Separating equivalence classes

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To the memory of Bohuslav Balcar

Abstract. Given a countable Borel equivalence relation, I introduce an invariant measuring how difficult it is to find Borel sets separating its equivalence classes. I evaluate these invariants in several standard generic extensions.

Keywords: countable Borel equivalence relation; forcing

Classification: 03E15, 03E17

1. Introduction

In this paper, I define and investigate a cardinal invariant motivated by the usual proof of the Glimm–Effros dichotomy, see [2, Theorem 10.4.1].

Definition 1.1. Let E be a countable Borel equivalence relation on a Polish space X. Define $\mathtt{sep}(E)$ to be the smallest possible cardinality of a family $\mathcal{B}$ of Borel subsets of X which *separates* E*-classes*: for any two E-unrelated elements $x_0, x_1 \in X$ there is a Borel set $B \in \mathcal{B}$ such that $[x_0]_E \subset B$ and $[x_1]_E \cap B = 0$.

I show that the cardinal invariant $\mathtt{sep}(E)$ respects the Borel reducibility between countable Borel equivalence relations (Theorem 2.1). I also provide basic ways of manipulating $\mathtt{sep}(E)$ for nonsmooth E: it is increased by the Silver forcings and its relatives (Theorem 3.1), while it is kept small by the Vitali forcing and its relatives (Corollary 5.7). The distinction appears to reside in the Ramsey theoretic features of a certain countable hypergraph of fundamental nature (Definition 4.1).

The main question remains open:

Question 1.2. Is $\mathtt{sep}(E) = \mathtt{sep}(F)$ provable in Zermelo–Fraenkel set theory (ZFC) whenever E, F are nonsmooth countable Borel equivalence relations?

In fact, in all models where I am able to evaluate the invariant $\mathtt{sep}(E)$, it happens to be equal to $\mathfrak{r}$, the splitting number. This leads to the following unlikely question.

Question 1.3. Is it provable in ZFC that $\mathtt{sep}(E) = \mathfrak{r}$ whenever E is a nonsmooth countable Borel equivalence relation?

DOI 10.14712/1213-7243.2015.267

The paper uses the set theoretic notational standard of [1]. For Borel equivalence relations E, F on respective Polish spaces X, Y, the relation E is said to be *reducible to* F if there is a Borel function, a *Borel reduction* $h\colon X \to Y$ such that for all $x_0, x_1 \in X$, $x_0\ E\ x_1 \leftrightarrow h(x_0)\ F\ h(x_1)$ holds. A Borel equivalence relation is *smooth* if it is reducible to the identity relation on 2^ω. The *Vitali equivalence relation* is the equivalence relation on 2^ω connecting binary sequences x_0, x_1 if they differ in at most finite number of entries. The well-known Glimm–Effros dichotomy [2, Theorem 10.4.1] says that every Borel equivalence relation E is either smooth or else the Vitali equivalence relation is reducible to it. The former alternative is equivalent to the statement that there is a countable collection of E-invariant Borel sets such that every pair of distinct E-classes can be separated by one of them. This observation is the initial motivation behind this paper.

2. Basic inequalities

In this section, I will provide the routine proof of the basic cardinal inequalities involving $\mathtt{sep}(E)$.

Theorem 2.1. *Let E, F be countable Borel equivalence relations on respective Polish spaces X, Y.*

(1) *If E is Borel reducible to F then* $\mathtt{sep}(E) \leq \mathtt{sep}(F)$;
(2) *if E is smooth then* $\mathtt{sep}(E) = \aleph_0$;
(3) *if E is not smooth then* $\mathtt{sep}(E) \geq \mathtt{cov}(\mathtt{meager}), \mathtt{cov}(\mathtt{null})$.

Proof: For (1), let $h\colon X \to Y$ be a Borel map reducing E to F. To see that $\mathtt{sep}(E) \leq \mathtt{sep}(F)$, suppose that $\mathcal{C}$ is a collection of Borel subsets of Y separating F-classes. Let $\mathcal{B} = \{h^{-1}B\colon B \in \mathcal{C}\}$, observe that $|\mathcal{B}| \leq |\mathcal{C}|$ and chase diagrams to show that $\mathcal{B}$ separates E-classes. (2) is just a restatement of the usual proof of the Glimm–Effros dichotomy [2, Theorem 10.4.1]. For (3), by (1) and the Glimm–Effros dichotomy it is enough to treat the case of the Vitali equivalence relation E on 2^ω, connecting points $x, y \in 2^\omega$ if they differ at a finite number of entries. For each point $x \in 2^\omega$ let $s(x)$ be the set of points in 2^ω which agree with x at only finite number of entries. Clearly, $s(x)$ is an E-class of points E-unrelated to x.

Claim 2.2. *Let $B \subset 2^\omega$ be a Borel set. The set $C_B = \{x \in B\colon s(x) \cap B = 0\}$ is meager and null.*

Proof: The set C_B is clearly Borel. Suppose first toward contradiction that it is not meager. Then there must be a binary string $t \in 2^{<\omega}$ such that C_B is comeager in $[t]$. Use standard arguments to find a point $x \in [t]$ such that $x \in C_B$ and the point $y \in 2^\omega$ which differs from x at exactly all entries past $|t|$, belongs to C_B as well. It follows that $x \in B$ and $y \in B$. At the same time, $y \in s(x)$. This contradicts the definition of the set C_B.

Suppose now that the set C_B is not null. Use the Lebesgue density theorem to find a binary string $t \in 2^{<\omega}$ such that $C_B \cap [t]$ has Lebesgue mass bigger than

$3\mu[t]/4$. Let $m = |t|$ and let $h\colon 2^\omega \to 2^\omega$ be the measure-preserving involution of 2^ω such that for every $x \in 2^\omega$, $h(x)$ differs from x at exactly all entries past $|t|$. Then $h(C_B) \cap C_B \cap [t]$ has nonzero Lebesgue mass. Pick a point x in $h(C_B) \cap C_B \cap [t]$. Clearly, $x, h(x) \in B$ and $h(x) \in s(x)$. This contradicts the definition of the set C_B. $\square$

Now suppose that κ is a cardinal smaller than one of $\mathtt{cov}(\mathtt{meager}), \mathtt{cov}(\mathtt{null})$ and let $\{B_\alpha\colon \alpha \in \kappa\}$ be a family of Borel subsets of X. The claim implies that there is $x \in 2^\omega$ such that for every $\alpha \in \kappa$, $x \notin C_{B_\alpha}$; in other words, no set B_α can separate x from the E-class $s(x)$. The proof is complete. $\square$

3. Making $\mathtt{sep}(E)$ large

In this section, I provide the routine proof that the Silver forcing and its products increase the invariant $\mathtt{sep}(E)$ for a nonsmooth equivalence relation E. In view of Theorem 2.1, this shows how to separate $\mathtt{sep}(E)$ from the covering of the meager and null ideals. Recall that the Silver forcing is the poset P of all partial functions $p\colon \omega \to 2$ with co-infinite domain, ordered by reverse extension.

Theorem 3.1. *Let κ be a regular cardinal greater than the continuum. Then in the extension given by the countable support product of κ-many copies of the Silver forcing, $\mathtt{sep}(E) \geq \kappa$ for all nonsmooth countable Borel equivalence relations E.*

Proof: Consider the equivalence relation E on 2^ω connecting points $x_0, x_1 \in 2^\omega$ if they differ at finitely many entries only, and the set $\{n \in \omega\colon x_0(n) \neq x_1(n)\}$ is of even size. It is immediate that this equivalence relation is nonsmooth and hyperfinite. By [2, Theorem 8.1.1], it is Borel bireducible with the Vitali equivalence relation and so by the Glimm–Effros dichotomy it is reducible to every nonsmooth Borel countable equivalence relation. Thus, it is enough to show that $\mathtt{sep}(E) \geq \kappa$ in the extension. Consider two Silver names for elements of 2^ω, $\dot{x}_0$ and $\dot{x}_1$: the former is simply the union of all conditions in the generic filter, and the latter is equal to $\dot{x}_0$ at all entries except for the 0th entry. Clearly, the points $\dot{x}_0, \dot{x}_1$ are forced to be E-unrelated. The main point of the proof is that the E-classes of these two points can be flipped in the following precise sense:

Claim 3.2. *Let $p \in P$ be a condition. There are conditions $p_0, p_1 \leq P$ and an isomorphism of $P \restriction p_0$ to $P \restriction p_1$ such that $p_0 \Vdash \dot{x}_0/\dot{G}\ E\ \dot{x}_1/\pi''\dot{G}$ and $\dot{x}_1/\dot{G}\ E\ \dot{x}_0/\pi''\dot{G}$.*

Proof: Let $n \in \omega$ be the first point in $\omega \setminus \mathrm{dom}(p)$, let $p_0 = p \cup \{\langle n, 0\rangle\}$, let $p_1 = p \cup \{\langle n, 1\rangle\}$, and let π be the map with domain equal to $P \restriction p_0$ which to each condition $q \leq p$ assigns the condition $\pi(q) \leq p_1$ which is equal to q at all entries except for the nth entry. This clearly works. $\square$

Now, let P_κ be the countable support product of κ-many copies of P. Suppose that $\mathcal{B}$ is a collection of size less than κ of Borel subsets of X in the P_κ-extension;

I must show that the family does not separate E-classes. By a chain condition argument, the collection $\mathcal{B}$ appears already in the extension given by P_λ, the countable support product of the first λ-many copies of P for some $\lambda < \kappa$. Let $x_0, x_1 \in X$ be the points derived from the λth generic on P; I claim that no set in $\mathcal{B}$ separates the E-equivalence classes of x_0, x_1. To see this, go back to the ground model, and towards a contradiction assume that $p \in P_\kappa$ is a condition, $\dot{B}$ is a name for some element of $\mathcal{B}$ and $p \Vdash [\dot{x}_0]_E \subset \dot{B}$ and $[\dot{x}_1]_E \cap \dot{B} = 0$. Use the claim on the λth coordinate to find an automorphism π of the poset P_κ which leaves P_λ unmoved and exchanges the E-classes of $\dot{x}_0$ and $\dot{x}_1$. Let $G \subset P_\kappa$ be a generic filter, and consider also the generic filter $\pi''G$. Both of these filters evaluate the set B in the same way and the switch the E-classes of $\dot{x}_0, \dot{x}_1$. This means that at least one of them is in violation of the forcing theorem. $\square$

4. The key hypergraph

For the purposes of keeping the invariant $\mathtt{sep}(E)$ small, I will consider a hypergraph of independent interest. Recall that a *hypergraph* on a set X is just a collection of subsets of X. The hypergraphs in this paper will always be *finitary*, i.e. they consist of finite subsets of X only; they will also be *analytic*, meaning that the underlying set X can be viewed as a Polish space and the hypergraph is then an analytic subset of the hyperspace $K(X)$.

Definition 4.1. Let n, m be natural numbers with $m \geq 2$. Let V_{nm} be the set of all partial functions from ω to m of size n. The hypergraph H_{nm} is the collection of finite sets $e \subset V_{nm}$ such that $\bigcup e$ is not a function.

For the Δ-system arguments in set theory, it seems to be of great concern which countable (hyper)graphs can be homomorphically mapped to H_{nm}. This is an interesting subject in itself; I provide only the most basic propositions in this direction.

Proposition 4.2. *If H is a hypergraph with chromatic number less than or equals to 2^n, then H can be homomorphically mapped to H_{n2}.*

Proof: Let V be the vertex set of H, let $g\colon V \to 2^n$ be a coloring with no monochromatic hyperedges. Viewing 2^n as a binary string, this is clearly a homomorphism of H to H_{n2}. $\square$

Proposition 4.3. *For every $n, m \in \omega$ with $m \geq 2$ there is $k \in \omega$ such that the clique of k vertices cannot be homomorphically mapped to H_{nm}.*

Proof: Let k be such that $k \to (m+1)^2_{n^2}$; I claim that k works. Let H be a clique on a set V of vertices of size k, and suppose that $f\colon V \to V_{nm}$ is a function; I must find a pair of vertices $\{v_0, v_1\}$ such that $f(v_0) \cup f(v_1)$ is a function. Suppose towards contradiction that such a pair does not exist, and define $g\colon [V]^2 \to n \times n$ by $g(v_0, v_1) = \langle m_0, m_1 \rangle$ if m_0th element of $\mathrm{dom}(f(v_0))$ is equal to m_1th element of $\mathrm{dom}(f(v_1))$ and the functions f_0, f_1 yield a different output on this number.

By the partition assumption on k, there must be a homogeneous set $W \subset V$ of size $m+1$, with a homogeneous color (m_0, m_1). A brief review of the definitions shows that $m_0 = m_1$ must hold. Let p be the common m_0th element of $\mathrm{dom}(f(v))$ for $v \in W$ and observe that $\{f(v)(p) \colon v \in W\}$ would have to be a collection of $m+1$ distinct numbers smaller than m, an impossibility. $\square$

Given a nonempty finitely branching tree $T \subset \omega^{<\omega}$, define the hypergraph G_T on T to contain those finite sets $e \subset T$ such that for some vertex $t \in T$, every element of e properly extends t and every immediate successor of t has exactly one extension in the set e.

Proposition 4.4. *Let $T \subset \omega^{<\omega}$ be a nonempty finitely branching tree with no end-nodes. Then G_T cannot be homomorphically mapped to H_{nm} for any $n, m \in \omega$.*

Proof: Let $n, m \in \omega$ be natural numbers with $m \geq 2$, and let $h \colon T \to V_{nm}$ be a function. I must produce an edge $e \in G_T$ such that $\bigcup_{t \in e} h(t)$ is a function. Suppose towards contradiction that such an edge does not exist.

Claim 4.5. *For every somewhere dense set $S \subset T$ there is $t \in S$ such that the set $\{s \in S \colon \{h(s), h(t)\} \in H_{nm}\}$ is again somewhere dense.*

Proof: If this failed for some somewhere dense set $S \subset T$, pick $u \in T$ such that S is dense below u. Let $\{u_j \colon j \in k\}$ be a list of immediate successors of the node u, and by induction on $j \in k$ pick nodes $t_j \in S$ extending u_j such that $\bigcup_{i \leq j} h(t_i)$ is a function, if possible. This construction would result in a G_T-edge whose h-image is not in H_{nm}, so for some $j \in k$ the node t_j cannot be found. This means that for each node $s \in S$ extending u_j, there is $i \in j$ such that $\{h(s), h(t_i)\} \in H_{nm}$. Thus, for some $i \in j$, the set $\{s \in S \colon \{h(s), h(t_i)\} \in H_{nm}\}$ is somewhere dense below u_j. The claim follows. $\square$

Now, by induction on $k \in \omega$ build somewhere dense sets $S_k \subset T$ and nodes $t_k \in S_k$ such that $S_0 = T$ and for all $s \in S_{k+1}$, $\{h(s), h(t_k)\} \in H_{nm}$. This is possible by the claim. However, in the end the values $h(t_k)$ for $k \in \omega$ would form an infinite clique in H_{nm}, an impossibility by Proposition 4.3. $\square$

5. Keeping $\mathtt{sep}(E)$ low

In this section, I show that in many product and iterated forcing extensions, the invariants $\mathtt{sep}(E)$ remain small. As usual, this is much more demanding than Theorem 3.1; the point of this paper is that the hypergraph forcing technology of [4] reduces the considerations to their combinatorial core. The relevant definitions:

Definition 5.1. Let Y be a Polish space and $\mathcal{G}$ a countable family of analytic hypergraphs on Y. The σ-ideal $I_{\mathcal{G}}$ is generated by Borel subsets of Y which are anticliques in at least one of the hypergraphs in $\mathcal{G}$. The poset $P_\mathcal{G}$ is the quotient poset of Borel $I_\mathcal{G}$-positive sets ordered by inclusion.

Numerous quotient posets of the form $P_\mathcal{G}$ are proper, and numerous definable proper forcing in pre-existing literature have a hypergraph presentation of the form $P_\mathcal{G}$. The countable support iteration and product can then be restated in terms of simple operations on hypergraphs. This makes many preservation theorems easy to state and prove. The strategy for this section is to show that iterations of hypergraphable forcings of a certain form do not increase the invariant $\mathtt{sep}(E)$. To state the instrumental iterable preservation property, recall that a poset $\langle P, \leq\rangle$ is *Suslin* if there is an ambient Polish space Z such that P is an analytic subset of Z, and so are the ordering and incompatibility relations on P.

Definition 5.2 ([4, Definition 3.2])**.** Let Y be a Polish space and $\mathcal{G}$ a countable family of finitary analytic hypergraphs on Y. Let P be a Suslin poset. The symbol $\mathcal{G} \not\perp P$ stands for the following: if $B \subset Y$ is a Borel $I_\mathcal{G}$-positive set, $G \in \mathcal{G}$ is a hypergraph and $f\colon B \to P$ is a Borel map, then there is an edge $e \in G$ consisting only of vertices in B such that the conditions $f(y)$ for $y \in e$ have a common lower bound in P.

The property $\mathcal{G} \not\perp P$ is preserved under the countable support iterations and certain products of hypergraphable ideals [4, Theorem 5.4], and this is the road we take to ensure that in the resulting forcing extensions $\mathtt{sep}(E) = \aleph_1$ holds. In order to use the machinery successfully, one needs to select a useful Suslin countable chain condition (ccc) forcing P. Let E be a countable Borel equivalence relation on a Polish space X. Let P be the partial order of finite partial functions from X to ω which assign equal values to E-equivalent points; the ordering is that of reverse inclusion. As a forcing, the poset P is quite uneventful; a moment's thought will show that if the Polish space X is uncountable then P is isomorphic to a finite support product of $\mathfrak{c}$ many Cohen forcings. However, the important feature of P is its Borel presentation rather than its forcing properties.

Theorem 5.3. *Let Y be a Polish space and $\mathcal{G}$ a countable family of finitary hypergraphs on Y such that the poset $P_\mathcal{G}$ is proper. If $\mathcal{G} \not\perp P$, then in the $P_\mathcal{G}$-extension, any two distinct E-classes can be separated by ground model coded Borel subsets of X.*

Proof: The argument needs a slight technical upgrade of the poset P: a Suslin ccc poset Q which adds a countable sequence $\langle \dot{B}_n \colon n \in \omega\rangle$ of Borel subsets of X which separate any pair of distinct ground model coded E-classes. Let $S \subset \mathcal{P}(\omega)$ be a perfect collection of pairwise almost disjoint infinite subsets of ω. Let $g\colon X \to S$ be a Borel injection. The poset Q^0 consists of triples $q = \langle a_q, b_q, c_q\rangle$ where $a_q \subset \omega$ and $b_q, c_q \subset X$ are finite sets such that $[b_q]_E \cap [c_q]_E = 0$. The ordering is defined by $r \leq q$ if $a_q \subset a_r, b_q \subset b_r$, $c_q \subset c_r$, and for each $x \in b_q$, $g(x) \cap a_r \setminus a_q = 0$. The generic filter adds an infinite set $\dot{a} \subset \omega$ which is the union of the first coordinates of the conditions in the generic filter, and the derived Borel set $\dot{B} \subset X$ defined as $\dot{B} = \{x \in X : g(x) \cap \dot{a} \text{ is finite}\}$. The following is easy to check:

Claim 5.4. *Let E be a countable Borel equivalence relation on a Polish space X.*

(1) *The poset Q^0 is Suslin ccc;*
(2) *for any two E-unrelated points $x_0, x_1 \in X$ the triple $\langle 0, \{x_0\}, \{x_1\}\rangle$ is a condition in Q^0 and it forces $[\check{x}_0]_E \subset \dot{B}$, $[\check{x}_1]_E \cap \dot{B} = 0$.*

Let Q be the finite support product of infinitely many copies of Q^0; this is again a Suslin ccc forcing. It adds sets $\dot{a}_n \subset \omega$ for every coordinate $n \in \omega$. Let Z be the Polish space $(\mathcal{P}(\omega))^\omega$, and let J be the σ-ideal generated by the analytic sets $A \subset Z$ such that $Q \Vdash \langle \dot{a}_n : n \in \omega\rangle \notin \dot{A}$. The following claim is the main point of the perpendicularity assumption $\mathcal{G} \not\perp P$.

Claim 5.5. *Whenever $B \subset Y$ is a Borel $I_{\mathcal{G}}$-positive set and $C \subset B \times Z$ is a Borel set, then either there is a vertical section of C which does not belong to J, or there is a horizontal section of the complement of C which does not belong to $I_{\mathcal{G}}$.*

Proof: By Theorem 3.3 of [4], it is enough to show that $\mathcal{G} \not\perp Q$ holds. This, however, follows mechanically from the assumption $\mathcal{G} \not\perp P$. Assume that $B \subset X$ is an $I_{\mathcal{G}}$-positive Borel set, $f : B \to Q$ is a Borel function and $G \in \mathcal{G}$ is a hypergraph; we must find an edge $e \in G$ consisting of points in B such that the set $f''e \subset Q$ has a lower bound.

To establish succinct notation, note that every condition $q \in Q$ is a function with finite domain $\mathrm{dom}(q) \subset \omega$, and for each $n \in \mathrm{dom}(q)$, $q(n)$ is an element of Q^0 and as such is a triple with coordinates $\langle a_{q(n)}, b_{q(n)}, c_{q(n)}\rangle$. Use the σ-additivity of the σ-ideal $I_{\mathcal{G}}$ to thin down the Borel set B if necessary so that all conditions $f(y)$ for $y \in B$ have the same domain $d \subset Q$, and for each $n \in d$ have the same coordinate $a_{f(y)(n)}$. Now, for each $y \in B$ let $h(y)$ be the condition in P such that $\mathrm{dom}(h(y)) = \bigcup\{b_{f(y)(n)} \cup c_{f(y)(n)} : n \in \mathrm{dom}(f(y))\}$, and for each $x \in \mathrm{dom}(h(y))$ let $h(y)(x)$ be (a natural number coding) the set $\{\langle n, 0\rangle :$ for some $x' \in \mathrm{dom}(f(y))$, $x' \mathrel{E} x$ and $x' \in b_{f(y)(n)}\} \cup \{\langle n, 1\rangle :$ for some $x' \in \mathrm{dom}(f(y))$, $x' \mathrel{E} x$ and $x' \in c_{f(y)(n)}\}$. By the assumption $\mathcal{G} \not\perp P$, there is an edge $e \in G$ such that the set $h''e$ has a common lower bound in P. A review of definitions shows that the set $f''e$ has a common lower bound in Q. $\square$

To conclude the proof of the theorem, suppose that $\dot{x}_0, \dot{x}_1$ are $P_{\mathcal{G}}$-names for E-unrelated elements of X and $B \in P_{\mathcal{G}}$ is a condition. By the Borel reading of names, it is possible to thin out the Borel set B to find Borel functions $f_0, f_1 : B \to X$ such that for every $y \in B$, $f_0(y) \mathrel{E} f_1(y)$ fails, and $B \Vdash \dot{x}_0 = \dot{f}_0(\dot{y}_{\mathrm{gen}})$ and $\dot{x}_1 = \dot{f}_1(\dot{y}_{\mathrm{gen}})$. Let $C \subset B \times Z$ be the Borel set of all tuples $\langle y, a_n : n \in \omega\rangle$ such that for all $n \in \omega$ there are $x'_0 \mathrel{E} f_0(y)$ and $x'_1 \mathrel{E} f_1(y)$ such that $a_n \cap g(x'_0)$ is infinite if and only if $a_n \cap g(x'_1)$ is infinite. The vertical sections of the Borel set C are J-small. The claim shows that there must be a tuple $\langle a_n : n \in \omega\rangle \in Z$ such that the Borel set $B' = \{y \in B : \langle y, a_n : n \in \omega\rangle \notin C\}$ is $I_{\mathcal{G}}$-positive. The Borel set B' forces that for some number $n \in \omega$, the ground model coded Borel set $\{x \in X : \dot{g}(x) \cap a_n$ is infinite$\}$ separates the equivalence classes $[\dot{x}_0]_E$ and $[\dot{x}_1]_E$. □

Proposition 5.6. *Let Y be a Polish space and $\mathcal{G}$ a countable family of finitary analytic hypergraphs on Y. If for every Borel $I_\mathcal{G}$-positive set $B \subset Y$, every hypergraph $G \in \mathcal{G}$, and every $n \in \omega$ there is a countable set $a \subset B$ such that $G \restriction [a]^{<\aleph_0}$ cannot be homomorphically mapped to H_{nm}, then $\mathcal{G} \not\perp P$.*

Proof: Let $B \subset Y$ be a Borel $I_\mathcal{G}$-positive set, $G \in \mathcal{G}$ be a hypergraph, and $f : B \to P$ be a Borel function. I need to find an edge $e \in G$ consisting of points in B such that the set $f''e \subset P$ has a lower bound in the poset P. To find the edge, first use a counting argument and a σ-additivity of the σ-ideal $I_\mathcal{G}$ to thin out the condition B if necessary so that there are numbers $n, m \in \omega$ such that for each $y \in B$, $|\mathrm{dom}(f(y))| \leq n$ and $\mathrm{rng}(f(y)) \subset m$ holds. Let $a \subset B$ be a countable set such that $G \restriction [a]^{<\aleph_0}$ cannot be homomorphically mapped to H_{nm}. Let b be the countable set of E-classes represented by elements of $\bigcup_{y\in a} \mathrm{dom}(f(y))$ and define the map h assigning to each point $y \in a$ the finite partial function $h(y) : b \to m$ defined by $h(y)(c) = i$ if there is $x \in c$ such that $f(y)(x) = i$. Since h is not a homomorphism from G to H_{nm}, there must be an edge $e \in G$ consisting of elements of the set a only such that $\bigcup_{y\in\mathrm{rng}(e)} h(y)$ is a function. This means that the conditions $f(y)$ for $y \in e$ have a common lower bound in P as desired. □

Corollary 5.7. *Let E be a nonsmooth countable equivalence relation on a Polish space X. In the iterated or product Vitali extension of a model of the Continuum hypothesis, $\mathtt{sep}(E) = \aleph_1$.*

Proof: Recall that the Vitali forcing is obtained from the family $\mathcal{G} = \{G\}$ where G is the Vitali equivalence relation on $Y = 2^\omega$, connecting two points just in case they differ in only a finite number of entries. If $B \subset Y$ is an $I_\mathcal{G}$-positive Borel set, the Glimm–Effros dichotomy yields a G-preserving continuous injection from Y to B. This immediately implies that B contains an infinite G-clique. By Propositions 5.6 and 4.3, it follows that $\mathcal{G} \not\perp P$ holds. Now, $\mathcal{G} \not\perp P$ is preserved by countable support iterations, and also by the countable support product of actionable hypergraph families by [4, Section 5]. It follows from Theorem 5.3 that in the resulting extension, any two distinct E-classes can be separated by ground model coded Borel subsets of X. □

Corollary 5.8. *Let $\mathcal{G}$ be a countable family of finitary open hypergraphs on a Polish space Y. In the countable support iterated $P_\mathcal{G}$-extension of a model of the Continuum hypothesis, $\mathtt{sep}(E) = \aleph_1$.*

The posets of the form discussed in Corollary 5.8 are always proper; they include Sacks forcing, $c_{\min}$ forcing and similar. The resulting σ-ideals are exactly the σ-ideals generated by a σ-compact family of compact sets, whose quotients were studied in [3, Theorem 4.1.8]. I do not know if the products of these forcings are hypergraphable, and as a consequence the corollary does not apply to the product extension.

Proof: Suppose that $B \subset Y$ is a Borel $I_{\mathcal{G}}$-positive set and $G \in \mathcal{G}$ is a hypergraph. It will be enough to show that there is a finitely branching tree $T \subset \omega^{<\omega}$ with no terminal nodes, and a homomorphism $h\colon T \to B$ of the hypergraph G_T to G. Then, look at the countable set $a = \operatorname{rng}(h) \subset B$. By Proposition 4.4, there is no homomorphism of $G \restriction a$ to H_{nm} for any natural number n. The corollary follows by Proposition 5.6 and Theorem 5.3 and the preservation theorems of [4, Section 5].

The homomorphism h is not difficult to produce. Let $C = B \setminus \bigcup\{B \cap O\colon O \subset Y$ is a basic open set such that $B \cap O \in I_{\mathcal{G}}\}$. The set C is Borel, $I_{\mathcal{G}}$-positive, and its intersection with every open set is either $I_{\mathcal{G}}$-positive or empty. By tree induction build finite trees $T_n \subset \omega^{<\omega}$ and maps g_n from T_n to basic open subsets of Y so that

(1) $t \subset s$ implies $g_n(s) \subset g_n(t)$, and $g_n(s) \cap C \neq 0$;
(2) T_{n+1} end-extends T_n and g_{n+1} extends g_n;
(3) whenever $t \in T_n$ is an endnode, writing $b \subset T_{n+1}$ for the set of immediate successors of t, then $\prod_{s \in a} g_{n+1}(s) \subset G$.

This is easy to do using the fact that the hypergraph G is open and its edges are finite. In the end, let $T = \bigcup_n T_n$, and let $h\colon T \to B$ be any map which assigns to a node $t \in T_n$ any point in $C \cap g_n(t)$. It is easy to check that the map h is a homomorphism of G_T to G. □

Corollary 5.9. *Let $\mathcal{H}$ be a countable family of finitary open hypergraphs on a Polish space Y, invariant under a continuous action of a countable group Γ. Let $\mathcal{G}$ be the family of hypergraphs obtained by intersecting the hypergraphs in $\mathcal{H}$ with the orbit equivalence relation. In the countable support iterated or product $P_{\mathcal{G}}$-extension of Continuum hypothesis,* $\mathtt{sep}(E) = \aleph_1$.

The posets of the form discussed in Corollary 5.9 are always proper, since they are actionable. They should be viewed as symmetric versions of the posets generated by finitary open hypergraphs. They have not appeared in the literature explicitly, but they are useful, among other things, in the study of the countable support products of the posets generated by finitary open hypergraphs.

Proof: Suppose that $B \subset Y$ is a Borel $I_{\mathcal{G}}$-positive set and $G \in \mathcal{G}$ is a hypergraph which is an intersection of some open hypergraph $H \in \mathcal{H}$ with the orbit equivalence relation. It will be enough to show that there is a finitely branching tree $T \subset \omega^{<\omega}$ with no terminal nodes, and a homomorphism $h\colon T \to B$ of the hypergraph G_T to G. Then, look at the countable set $a = \operatorname{rng}(h) \subset B$. By Proposition 4.4, there is no homomorphism of $G \restriction a$ to H_{nm} for any natural number n. The corollary follows by Proposition 5.6 and Theorem 5.3 and the preservation theorems of [4, Section 5].

To construct T and h, enumerate the set $\omega^{<\omega}$ as $\langle t_n\colon n \in \omega\rangle$ with infinite repetitions and by induction on $n \in \omega$ build finite trees $T_n \subset \omega^{<\omega}$, $I_{\mathcal{G}}$-positive compact sets C_n and group elements γ_t for $t \in T_n$ as follows:

(1) $T_0 = \{0\}$, $C_0 \subset B$ is an arbitrary compact $I_{\mathcal{G}}$-positive set, and $\gamma_0 = 1_\Gamma$;

(2) if t_n is not an endnode of T_n then $T_{n+1} = T_n$ and $C_{n+1} = C_n$;
(3) if t_n is an endnode of T_n then $T_{n+1} = T_n \cup a_n$ where $a_n \subset \omega^{<\omega}$ is a finite set of immediate successors of t_n, $C_{n+1} \subset C_n$ is a set of metric diameter less than 2^{-n} and γ_s for $s \in a_n$ are chosen so that $\prod_{s\in a_n} \gamma_s \cdot C_{n+1} \subset H$ and $\gamma_s \cdot C_{n+1} \subset \gamma_{t_n} \cdot C_n$.

To start the induction note that the generating hypergraphs are finitary and therefore the poset $P_{\mathcal{G}}$ is bounding [4, Corollary 3.18] and so compact sets are dense in it [3, Theorem 3.3.2]. To perform the induction step in case (3), write $t = t_n$. There must be a finite set $b \subset \Gamma$ and a function o assigning to each element of b a basic open subset of Y such that $\prod_{\gamma\in b} o(\gamma) \subset H$ and the set $D_{bo} = \{y \in C_n \colon \forall \gamma \in b \ \ \gamma \cdot y \in o(\gamma) \cap \gamma_t \cdot C_n\}$ must be $I_{\mathcal{G}}$-positive. (Otherwise the positive set $\gamma_t \cdot C_n$ would be covered by all the countably many $I_{\mathcal{G}}$-small sets $\gamma_t \cdot D_{bo}$ and a G-anticlique, an impossibility.) Let C_{n+1} be some compact $I_{\mathcal{G}}$-positive subset of D_{bo} of diameter less than 2^{-n}, select a set $a_n \subset \omega^{<\omega}$ of immediate successors of t, a complete list $\langle \gamma_s \colon s \in a_n \rangle$ of elements of the set b, and let $T_{n+1} = T_n \cup a_n$. This concludes the induction step.

In the end, let $y \in Y$ be the unique point in the intersection $\bigcap_n C_n$. Let $T = \bigcup_n T_n$, and let $h \colon T \to Y$ be the map defined by $h(t) = \gamma_t \cdot y$. The induction procedure guarantees that h is a homomorphism of the hypergraph G_T to H; since the range of h consists of points in a single Γ-orbit, it is also a homomorphism of G_T to G. □

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(*Received* December 23, 2017, *revised* August 16, 2018)