $A(X) = C(X)$ where X is not a P -space, then there exists $f \in C(X)$ and $p \in X$ such that $f \in M^p \setminus O^p$. Also, we clearly have $\mathfrak{S}_C^{-1}(\mathcal{F}) = Z^{-1}(\mathcal{F})$ for each z -filter \mathcal{F} . It follows that $\mathfrak{S}_C^{-1}\mathcal{Z}_C(M^p) = Z^{-1}\mathcal{Z}_C(M^p) = Z^{-1}Z(O^p) = O^p \neq M^p$. Recall that $\mathcal{Z}_A(M_A^p) = \mathcal{Z}_A(O_A^p)$ and $\mathcal{Z}_A(M_A^p) = \{E \in Z(X) : p \in \text{int}_{\beta X} \text{cl}_{\beta X} E\}$. Hence, $\mathcal{Z}_C(M^p) = Z(O^p)$ for each $p \in \beta X$.

The following question has been raised in [13].

Question 3. Is it the case that if \mathcal{F} is a z -filter on X , then $\mathfrak{S}_A^{-1}(\mathcal{F})$ is an ideal in $A(X)$?

We answer Question 3 by using Proposition 3.2.

ANSWER TO QUESTION 3: Let \mathcal{F} be a z -filter on X , $f, g \in \mathfrak{S}_A^{-1}(\mathcal{F})$ and $h \in A(X)$. Then, $\mathfrak{S}_A(fh) = \mathfrak{S}_A(f) \wedge \mathfrak{S}_A(h) \subseteq \mathcal{F}$, since, $f \in \mathfrak{S}_A^{-1}(\mathcal{F})$. Hence, $fh \in \mathfrak{S}_A^{-1}(\mathcal{F})$. Moreover, $\mathfrak{S}_A(f + g) \subseteq \mathfrak{S}_A(f) \vee \mathfrak{S}_A(g) \subseteq \mathcal{F}$, since, $f, g \in \mathfrak{S}_A^{-1}(\mathcal{F})$. Thus, $f + g \in \mathfrak{S}_A^{-1}(\mathcal{F})$. Therefore, $\mathfrak{S}_A^{-1}(\mathcal{F})$ is an ideal in $A(X)$ for each z -filter \mathcal{F} on X .

The following question also has been raised in [13].

Question 4: Is it the case that $A(X) = C(X)$ if and only if $\mathfrak{S}_A(M)$ is a z -ultrafilter for every maximal ideal M ?

ANSWER TO QUESTION 4. If $A(X) = C(X)$, then for each maximal ideal M of $A(X)$, $\mathfrak{S}_A(M) = Z(M)$ is a z -ultrafilter on X . On the other hand, it is easily inferred from [7, Theorem 2.10] that if X is a non pseudocompact P -space, then there exists an intermediate ring $B(X) \subsetneq C(X)$ for which for any maximal ideal M of $B(X)$, $\mathfrak{S}_B(M)$ is a z -ultrafilter on X .

The following question has been raised in [13].

Question 5. Is it the case that $A(X) = C(X)$ if and only if every z -filter on X is a \mathfrak{S}_A -filter?

The answer to Question 5 is negative as it could be inferred from [7, Example 2.13].

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