Balcar's theorem on supports

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To the memory of Bohuslav Balcar

Abstract. In *A theorem on supports in the theory of semisets* [Comment. Math. Univ. Carolinae **14** (1973), no. 1, 1–6] B. Balcar showed that if $\sigma \subseteq D \in M$ is a support, M being an inner model of ZFC, and $\mathcal{P}(D \setminus \sigma) \cap M = r``\sigma$ with $r \in M$, then r determines a preorder "$\preceq$" of D such that σ becomes a filter on $(D, \preceq)$ generic over M. We show that if the relation r is replaced by a function $\mathcal{P}(D \setminus \sigma) \cap M = f_{-1}(\sigma)$, then there exists an equivalence relation "$\sim$" on D and a partial order on $D/\!\sim$ such that $D/\!\sim$ is a complete Boolean algebra, $\sigma/\!\sim$ is a generic filter and $[f(u)]_\sim = -\sum(u/\!\sim)$ for any $u \subseteq D$, $u \in M$.

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1. Introduction

B. Balcar in [1] has found very nice and important proof of a theorem on supports by P. Vopěnka in [4]. Actually, B. Balcar proved a theorem of the theory of semisets. The translations of the theorem to the set theory is rather immediate. Following ideas of this proof we show a related result.

We shall follow the terminology and notations of T. Jech in [3].

If M is an inner model then a set $\sigma \subseteq D \in M$, $\sigma \notin M$ is a *support over* M if for any binary relations $r_1, r_2 \in M$ there exists a binary relation $r \in M$ such that

$$r``\sigma = r_1``\sigma \setminus r_2``\sigma.$$

Note that $r``\sigma = \{y \in \operatorname{ran}(r) \colon \exists\, x \in \sigma \ [x, y] \in r\}$. If f is a function, then $f_{-1}(\sigma) = \{x \in \operatorname{dom}(f) \colon f(x) \in \sigma\}$. If "$\sim$" is an equivalence relation on a set A, then for any $x \in A$ we denote the equivalence class of x by $[x]_\sim$. If $B \subseteq A$, then $B/\!\sim = \{[x]_\sim \colon x \in B\}$.

By Balcar's proof in [1], see also [2, pages 365–370], we obtain

Theorem 1. *Let M be an inner model. Let σ be a subset of a set $D \in M$, $\sigma \notin M$. Then the following are equivalent*

a) *σ is a support over M;*
b) *there exists a binary relation $r \in M$ such that $P(D \setminus \sigma) \cap M = r``\sigma$;*

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c) *there exists a preorder "$\preceq$" on D such that σ is a filter on $(D, \preceq)$ generic over M;*
d) *there exist a Boolean algebra $B \in M$ complete in M, a filter $G \subseteq B$ generic over M, a binary relation $r \in M$ and a function $f \in M$ such that $G = r``\sigma$ and $\sigma = f_{-1}(G)$.*

P. Vopěnka in [4] has proved the implication a) $\Rightarrow$ d). B. Balcar in [1] has proved the implication b) $\Rightarrow$ c) of Theorem 1. The other implications are known from the theory of semisets and from the theory of Boolean valued models.

The condition c) of Theorem 1 is equivalent to the following:

- there exist an equivalence relation $\sim \in M$ on D
- and a partial order $\leq \in M$ on $D/\!\sim$ such that
- $\sigma/\!\sim$ is a filter on $(D/\!\sim, \leq)$ generic over M.

If $B \in M$ is a Boolean algebra complete in M and $G \subseteq B$ is a filter generic over M, we define a function $f\colon \mathcal{P}(B) \cap M \longrightarrow B$ as

$$f(u) = -\sum u \qquad \text{for } u \subseteq B,\ u \in M. \tag{1}$$

Then the condition b) of Theorem 1 holds true with f_{-1} instead a binary relation as

$$P(B \setminus G) \cap M = f_{-1}(G). \tag{2}$$

We shall study how we can change the assertion c) if we replace the relation r in b) by the inverse of a function f_{-1} as above. We show

Theorem 2. *Assume that M is an inner model. Let σ be a subset of a set $D \in M$, $\sigma \notin M$, and let $f\colon \mathcal{P}(D) \cap M \longrightarrow D$ be a function in the model M. Then the following are equivalent*

a) *$P(D \setminus \sigma) \cap M = f_{-1}(\sigma)$;*
b) *there exist an equivalence relation $\sim \in M$ on D and a partial order $\leq \in M$ on $D/\sim$ such that $(D/\sim, \leq)$ is a Boolean algebra complete in M, $\sigma/\sim$ is a filter on $(D/\sim, \leq)$ generic over M, and for any $u \subseteq D$, $u \in M$, we have*

$$[f(u)]_\sim = -\sum \{[x]_\sim : x \in u\}.$$

2. Getting a partial order

In the next we assume that

$$D \in M,\ \sigma \subseteq D, \quad f\colon P(D) \cap M \longrightarrow D, \qquad f \in M, \quad P(D \setminus \sigma) \cap M = f_{-1}(\sigma).$$

We set

$$s(x) = \{y \in D\colon \exists\, u \in P(D) \cap M \ \ u \cap \{x, y\} \neq \emptyset \wedge f(u) \in \{x, y\}\}. \tag{3}$$

The intended interpretation is that $s(x)$ is the set of elements of the partially preordered set D incompatible with x.

Immediately from the definition we obtain

$$y \in s(x) \to x \in s(y) \qquad \text{for any } x, y \in D,$$

and

$$u \subseteq s(f(u)) \qquad \text{for any } u \subseteq D,\ u \in M. \tag{4}$$

If $x \in s(x)$, then by definition there is $u \in P(D) \cap M$ such that $x \in u$ and $f(u) = x$. Hence $x \notin \sigma$. Moreover then $u \cap \{x, y\} \neq \emptyset$ and $f(u) \in \{x, y\}$ for every $y \in D$. Thus

$$\text{if } \ x \in s(x),\ \text{then } \ x \notin \sigma, \tag{5}$$

and

$$x \in s(x) \ \text{ if and only if } \ s(x) = D. \tag{6}$$

If we take $u = \{x\}$ in the definition (3), we obtain

$$x \in s(f(\{x\})) \quad \text{and} \quad f(\{x\}) \in s(x) \qquad \text{for any } x \in D.$$

We define a preorder "$\preceq$" of the set D setting

$$x \preceq y \ \text{ if and only if } \ s(y) \subseteq s(x).$$

By (6), any $x \in D$ such that $x \in s(x)$ plays the role of the least element in this preorder.

The preorder "$\preceq$" induces an equivalence relation "$\approx$" defined by

$$x \approx y \ \text{ if and only if } \ s(x) = s(y).$$

The preorder "$\preceq$" becomes a partial order on the quotient set $D/\approx$. We identify elements of D with their equivalence classes and subsets of D with the set of corresponding equivalence classes. So, speaking about $x \in D$ we mean the equivalence class $[x]_\approx$ in $D/\approx$. Similarly for a subset of D.

Let

$$A_0 = \{x \in D \colon x \in s(x)\}, \qquad D_0 = D \setminus A_0.$$

It is easy to see that A_0 is hereditary downward, i.e.,

$$(y \preceq x \wedge x \in A_0) \to y \in A_0. \tag{7}$$

Since $f(D) \in D = s(f(D))$ we have

$$D \setminus D_0 \neq \emptyset. \tag{8}$$

By (6), A_0 is the $\preceq$-least equivalence class of $D/\approx$ and $D_0/\approx$ is the set of nonzero elements of $D/\approx$.

Note the following:

$$\text{if } \ z \preceq y \ \text{ and } \ y \in s(x), \text{ then } \ z \in s(x).$$

Indeed, if $z \preceq y$ and $y \in s(x)$, then $x \in s(y) \subseteq s(z)$. Hence $z \in s(x)$.

Let $X \subseteq D$. We say that elements $x, y \in D$ are *incompatible in* X, if for every $z \preceq x$, $z \preceq y$ we have $z \notin X$. If $x, y \in D_0$ are incompatible in D_0 we shall write $x \perp y$.

We show that for any $x \in D_0$, every element of the set $s(x) \cap D_0$ is incompatible with x.

$$\text{If } \ y \in s(x), \ z \preceq x, \ z \preceq y, \ x, y \in D_0, \ \text{ then } \ z \notin D_0.$$

So, assume that $y \in s(x)$ and $s(x) \subseteq s(z)$, $s(y) \subseteq s(z)$. Then $y \in s(z)$ and therefore also $z \in s(y)$. Thus $z \in s(z)$. Hence

$$\text{if } \ y \in s(x), \text{ then } \ x \perp y \ \text{ for any } \ x, y \in D_0. \tag{9}$$

By (4) we obtain that

$$\text{if } \ s(x) \cup s(f(u)) \subseteq s(y) \ \text{ for some } \ x \in u, \text{ then } \ y \in s(y). \tag{10}$$

In particular,

$$\text{if } \ s(x) \cup s(f(\{x\})) \subseteq s(y) \ \text{ for some } \ x, \text{ then } \ y \in s(y).$$

Lemma 3. a) *$s(x) \subseteq D \setminus \sigma$ for each $x \in \sigma$.*
b) *If $u \subseteq D \setminus \sigma$, $u \in M$, then there exists an $x \in \sigma$ such that $u \subseteq s(x)$.*

Proof: a) Let $x \in \sigma$. If $y \in s(x)$, then by (3) there exists a set $u \in P(D) \cap M$ such that

$$u \cap \{x, y\} \neq \emptyset \wedge f(u) \in \{x, y\}.$$

We have four possibilities. If $x \in u$ and $f(u) = x$ then $u \subseteq D \setminus \sigma$, a contradiction. If $x \in u$ and $f(u) = y$ then $u \nsubseteq D \setminus \sigma$, hence we obtain that $y = f(u) \notin \sigma$. If $y \in u$ and $f(u) = x$ then $u \subseteq D \setminus \sigma$, therefore $y \notin \sigma$. If $y \in u$ and $f(u) = y$, then $y \in \sigma$ implies that $u \subseteq D \setminus \sigma$, a contradiction. Thus $y \notin \sigma$.

b) Let $u \subseteq D \setminus \sigma$, $u \in M$. Then $x = f(u) \in \sigma$. By (4) we obtain $u \subseteq s(x)$. □

Lemma 4. *The set σ is a filter on $(D_0, \preceq)$ generic over M.*

Proof: By (5) we have $\sigma \subseteq D_0$.

Let $x \in \sigma$, $x \preceq y$. Assume that $y \notin \sigma$. Then $s(x) \cup \{y\} \subseteq D_0 \setminus \sigma$, hence by Lemma 3 b), there exists a $z \in \sigma$ such that $s(x) \cup \{y\} \subseteq s(z)$. Since $y \in s(z)$ also $z \in s(y) \subseteq s(x) \subseteq D_0 \setminus \sigma$, a contradiction.

Let x, $y \in \sigma$. Then by Lemma 3 a) $s(x) \cup s(y) \subseteq D \setminus \sigma$. Thus, by Lemma 3 b), there exists a $z \in \sigma$ such that $s(x) \cup s(y) \subseteq s(z)$. Then $z \preceq x$ and $z \preceq y$.

Assume that $u \subseteq D_0 \setminus \sigma$, $u \in M$. By Lemma 3 b), there exists an $x \in \sigma$ such that $u \subseteq s(x)$. By (9) every element of u is incompatible with x, therefore u is not dense in D_0. □

We denote by q_1 the quotient mapping $q_1\colon D_0 \longrightarrow D_0/\approx$ defined as $q_1(x) = [x]_\approx$. Since

$$(x \approx y \wedge x \in \sigma) \to y \in \sigma,$$

$q_1(\sigma)$ is a filter on $\langle D_0/\approx, \preceq\rangle$ generic over M.

The function $f\colon \mathcal{P}(B) \cap M \to B$ defined by (1) may be easily changed still keeping (2) true. E.g. take $u, v \subseteq B$ such that $f(u), f(v) \in G$, $f(u) \neq f(v)$. If you exchange the values $f(u)$ and $f(v)$, (2) is true and (1) fails. However the restriction of f to $\mathcal{P}(B|(f(u) \cdot f(v)))$ will satisfy (1).

So we must consider some "inconvenient" elements in D which we must omit. It turns out that none of those elements is in σ. We show that

$$\text{if } x \perp y \text{ for every } y \in u \text{ and } x \perp f(u), \text{ then } x \notin \sigma. \tag{11}$$

Assume that $x \in \sigma$. Then by Lemma 4 we obtain $u \subseteq D \setminus \sigma$. Therefore $f(u) \in \sigma$, which is a contradiction with $x \perp f(u)$.

In particular,

$$\text{if } x \perp y \text{ and } x \perp f(\{y\}), \text{ then } x \notin \sigma.$$

3. Getting a complete Boolean algebra

There exist an equivalence relation "$\asymp$" on $D_0/\approx$ and the quotient mapping $q_2\colon D_0/\approx \to (D_0/\approx)/\asymp$, see [3, page 205], preserving inequalities "$\preceq$" and $\preceq/\asymp$ and compatibility of elements in both sides, such that $((D_0/\approx)/\asymp, \preceq/\asymp)$ is a separative partially ordered set. We denote by "$\sim$" the equivalence relation on D_0 defined as

$$x \sim y \text{ if and only if } [x]_\approx \asymp [y]_\approx,$$

and the partial order "$\leq$" on $D_0/\sim$, compare [3, page 205], defined as

$$[x]_\sim \leq [y]_\sim \equiv \forall z \preceq x \quad [z]_\approx \text{ is compatible with } [y]_\approx \text{ in } D_0/\approx .$$

Then $(D_0/\sim, \leq)$ is a separative partially ordered set. We denote

$$q = q_1 * q_2\colon D_0 \longrightarrow D_0/\sim .$$

Then $q(\sigma)$ is a filter on $D_0/\sim$ generic over M.

Hence there exists a Boolean algebra $B_0 \in M$ complete in M and a mapping $e\colon D_0 \longrightarrow B_0$ such that $e``D_0$ is dense in B_0, and

$$\forall x, y \in D_0 \qquad x \preceq y \to e(x) \leq e(y), \tag{12}$$

$$\forall x, y \in D_0 \qquad x \perp y \equiv e(x), e(y) \text{ are incompatible in } B_0. \tag{13}$$

We can assume that $D_0/\sim\, \subseteq B_0$ and the partial order "$\leq$" coincides with the order of the Boolean algebra B_0. Then

$$e(x) = [x]_\sim \qquad \text{for any } x \in D_0.$$

We set

$$A = \{x \in D \colon \exists\, u \subseteq D,\, u \in M \quad (x \perp f(u) \ \wedge\ \forall\, y \in u \ \ x \perp y)\},$$
$$C = D \setminus A.$$

Evidently $A_0 \subseteq A$ and therefore $C = D_0 \setminus A$. By (11) we have

$$\sigma \subseteq C.$$

By (7) and by the definition, the set A is hereditary downward.

Let $a = e(f(A)) \in B_0$. We denote $B = B_0|a = \{x \in B_0 \colon x \leq a\}$. We set

$$h(x) = \begin{cases} e(x) \cdot a & \text{if } \ x \in C, \\ 0 & \text{if } \ x \in D \setminus C. \end{cases}$$

We show that $h``C$ is dense in $B \setminus \{0\}$. So let $b \in B$, $b \neq 0$. Then $b \leq a$. Since $e``D_0$ is dense in $B_0 \setminus \{0\}$, there exists $z \in D_0$ such that $0 \neq e(z) \leq b$. Assume that $z \in A$. Then $z \in s(f(A))$, hence $z \perp f(A)$. Hence $e(z) = e(z) \cdot a = 0$, a contradiction. Thus $z \in C$.

Evidently

$$\forall\, x, y \in D \qquad x \preceq y \ \rightarrow\ h(x) \leq h(y).$$

Let $u \subseteq C$, $u \in M$. By (10) we obtain that for any $x \in u$, the elements x and $f(u)$ are incompatible in $(D_0, \preceq)$. Hence by (13) we obtain $h(f(u)) \cdot \sum h``u = 0$.

By the definition of A, we have $\sum e``u + e(f(u)) \geq a$ in $B_0$. If $f(u) \in C$, then $\sum h``u + h(f(u)) = 1$ in B and

$$h(f(u)) = -\sum h``u \qquad \text{in } B.$$

If $f(u) \notin C$ then $\sum e``u \geq a$, i.e., $\sum h``u = 1$ in B and $-\sum h``u = 0 = h(f(u))$.

In particular we obtain that $h(f(\{x\})) = -h(x)$ for any $x \in C$.

If $x \in B$, $x \neq 0$ then $x = \sum h''u$, where $u = \{y \in C \colon h(y) \leq x\}$. Since $\sum h''u = h(f(\{f(u)\}))$, we obtain that $B \setminus \{0\} \subseteq h``C$. By (8), $D \setminus C \neq \emptyset$, hence $0 \in h``D$. Thus h is a surjection.

Since $\sigma \subseteq C$, we obtain that

$$\sigma/\!\sim\; = q(\sigma) \qquad \text{is a filter on } B \text{ generic over } M.$$

If we redefine the equivalence relation "$\sim$" as

$$x \sim y \ \text{ if and only if } \ h(x) = h(y)$$

for any $x, y \in D$, we obtain the assertion of the theorem.

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