Comment.Math.Univ.Carolin. 60,1 (2019) 39–47

A combinatorial proof of the extension property for partial isometries

Jan Hubička, Matěj Konečný, Jaroslav Nešetřil

In memory of Bohuslav Balcar

Abstract. We present a short and self-contained proof of the extension property for partial isometries of the class of all finite metric spaces.

Keywords: metric space; Hrushovski property; extension property for partial automorphisms; homogeneous structure; amalgamation class

Classification: 20B27, 05E18, 54E35, 20F05, 22F50, 37B05

1. Introduction

A class of metric spaces $\mathcal{C}$ has the *extension property for partial isometries* if for every $\mathbf{A} \in \mathcal{C}$ there exists $\mathbf{B} \in \mathcal{C}$ containing $\mathbf{A}$ as a subspace with the property that every isometry of two subspaces of $\mathbf{A}$ extends to an isometry of $\mathbf{B}$. (By isometry we mean a bijective distance-preserving function.) In this note we give a self-contained combinatorial proof of the following theorem:

Theorem 1.1 (S. Solecki [23], A. M. Vershik [25])**.** *The class of all finite metric spaces has the extension property for partial isometries.*

This result is important from the point of view of combinatorics, model theory as well as topological dynamics. It has several proofs, see [23], [18], [19], [21, Theorem 8.3], which are based on deep group-theoretic results (the Hall theorem in [4], the Herwig–Lascar theorem in [5], [17], [22], the Ribes–Zalesskiĭ theorem in [20] or Mackey's construction in [14]). A. M. Vershik announced an elementary proof in [25] which remains unpublished and differs from the approach presented here (this information is gained from personal communication on July 28, 2018).

Our construction is elementary. We follow a general strategy analogous to the corresponding results about the existence of Ramsey expansions of the class of finite metric spaces developed in series of papers [16], [15], [11]. We proceed in two steps.

First, given a metric space $\mathbf{A}$, we find an edge-labelled graph $\mathbf{B}_0$ which extends all partial isometries of $\mathbf{A}$, but does not define all distances between vertices and

DOI 10.14712/1213-7243.2015.275

Jan Hubička and Matěj Konečný are supported by project 18-13685Y of the Czech Science Foundation (GAČR). Jan Hubička is also supported by Charles University project Progres Q48.

may not have a completion to metric space (for example, it may contain nonmetric triangles). This step is analogous to the easy combinatorial proof of Hrushovski's theorem by B. Herwig and D. Lascar, see [5].

In the second step we further expand and "sparsify" $\mathbf{B}_0$ in order to remove all obstacles which prevent us from being able to define the missing distances and get a metric space. Once all such obstacles are eliminated, we can complete the edge-labelled graph to a metric space $\mathbf{B}$ by assigning every pair of vertices a distance corresponding to the shortest path connecting them. This part is inspired by a clique-faithful EPPA construction of I. Hodkinson and M. Otto in [7] (see also Hodkinson's exposition in [6]).

Similarly to the Ramsey constructions which were developed to work under rather general structural conditions in [11], our technique generalises further to classes described by forbidden homomorphisms as well as to the classes with algebraic closures (in the sense of [3]) and antipodal metric spaces (as shown in [2]). These strengthenings are going to appear elsewhere, see [10].

2. Notation and preliminaries

Given a set of labels L, an *L-edge-labelled graph* is an (undirected) graph where every edge has a unique *label* $l \in L$. In our proof we use "partial" metric spaces (where some distances are not known) and thus we will consider metric spaces as a special case of $\mathbb{R}^{>0}$-edge-labelled graphs where $\mathbb{R}^{>0}$ is the set of positive reals: an $\mathbb{R}^{>0}$-edge-labelled graph is then a *metric space* if it is complete (that is, every pair of vertices is connected by an edge) and for every triple of distinct vertices x, y, z the labels of edges $\{x, y\}$, $\{y, z\}$ and $\{x, y\}$ satisfy the triangle inequality.

While we need to work with edge-labelled graphs to represent intermediate objects in our construction, we find it useful to adopt standard terminology of metric spaces. If vertices x and y of an edge-labelled graph $\mathbf{A}$ form an edge with label l, we will also say that the edge $\{x, y\}$ has *length* l, or write $d_{\mathbf{A}}(x, y) = d_{\mathbf{A}}(y, x) = l$ and say that l is the *distance* between x and y.

We will use bold letters such as $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ to denote edge-labelled graphs and the corresponding normal letters $(A, B, C, \ldots)$ to denote the corresponding vertex sets.

Given two L-edge-labelled graphs $\mathbf{A}$ and $\mathbf{B}$, a function $\varphi\colon A \to B$ is a *homomorphism* if for every pair of vertices $x, y \in A$ which forms an edge with label l in $\mathbf{A}$ it holds that $\varphi(x), \varphi(y)$ is an edge with label l in $\mathbf{B}$. If φ is injective, it is a *monomorphism.* A monomorphism where for every $x, y \in A$ it holds that x, y form an edge with label l if and only if $\varphi(x), \varphi(y)$ form an edge with the same label l is called *embedding.* If $A \subseteq B$ and the inclusion map is a monomorphism, we say that $\mathbf{A}$ is a *subgraph* of $\mathbf{B}$. A subgraph is *induced* if the inclusion map is an embedding. A bijective embedding is an *isomorphism* and an isomorphism $\mathbf{A} \to \mathbf{A}$ is an *automorphism.* A *partial automorphism* of $\mathbf{A}$ is any isomorphism of two induced subgraphs of $\mathbf{A}$. In the context of metric spaces we sometimes say *isometry* instead of isomorphism.

A *walk* in an edge-labelled graph $\mathbf{A}$ connecting $x_1 \in A$ and $x_n \in A$ is any sequence of vertices $x_1, x_2, \dots, x_n$ such that for every $1 \leq i < n$ there is an edge connecting x_i and x_{i+1}. The *length of this walk* is $\sum_{1\leq i<n} d_{\mathbf{A}}(x_i, x_{i+1})$. A *path* is a walk which contains no repeated vertices. If there is a path $x_1, \dots, x_n$ with $n \geq 3$ and there is an edge connecting x_1 and x_n then $x_1, x_2, \dots, x_n$ is a *cycle.* A cycle is *nonmetric* if it contains a (unique) edge with label l which is greater than the sum of the labels of all the remaining edges. We will call this edge the *long edge* of the nonmetric cycle. An $\mathbb{R}^{>0}$-edge-labelled graph $\mathbf{A}$ is *connected* if for every $x, y \in A$ there exists a path connecting x and y.

Given a connected $\mathbb{R}^{>0}$-edge-labelled graph $\mathbf{G}$, its *shortest path completion* is the complete $\mathbb{R}^{>0}$-edge-labelled graph $\overline{\mathbf{G}}$ on the same vertex set as $\mathbf{G}$ such that the label of x, y in $\overline{\mathbf{G}}$ is the minimal length of a path connecting x and y in $\mathbf{G}$. We will need the following fact about the shortest path completion.

Observation 2.1. *For every connected $\mathbb{R}^{>0}$-edge-labelled graph $\mathbf{G}$, its shortest path completion $\overline{\mathbf{G}}$ is a metric space. The graph $\mathbf{G}$ is a (not necessarily induced) subgraph of $\overline{\mathbf{G}}$ if and only if it contains no induced nonmetric cycles (that is, no induced subgraphs isomorphic to a nonmetric cycle). Moreover, every automorphism of $\mathbf{G}$ is also an automorphism of $\overline{\mathbf{G}}$.*

Proof: For any triple of vertices $x, y, z \in \overline{\mathbf{G}}$ there are, by definition, paths $x = x_1, x_2, \dots, x_n = y$ and $y = x_n, x_{n+1}, \dots, x_m = z$ in $\mathbf{G}$ witnessing the distances $d_{\overline{\mathbf{G}}}(x, y)$ and $d_{\overline{\mathbf{G}}}(y, z)$, respectively. It follows that $x_1, x_2, \dots, x_m$ is a walk in $\mathbf{G}$ containing a path connecting x and z of length no greater than $d_{\overline{\mathbf{G}}}(x, y) + d_{\overline{\mathbf{G}}}(y, z)$. We thus conclude that $d_{\overline{\mathbf{G}}}(x, z) \leq d_{\overline{\mathbf{G}}}(x, y) + d_{\overline{\mathbf{G}}}(y, z)$, that is, the triangle inequality holds, and thus $\overline{\mathbf{G}}$ indeed is a metric space.

If $\mathbf{G}$ contains a nonmetric cycle with the longest edge between x and y, it is easy to see that distance of x, y in $\overline{\mathbf{G}}$ is strictly smaller than the distance of x and y in $\mathbf{G}$. Therefore $\mathbf{G}$ is not a subgraph of $\overline{\mathbf{G}}$.

Next we show that if $\mathbf{G}$ contains no induced nonmetric cycles then it is a subgraph of $\overline{\mathbf{G}}$. Assume, to the contrary, that there is a pair of vertices x, y connected by an edge in $\mathbf{G}$ where the labels differ. Because x, y is also a path connecting x and y in $\mathbf{G}$, we know that the label of x, y in $\mathbf{G}$ is greater than the length of shortest path connecting x, y, hence they together form a nonmetric cycle. This cycle is not necessarily induced but adding an edge to a nonmetric cycle splits it to two cycles where at least one is necessarily also nonmetric.

Finally, to verify that the shortest path completion preserves all automorphisms observe that every distance in $\overline{\mathbf{G}}$ corresponds to a path in $\mathbf{G}$ (and to a lack of any shorter path) and paths are preserved by every automorphism of $\mathbf{G}$. □

3. Extending partial automorphisms of $\mathbb{R}^{>0}$-edge-labelled graphs

Proposition 3.1. *For every finite $\mathbb{R}^{>0}$-edge-labelled graph $\mathbf{A}$ there exists a finite $\mathbb{R}^{>0}$-edge-labelled graph $\mathbf{B}$ containing $\mathbf{A}$ as an induced subgraph such that every partial automorphism of $\mathbf{A}$ extends to an automorphism of $\mathbf{B}$.*

What follows is a variant of the easy proof of the extension property for partial automorphisms for graphs in [5].

Proof: Fix $\mathbf{A}$ and let $S = \{s_1, s_2, \ldots, s_n\} \subseteq \mathbb{R}^{>0}$ be the finite subset of $\mathbb{R}^{>0}$ consisting of all distances in $\mathbf{A}$ (the *spectrum* of $\mathbf{A}$). First we assign every vertex $x \in A$ the set $\psi(x)$ such that for some fixed k the following is satisfied:

(1) For every $x \neq y \in A$ such that $d_{\mathbf{A}}(x,y) = s_j$ and integer i it holds that $(\{x,y\}, i) \in \psi(x)$ if and only if $1 \leq i \leq j$.
(2) For every $x \in A$ it holds that $|\psi(x)| = k$.
(3) For every $x \neq y \in A$ it holds that $\psi(x) \cap \psi(y) = \{(\{x,y\}, i)\colon 1 \leq i \leq j\}$, where $d_{\mathbf{A}}(x,y) = s_j$.

Such a function ψ is easy to build. Assign elements to sets to satisfy (1) and then extend the sets by arbitrary new elements (for example, natural numbers) to satisfy (2) where every new element belongs to precisely one set so that (3) holds.

Put

$$U = \bigcup_{x \in A} \psi(x)$$

to be the universe of our representation. We construct $\mathbf{B}$ as follows.

- The vertex set B of $\mathbf{B}$ consists of all subsets of U of size k (we will denote them by upper case letters X and Y).
- A pair of vertices $X, Y \in B$ is connected by an edge of length s_i if and only if $X \neq Y$ and $|X \cap Y| = i$. Otherwise X, Y is a non-edge.

It is easy to verify that the structure $\mathbf{A}'$ induced by $\mathbf{B}$ on $\{\psi(x)\colon x \in A\}$ is isomorphic to $\mathbf{A}$, that is, ψ is an embedding of $\mathbf{A}$ into $\mathbf{B}$. We claim that every partial automorphism of $\mathbf{A}'$ extends to an automorphism of $\mathbf{B}$. Fix such a partial automorphism φ' of $\mathbf{A}'$. By φ we denote the partial automorphism induced by φ' on $\mathbf{A}$, i.e. $\varphi = \psi^{-1} \circ \varphi' \circ \psi$. Note that every permutation of U gives rise to an automorphism of $\mathbf{B}$. We are going to construct an automorphism $\widehat{\varphi}$ of $\mathbf{B}$ which extends φ' by finding the right permutation π by the following procedure:

(1) Start with the partial permutation π that maps $(\{x,y\}, i) \mapsto (\{\varphi(x), \varphi(y)\}, i)$ for every $x \neq y \in \mathrm{Dom}(\varphi)$ and $1 \leq i \leq j$ where $d_{\mathbf{A}}(x,y) = s_j$.
(2) Consider every choice of $x \in \mathrm{Dom}(\varphi)$. Let e be element of $\psi(x)$ such that $\pi(e)$ is not defined and put $\pi(e)$ to be any element of $\psi(\varphi(x))$ which is not in the image of π yet. This is always possible because all the sets have the same size and are disjoint except for elements which were dealt with in the previous point.
(3) The partial permutation π can then be extended to a full permutation in an arbitrary way.

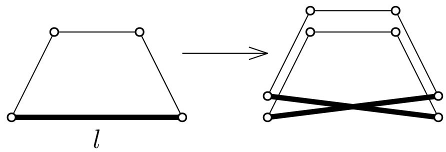

FIGURE 1. Expansion of a nonmetric cycle with longest edge l to a "Möbius strip".

It is easy to see that π induces an automorphism $\widehat{\varphi}$ of $\mathbf{B}$ and that this automorphism extends φ'. □

4. Proof of the main result

Now we are ready to prove Theorem 1.1. Similarly as in the proof of Hodkinson–Otto, see [7], we use Proposition 3.1 to obtain an $\mathbb{R}^{>0}$-edge-labelled graph $\mathbf{B}$. We then consider all "bad" substructures of $\mathbf{B}$ (namely the nonmetric cycles) and eliminate each one independently while preserving all necessary symmetries and a projection (in fact, a homomorphism) to the original structure. The resulting structure is then a product of all these constructions (however, we will define it explicitly). The extension property for partial automorphisms then follows from the fact that automorphisms of $\mathbf{B}$ are mapping bad substructures to their isomorphic copies and we repaired both of them in the same way.

To simplify the construction, we proceed by induction on the size of the nonmetric cycles (we start by fixing triangles, then four-cycles and so on). This will make all nonmetric cycles considered in each step of the construction induced. Because $\mathbf{A}$ is a metric space and thus a complete graph, we will only need to consider partial automorphisms of the nonmetric cycles which move at most two vertices. This makes it easy to fix every nonmetric cycle by unwinding it to a "Möbius strip" as depicted at Figure 1.

PROOF OF THEOREM 1.1: Given a metric space $\mathbf{A}$, let N be an integer greater than the ratio of the largest distance in $\mathbf{A}$ and the smallest distance in $\mathbf{A}$.

Let $\mathbf{C}_2$ be the $\mathbb{R}^{>0}$-edge-labelled graph given by Proposition 3.1 applied on $\mathbf{A}$ and let $\mathbf{A}_2$ be the copy of $\mathbf{A}$ in $\mathbf{C}_2$. We then build a sequence of $\mathbb{R}^{>0}$-edge-labelled graphs $\mathbf{C}_3, \mathbf{C}_4, \ldots, \mathbf{C}_N$ such that for every $2 \leq i \leq N$ the following conditions are satisfied:

(I) the graph $\mathbf{C}_i$ contains an isomorphic copy $\mathbf{A}_i$ of $\mathbf{A}$ as a subgraph,
(II) every partial automorphism of $\mathbf{A}_i$ extends to an automorphism of $\mathbf{C}_i$, and,
(III) the graph $\mathbf{C}_i$ contains no nonmetric cycles with at most i vertices.

First we show that from the existence of $\mathbf{C}_N$ the theorem follows. Observe that by the choice of N every nonmetric cycle using only distances used ind $\mathbf{A}$ has fewer than N vertices and thus $\mathbf{C}_N$ contains no nonmetric cycles. Without loss of generality we can assume that $\mathbf{C}_N$ is connected (otherwise we simply take the connected component of $\mathbf{C}_N$ containing $\mathbf{A}_N$) and thus we can apply Observation 2.1. Let $\mathbf{C}$ be the shortest path completion of $\mathbf{C}_N$. Because every automorphism of $\mathbf{C}_N$ is also automorphism of $\mathbf{C}$ and $\mathbf{A}_N$ is a subgraph of $\mathbf{C}$ we get that $\mathbf{C}$ extends all partial isometries of $\mathbf{A}_N$ (which is isomorphic to $\mathbf{A}$).

It remains to give the construction of $\mathbf{C}_{i+1}$ from $\mathbf{C}_i$ satisfying conditions (I)–(III). A subset M of C_i is called *bad* if $|M| = i+1$ and $\mathbf{C}_i$ induces a nonmetric cycle on M. For $x \in C_i$ denote by $U(x)$ the family of all bad sets M containing x.

We construct $\mathbf{C}_{i+1}$ as follows:

- Vertices of $\mathbf{C}_{i+1}$ are pairs (x, χ_x) where $x \in C_i$ and χ_x is a function from $U(x)$ to $\{0,1\}$. We call such χ_x *valuation function.*
- Pairs (x, χ_x) and (y, χ_y) are connected by an edge of length l if and only if $d_{\mathbf{C}_i}(x, y) = l$ and for every $M \in U(x) \cap U(y)$ one of the following holds:
  (a) the edge $\{x, y\}$ is the longest edge of the nonmetric cycle induced on M and $\chi_x(M) \neq \chi_y(M)$, or
  (b) the edge $\{x, y\}$ is not the longest edge of the nonmetric cycle induced on M and $\chi_x(M) = \chi_y(M)$.

  (These rules describe the "Möbius strip" of every bad set.)

There are no other edges in $\mathbf{C}_{i+1}$. This finishes the construction of $\mathbf{C}_{i+1}$. We now verify that $\mathbf{C}_{i+1}$ satisfies conditions (I)–(III).

(I): We give an explicit description of an embedding ψ of $\mathbf{A}_i$ to $\mathbf{C}_{i+1}$ and put $\mathbf{A}_{i+1}$ to be the structure induced by $\mathbf{C}_{i+1}$ on $\{\psi(x) \colon x \in A_i\}$.

For every bad set $M \subseteq C_i$ such that $M \cap A_i \neq \emptyset$ we define a function $\chi_M : M \cap A_i \to \{0,1\}$. By definition, M is bad because $\mathbf{C}_i$ induces a nonmetric cycle on M. Since $\mathbf{A}$ is complete and it is a metric space (hence contains no nonmetric triangles), it follows that $M \cap A$ consists either of one vertex or two vertices connected by an edge of the cycle. Consider now two cases:

(1) The set $M \cap A = \{x, y\}$ where $\{x, y\}$ is the long edge of the nonmetric cycle induced on M. In this case we put $\chi_M(x) = 0$ and $\chi_M(y) = 1$. (Notice that this step is not uniquely defined because the choice of x and y can be exchanged and it is indeed the purpose of the function χ_M to fix this choice.)

(2) The set M does not intersect with A by a long edge. In this case put $\chi_M(x) = 0$ for all $x \in M \cap A$.

Now we define a mapping ψ from A_i to C_{i+1} by putting $\psi(x) = (x, \chi_x)$ where $\chi_x(M) = \chi_M(x)$ and put $A_{i+1} = \psi(A_i)$. It is easy check that ψ is an embedding $\mathbf{A}_i \to \mathbf{C}_{i+1}$ because we chose functions χ_M in a way so that all edges are preserved. This verifies condition (I).

(II): We show that $\mathbf{C}_{i+1}$ extends all partial automorphisms of $\mathbf{A}_{i+1}$.

Consider any partial automorphism φ of $\mathbf{A}_{i+1}$. Define $p \colon C_{i+1} \to C_i$ to be the *projection* which maps every $(x, \chi_x) \in C_{i+1}$ to $x \in C_i$. By p we project the partial automorphism φ of $\mathbf{A}_{i+1}$ to a partial automorphism $p \circ \varphi \circ p^{-1}$ of $\mathbf{A}_i$. Denote by

$\widehat{\varphi}$ an extension of the partial automorphism $p \circ \varphi \circ p^{-1}$ of $\mathbf{A}_i$ to an automorphism of $\mathbf{C}_i$ (which always exist by the induction hypothesis).

Let F consist of all bad sets $M \subseteq C_i$ with the property that $M \cap A_i \neq \emptyset$ and there exists $x \in M$, such that $(x, \chi_x) = \psi(x) \in \mathrm{Dom}(\varphi)$ and $\chi_x(M) \neq \chi_y(\widehat{\varphi}(M))$ where (y, χ_y) is such that $\varphi((x, \chi_x)) = (y, \chi_y)$ (these are bad sets whose valuations are *flipped* by φ).

We build an automorphism θ of $\mathbf{C}_{i+1}$ by putting $\theta((x, \chi_x)) = (\widehat{\varphi}(x), \chi')$ where $\chi'(\widehat{\varphi}(M)) = \chi_x(M)$ if $M \notin F$ and $1 - \chi_x(M)$ if $M \in F$. To verify that θ is indeed an automorphism first check that θ is one-to-one because it is possible to construct its inverse. Because the action of θ on the valuation functions does not affect the outcome of conditions for edges in the construction of $\mathbf{C}_{i+1}$, we get that θ is an isomorphism.

It remains to verify that θ extends φ. This follows from the fact that for every bad set M it holds that $|M \cap \mathrm{Dom}(\varphi)| \leq 2$. Moreover, whenever $M \cap \mathrm{Dom}(\varphi) = \{x, y\}$, $x \neq y$, $\varphi(x) = (x', \chi_{x'})$, $\varphi(y) = (y', \chi_{y'})$ then $\chi_x(M) = \chi_{x'}(\widehat{\varphi}(M))$ if and only if $\chi_y(M) = \chi_{y'}(\widehat{\varphi}(M))$. This finishes the proof of condition (II).

(III): Consider any set $M \subseteq \mathbf{C}_{i+1}$ such that $|M| \leq i+1$ and the subgraph induced by $\mathbf{C}_{i+1}$ on M contains a nonmetric cycle as a subgraph. It follows that its projection $p(M)$ contains a nonmetric cycle in $\mathbf{C}_i$. By the induction hypothesis we thus know that $|M| = i + 1$ and $p(M)$ is a bad set (that is, $\mathbf{C}_i$ induces a nonmetric cycle on $p(M)$). Because of the projection of $\mathbf{C}_{i+1}$ to $\mathbf{C}$ it follows that $\mathbf{C}_{i+1}$ induces a nonmetric cycle on M. Let $(x, \chi_x), (y, \chi_y)$ be longest edge of this nonmetric cycle. From the definition of the edges of $\mathbf{C}_{i+1}$ we know that $\chi_x(M) \neq \chi_y(M)$. Following the short edges of the cycle, we however get $\chi_x(M) = \chi_y(M)$ a contradiction. □

Remark 4.1. We in fact prove that the class of all finite metric spaces has the coherent extension property for partial isometries as defined by S. Solecki and D. Siniora, see [24], [22]: In Proposition 3.1 it is enough to fix a linear order on U and extend the permutation in an order-preserving way. The coherency then goes through the proof of Theorem 1.1, it is enough to realise that "flips compose".

Remark 4.2. This proof generalises to many known binary and general classes which are known to have the extension property for partial automorphisms (see [9], [13], [1] for examples of classes of structures having a variant of shortest path completion). This is going to appear in [2], [10].

There are classes for which it is unknown whether they have EPPA or not. Prominent among them are the class of all finite tournaments (see [8] for partial results) and the class of all finite partial Steiner triple systems, see [12].

Acknowledgment. We would like to thank the anonymous referee for remarks and corrections that improved presentation of this paper and for the incredible speed in which they were delivered.

References

[1] Aranda A., Bradley-Williams D., Hubička J., Karamanlis M., Kompatscher M., Konečný M., Pawliuk M., *Ramsey Expansions of Metrically Homogeneous Graphs*, available at arXiv:1707.02612 [math.CO], 2017.

[2] Evans D., Hubička J., Konečný M., Nešetřil J., *EPPA for two-graphs and antipodal metric spaces*, available at arXiv:1812.11157 [math.CO] (2018), 13 pages.

[3] Evans D. M., Hubička J., Nešetřil J., *Ramsey properties and extending partial automorphisms for classes of finite structures*, available at arXiv:1705.02379 [math.CO] (2017), 33 pages.

[4] Hall M. Jr., *Coset representations in free groups*, Trans. Amer. Math. Soc. **67** (1949), 421–432.

[5] Herwig B., Lascar D., *Extending partial automorphisms and the profinite topology on free groups*, Trans. Amer. Math. Soc. **352** (2000), no. 5, 1985–2021.

[6] Hodkinson I., *Finite model property for guarded fragments*, 2012, slides available at `http://www.cllc.vuw.ac.nz/LandCtalks/imhslides.pdf`.

[7] Hodkinson I., Otto M., *Finite conformal hypergraph covers and Gaifman cliques in finite structures*, Bull. Symbolic Logic **9** (2003), no. 3, 387–405.

[8] Huang J., Pawliuk M., Sabok M., Wise D., *The Hrushovski property for hypertournaments and profinite topologies*, available at arXiv:1809.06435 [math.LO] (2018), 20 pages.

[9] Hubička J., Konečný M., Nešetřil J., *Conant's generalised metric spaces are Ramsey*, available at arXiv:1710.04690 [math.CO] (2017), 22 pages.

[10] Hubička J., Konečný M., Nešetřil J., *All those EPPA classes (Strengthenings of the Herwig–Lascar theorem)*, available at arXiv:1902.03855 [math.CO] (2019), 27 pages.

[11] Hubička J., Nešetřil J., *All those Ramsey classes (Ramsey classes with closures and forbidden homomorphisms)*, available at arXiv:1606.07979 [math.CO] (2016), 59 pages.

[12] Hubička J., Nešetřil J., *Ramsey theorem for designs*, The Ninth European Conf. on Combinatorics, Graph Theory and Applications (EuroComb 2017), Viena, 2017, Electronic Notes in Discrete Mathematics **61** (2017), 623–629.

[13] Konečný M., *Semigroup-valued Metric Spaces*, Master thesis in preparation available at arXiv:1810.08963 [math.CO], 2018.

[14] Mackey G. W., *Ergodic theory and virtual groups*, Math. Ann. **166** (1966), no. 3, 187–207.

[15] Nešetřil J., *Metric spaces are Ramsey*, European J. Comb. **28** (2007), no. 1, 457–468.

[16] Nešetřil J., Rödl V., *A structural generalization of the Ramsey theorem*, Bull. Amer. Math. Soc. **83** (1977), no. 1, 127–128.

[17] Otto M., *Amalgamation and symmetry: From local to global consistency in the finite*, available at arXiv:1709.00031 [math.CO] (2017), 49 pages.

[18] Pestov V. G., *A theorem of Hrushovski-Solecki-Vershik applied to uniform and coarse embeddings of the Urysohn metric space*, Topology Appl. **155** (2008), no. 14, 1561–1575.

[19] Rosendal Ch., *Finitely approximate groups and actions. Part I: The Ribes-Zalesskiĭ property*, J. Symbolic Logic **76** (2011), no. 4, 1297–1306.

[20] Ribes L., Zalesskii P. A., *On the profinite topology on a free group*, Bull. London Math. Soc. **25** (1993), no. 1, 37–43.

[21] Sabok M., *Automatic continuity for isometry groups*, J. Inst. Math. Jussieu (online 2017), 30 pages.

[22] Siniora D., Solecki S., *Coherent extension of partial automorphisms, free amalgamation, and automorphism groups*, available at arXiv:1705.01888v3 [math.LO] (2018), 29 pages.

[23] Solecki S., *Extending partial isometries*, Israel J. Math. **150** (2005), no. 1, 315–331.

[24] Solecki S., *Notes on a strengthening of the Herwig–Lascar extension theorem*, available at `http://www.math.uiuc.edu/~ssolecki/papers/HervLascfin.pdf` (2009), 16 pages.

[25] Vershik A. M., *Globalization of the partial isometries of metric spaces and local approximation of the group of isometries of Urysohn space*, Topology Appl. **155** (2008), no. 14, 1618–1626.

J. Hubička, M. Konečný:
Charles University, Faculty of Mathematics and Physics,
Department of Applied Mathematics (KAM), Malostranské nám. 2/25,
118 00 Praha 1, Czech Republic

E-mail: hubicka@kam.mff.cuni.cz

E-mail: matej@kam.mff.cuni.cz

J. Nešetřil:
Charles University, Faculty of Mathematics and Physics,
Computer Science Institute (IU UK), Malostranské nám. 2/25,
118 00 Praha 1, Czech Republic

E-mail: nesetril@iuuk.mff.cuni.cz

(*Received* July 31, 2018*, revised* August 28, 2018)