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# Another ordering of the ten cardinal characteristics in Cichoń's diagram

Jakob Kellner, Saharon Shelah, Anda R. Tănasie

*Dedicated to the memory of Bohuslav Balcar (1943–2017)*

*Abstract.* It is consistent that

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}.$$

Assuming four strongly compact cardinals, it is consistent that

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M})$$
$$< \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{M}) = \mathfrak{d} < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

*Keywords:* set theory of the reals; Cichoń's diagram; forcing; compact cardinal

*Classification:* 03E17

## Introduction

We assume that the reader is familiar with basic properties of Amoeba, Hechler, random and Cohen forcing, and with the cardinal characteristics in Cichoń's diagram, given in Figure 1: An arrow between $\mathfrak{x}$ and $\mathfrak{y}$ indicates that Zermelo–Fraenkel set theory (ZFC) proves $\mathfrak{x} \leq \mathfrak{y}$. Moreover, $\max(\mathfrak{d}, \mathrm{non}(\mathcal{M})) = \mathrm{cof}(\mathcal{M})$ and $\min(\mathfrak{b}, \mathrm{cov}(\mathcal{M})) = \mathrm{add}(\mathcal{M})$. These (in)equalities are the only one provable. More precisely, all assignments of the values $\aleph_1$ and $\aleph_2$ to the characteristics in Cichoń's diagram are consistent, provided they do not contradict the above (in)equalities. (A complete proof can be found in [2, Chapter 7].)

In the following, we will only deal with the ten "independent" characteristics listed in Figure 2 (they determine $\mathrm{cof}(\mathcal{M})$ and $\mathrm{add}(\mathcal{M})$).

**Regarding the left hand side,** it was shown in [8] that consistently

$$(\mathrm{left}_{\mathrm{old}}) \quad \aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}.$$

(This corresponds to $\lambda_1$ to $\lambda_5$ in Figure 3.) The proof is repeated in [7], in a slightly different form which is more convenient for our purpose. Let us call this construction the "old construction".

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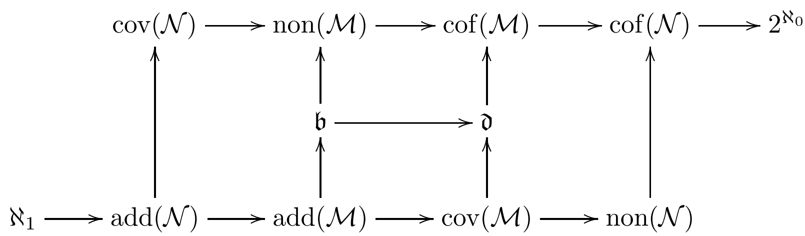


FIGURE 1. Cichoń's diagram.

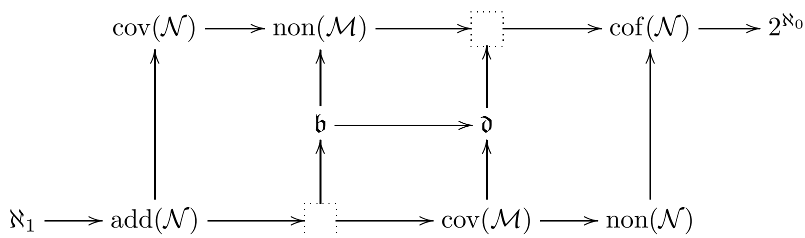


FIGURE 2. The ten "independent" characteristics.

In this paper, building on [16], we give a construction to get a different order for these characteristics, where we swap $\mathrm{cov}(\mathcal{N})$ and $\mathfrak{b}$:

$$(\mathrm{left}_{\mathrm{new}}) \quad \aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}.$$

(This corresponds to $\lambda_1$ to $\lambda_5$ in Figure 4.)

This construction is more complicated than the old one. Let us briefly describe the reason: In both constructions, we assign to each of the cardinal characteristics of the left hand side a relation R. E.g., we use the "eventually different" relation $\mathrm{R}_4 \subseteq \omega^\omega \times \omega^\omega$ for $\mathrm{non}(\mathcal{M})$. We can then show that the characteristic remains "small" (i.e., is at most the intended value $\lambda$ in the final model), because all single forcings we use in the iterations are either small (i.e., smaller than $\lambda$) or are "R-good". However, $\mathfrak{b}$ (with the "eventually dominating" relation $\mathrm{R}_2 \subseteq \omega^\omega \times \omega^\omega$) is an exception: We do not know any variant of an eventually different forcing (which we need to increase $\mathrm{non}(\mathcal{M})$) which satisfies that all of its subalgebras are $\mathrm{R}_2$-good. Accordingly, the main effort (in both constructions) is to show that $\mathfrak{b}$ remains small.

In the old construction, each non-small forcing is a ($\sigma$-centered) subalgebra of the eventually different forcing $\mathbb{E}$. To deal with such forcings, ultrafilter limits of sequences of $\mathbb{E}$-conditions are introduced and used (and we require that all

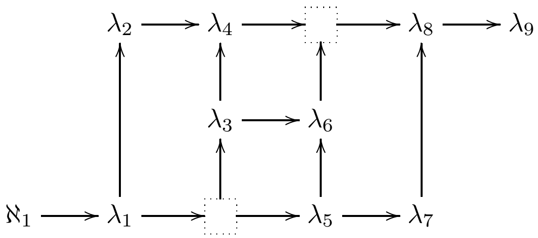


FIGURE 3. The old order.

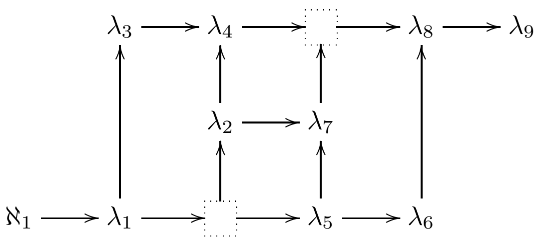


FIGURE 4. The new order.

$\mathbb{E}$-subforcings are basically $\mathbb{E}$ intersected with some model, and thus closed under limits of sequences in the model). In the new construction, we have to deal with an additional kind of "large" forcing: (subforcings of) random forcing. Ultrafilter limits do not work any more, but, similarly to [16], we can use finite additive measures (FAMs) and interval-FAM-limits of random conditions. But now $\mathbb{E}$ doesn't seem to work with interval-FAM-limits any more, so we replace it with a creature forcing notion $\widetilde{\mathbb{E}}$.

We also have to show that $\mathrm{cov}(\mathcal{N})$ remains small. In the old construction, we could use a rather simple (and well understood) relation $\mathrm{R}^{\mathrm{old}}$ and use the fact that all $\sigma$-centered forcings are $\mathrm{R}^{\mathrm{old}}$-good: As all large forcings are subalgebras of either eventually different forcing or of Hechler forcing, they are all $\sigma$-centered. In the new construction, the large forcings we have to deal with are subforcings of $\widetilde{\mathbb{E}}$. But $\widetilde{\mathbb{E}}$ is not $\sigma$-centered, just $(\varrho,\pi)$-linked for a suitable pair $(\varrho,\pi)$ (a property between $\sigma$-centered and $\sigma$-linked, first defined in [15], see Definition 1.18). So we use a different (and more cumbersome) relation $\mathrm{R}_3$, introduced in [15], where it is also shown that $(\varrho,\pi)$-linked forcings are $\mathrm{R}_3$-good.

**Regarding the whole diagram,** in [7], starting with the iteration for ($\mathrm{left}_{\mathrm{old}}$), a new iteration is constructed to get simultaneously different values for all characteristics: Assuming four strongly compact cardinals, the following is consistent (cf. Figure 3):

$$\aleph_1 < \text{add}(\mathcal{N}) < \text{cov}(\mathcal{N}) < \mathfrak{b} < \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) < \mathfrak{d} < \text{non}(\mathcal{N}) < \text{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

The essential ingredient is the concept of the Boolean ultrapower of a forcing notion.

In exactly the same way we can expand our new version ($\text{left}_{\text{new}}$) to the right hand side, where also the characteristics dual to $\mathfrak{b}$ and $\text{cov}(\mathcal{N})$ are swapped. So we get: If four strongly compact cardinals are consistent, then so is the following (cf. Figure 4):

$$\aleph_1 < \text{add}(\mathcal{N}) < \mathfrak{b} < \text{cov}(\mathcal{N}) < \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) < \text{non}(\mathcal{N}) < \mathfrak{d} < \text{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

We closely follow the presentation of [7]. Several times, we refer to [7] and to [16] for details in definitions or proofs. We thank M. Goldstern and D. A. Mejía for valuable discussions, and an anonymous referee for a very detailed and helpful report pointing out (and even fixing) several mistakes in the first version of the paper.

## 1. Finitely additive measure limits and the $\widetilde{\mathbb{E}}$-forcing

**1.1 FAM-limits and random forcing.** We briefly list some basic notation and facts around finite additive measures. (A bit more details can be found in Section 1 of [16].)

**Definition 1.1.**
- A "partial FAM" (finitely additive measure) $\Xi'$ is a finitely additive probability measure on a sub-Boolean algebra $\mathcal{B}$ of $\mathcal{P}(\omega)$, the power set of $\omega$, such that $\{n\} \in \mathcal{B}$ and $\Xi'(\{n\}) = 0$ for all $n \in \omega$. We set $\text{dom}(\Xi') = \mathcal{B}$.
- $\Xi$ is a FAM if it is a partial FAM with $\text{dom}(\Xi) = \mathcal{P}(\omega)$.
- For every FAM $\Xi$ and bounded sequence of non-negative reals $\bar{a} = (a_n)_{n\in\omega}$ we can define in the natural way the average (or: integral) $\text{Av}_\Xi(\bar{a})$, a non-negative real number.

[16, 1.2] lists several results that informally say:

(∗) There is a FAM $\Xi$ that assigns the values $a_i$ to the sets $A_i$ (for all $i$ in some index set $I$) if and only if for each $I' \subseteq I$ finite and $\varepsilon > 0$ there is an arbitrary large[1] finite $u \subseteq \omega$ such that the counting measure on $u$ for $A_i$ approximates $a_i$ with an error of at most $\varepsilon$ for all $i \in I'$.

[1] Equivalently: "a finite $u$ with arbitrary large minimum", which is the formulation actually used in most of the results.

For the size of such an "$\varepsilon$-good approximation" $u$ to some FAM $\Xi$ we can give an upper bound for $|u|$ which only depends on $|I'|$ and $\varepsilon$ (and not on $\Xi$):

**Lemma 1.2.** *Given $N, k^* \in \omega$ and $\varepsilon > 0$, there is an $M \in \omega$ such that: For all* FAMs $\Xi$ *and $(A_n)_{n<N}$ there is a nonempty $u \subseteq \omega$ of size less than or equal to $M$ such that $\min(u) > k^*$ and $\Xi(A_n) - \varepsilon < |A_n \cap u|/|u| < \Xi(A_n) + \varepsilon$ for all $n < N$.*

Proof: We can assume that $\varepsilon = 1/L$ for an integer $L$. The set $\{A_n : n \in N\}$ generates the set algebra $\mathfrak{B} \subseteq \mathcal{P}(\omega)$. Let $\mathcal{X}$ be the set of atoms of $\mathfrak{B}$. So $\mathcal{X}$ is a partition of $\omega$ of size less than or equal to $2^N$. Set $\mathcal{X}' = \{x \in \mathcal{X} : \Xi(x) > 0\}$. Every $x \in \mathcal{X}'$ is infinite, and $\sum_{x \in \mathcal{X}'} \Xi(x) = 1$.

Round $\Xi(x)$ to some number $\Xi^\varepsilon(x) = l_x/(L \cdot 2^N)$ for some integer $0 \leq l_x \leq L \cdot 2^N$, such that $|\Xi(x) - \Xi^\varepsilon(x)| < (L \cdot 2^N)^{-1}$ and $\sum_{x \in \mathcal{X}'} \Xi^\varepsilon(x)$ is still 1. So $\sum_{x \in \mathcal{X}'} l_x = L \cdot 2^N$, and we construct $u$ consisting of $l_x$ many points that are bigger than $k^*$ and in $x$ (for each $x \in \mathcal{X}'$). □

We will use the following variants of $(*)$, regarding the possibility to extend a partial FAM $\Xi'$ to a FAM $\Xi$. The straightforward, if somewhat tedious, proofs are given in [16, 1.3 (G) and 1.7].

**Fact 1.3.** Let $\Xi'$ be a partial FAM, and $I$ some index set.

(a) Fix for each $i \in I$ some $A_i \subseteq \omega$.
*If* $A \cap \bigcap_{i \in I'} A_i \neq \emptyset$ for all $I' \subseteq I$ finite and $A \in \mathrm{dom}(\Xi')$ with $\Xi'(A) > 0$, *then* $\Xi'$ can be extended to a FAM $\Xi$ such that $\Xi(A_i) = 1$ for all $i \in I$.

(b) Fix for each $i \in I$ some real $b^i$ and some bounded sequence of non-negative reals $\bar{a}^i = (a^i_k)_{k \in \omega}$.
*If* for each finite partition $(B_m)_{m<m^*}$ of $\omega$ into elements of $\mathrm{dom}(\Xi')$ for each $\varepsilon > 0$, $k^* \in \omega$, and $I' \subseteq I$ finite there is a finite $u \subseteq \omega \setminus k^*$ such that
  - for all $m < m^*$, $\Xi'(B_m) - \varepsilon \leq |B_m \cap u|/|u| \leq \Xi'(B_m) + \varepsilon$, and
  - for all $i \in I'$, $|u|^{-1} \sum_{k \in u} a^i_k \geq b^i - \varepsilon$,

*then* $\Xi'$ can be extended to a FAM $\Xi$ such that $\mathrm{Av}_\Xi(\bar{a}^i) \geq b^i$ for all $i \in I$.

We first define what it means for a forcing $Q$ to have FAM limits.

**Remark 1.4.** Intuitively, this means (in the simplest version): Fix a FAM $\Xi$. We can define for each sequence $q_k$ of conditions that are all "similar" (e.g., have the same stem and measure) a limit $\lim_\Xi \bar{q}$. And we find in the $Q$-extension a FAM $\Xi'$ extending $\Xi$, such that $\lim_\Xi(\bar{q})$ forces that the set of $k$ satisfying $P(k) \equiv$ "$q_k \in G$" has "large" $\Xi'$-measure. Up to here, we get the notion used in [8] and [7] (but there we use ultrafilters instead of FAMs, and "large" means being in the ultrafilter). However, we need a modification: Instead of single conditions $q_k$ we use a finite sequence $(p_l)_{l \in I_k}$ (where $I_k$ is a fixed, finite interval); and the condition $P(k)$, which we want to satisfy on a large set, now is "$|\{l \in I_k : p_l \in G\}|/|I_k| > b$" for some suitable $b$. This is the notion used implicitly in [16].

**Notation.** Let $T^*$ be a compact subtree of $\omega^{<\omega}$, for example $T^* = 2^{<\omega}$. Let $s, t \in T^*$. Let $S$ be a subtree of $T^*$.

- $t \rhd s$ means "$t$ is immediate successor of $s$".
- $|s|$ is the length of $s$ (i.e.: the height, or level, of $s$).
- $[t]$ is the set of nodes in $T^*$ comparable with $t$.
- We set $\lim(S) = \{x \in \omega^\omega : (\forall\, n \in \omega)\ x \restriction n \in S\}$.
- $\operatorname{trunk}(S)$ is the smallest splitting node of $S$. With "$t \in S$ above the stem" we mean that $t \in S$ and $t \geq \operatorname{trunk}(S)$; or equivalently: $t \in S$ and $|t| \geq |\operatorname{trunk}(S)|$.
- Leb is the canonical measure on the Borel subsets of $\lim(T^*)$. We also write $\operatorname{Leb}(S)$ instead of $\operatorname{Leb}(\lim(S))$.[2]

We fix for the rest of the paper an interval partition $\bar{I} = (I_k)_{k\in\omega}$ of $\omega$ such that $|I_k|$ converges to infinity. We will use forcing notions $Q$ satisfying the following setup:

**Assumption 1.5.**
- $Q' \subseteq Q$ is dense and the domain of functions trunk and loss, where $\operatorname{trunk}(q) \in H(\aleph_0)$ and $\operatorname{loss}(q)$ is a non-negative rational.
- For each $\varepsilon > 0$ the set $\{q \in Q' : \operatorname{loss}(q) < \varepsilon\}$ is dense (in $Q'$ and thus in $Q$).
- $\{p \in Q' : (\operatorname{trunk}(p), \operatorname{loss}(p)) = (\operatorname{trunk}^*, \operatorname{loss}^*)\}$ is $\lfloor 1/\operatorname{loss}^* \rfloor$-linked. I.e., each $\lfloor 1/\operatorname{loss}^* \rfloor$ many such conditions are compatible.[3]

In this paper, $Q$ will be one of the following two forcing notions: random forcing, or $\widetilde{\mathbb{E}}$ (as defined in Definition 1.12). We will now specify the instance of random forcing that we will use:

**Definition 1.6.**
- A random condition is a tree $T \subseteq 2^{<\omega}$ such that the measure $\operatorname{Leb}(T \cap [t]) > 0$ for all $t \in T$.
- $\operatorname{trunk}(T)$ is the stem of $T$ (i.e., the shortest splitting node).
- If $\operatorname{Leb}(T) = \operatorname{Leb}([\operatorname{trunk}(T)])$, we set $\operatorname{loss}(T) = 0$. Otherwise, let $m$ be the maximal natural number such that

  $$\operatorname{Leb}(T) > \operatorname{Leb}([\operatorname{trunk}(T)])\Big(1 - \frac{1}{m}\Big)$$

  and set[4] $\operatorname{loss}(T) = 1/m$.

Note that $\operatorname{Leb}(T) \geq 2^{-|\operatorname{trunk}(T)|}(1 - \operatorname{loss}(T))$ (and the inequality is strict if $\operatorname{loss}(T) > 0$).

Note that this definition of random forcing satisfies Assumption 1.5 (with $Q' = Q$).

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[2] I.e., we define $\operatorname{Leb}([s])$ by induction on the height of $s \in T^*$ as follows: $\operatorname{Leb}(T^*) = 1$, and if $s$ has $n$ many immediate successors in $T^*$, then $\operatorname{Leb}([t]) = \operatorname{Leb}([s])/n$ for any such successor. This defines a measure on each basic clopen set, which in turn defines a (probability) measure on the Borel subsets of $\lim(T^*)$ (a closed subset of $\omega^\omega$).

[3] In [16, 2.9], trunk and loss are called $h_2$ and $h_1$; and instead of $I_k$ the interval is called $[n^*_k, n^*_{k+1} - 1]$. Moreover, in [16] the sequence $(n^*_k)_{k\in\omega}$ is one of the parameters of a "blueprint", whereas we assume that the $I_k$ are fixed.

[4] In [16], this is implicit in 2.11 (f).

**Definition 1.7.** Fix $Q$ and functions $(\text{trunk}, \text{loss})$ as in Assumption 1.5, a FAM $\Xi$ and a function $\lim_\Xi \colon Q^\omega \to Q$. Let us call the objects mentioned so far a "limit setup". Let a $(\text{trunk}^*, \text{loss}^*)$-sequence be a sequence $(q_l)_{l\in\omega}$ of $Q$-conditions such that $\text{trunk}(q_l) = \text{trunk}^*$ and $\text{loss}(q_l) = \text{loss}^*$ for all $l \in \omega$.

We say "$\lim_\Xi$ is a strong FAM limit for intervals", if the following is satisfied: Given

- a pair $(\text{trunk}^*, \text{loss}^*)$, $j^* \in \omega$, and $(\text{trunk}^*, \text{loss}^*)$-sequences $\bar{q}^j$ for $j < j^*$;
- $\varepsilon > 0$, $k^* \in \omega$;
- $m^* \in \omega$ and a partition of $\omega$ into sets $B_m$, $m \in m^*$; and
- a condition $q$ stronger than all $\lim_\Xi(\bar{q}^j)$ for all $j < j^*$;

there is a finite $u \subseteq \omega \setminus k^*$ and a $q'$ stronger than $q$ such that

- $\Xi(B_m) - \varepsilon < |u \cap B_m|/|u| < \Xi(B_m) + \varepsilon$ for $m < m^*$;
- $|u|^{-1} \sum_{k\in u} |\{l \in I_k \colon q' \leq q^j_l\}|/|I_k| \;\geq\; 1 - \text{loss}^* - \varepsilon$ for $j < j^*$.

(We are only interested in $\lim_\Xi(\bar{q})$ for $\bar{q}$ as above, so we can set $\lim_\Xi(\bar{q})$ to be undefined or some arbitrary value for other $\bar{q} \in Q^\omega$.)

The motivation for this definition is the following:

**Lemma 1.8.** *Assume that* $\lim_\Xi$ *is such a limit. Then there is a* $Q$*-name* $\Xi^+$ *such that for every* $(\text{trunk}^*, \text{loss}^*)$*-sequence* $\bar{q}$ *the limit* $\lim_\Xi(\bar{q})$ *forces* $\Xi^+(A_{\bar{q}}) \geq 1 - \sqrt{\text{loss}^*}$, *where*

$$A_{\bar{q}} = \{k \in \omega \colon |\{l \in I_k \colon q_l \in G\}| \geq |I_k|(1 - \sqrt{\text{loss}^*})\}. \tag{1.9}$$

Proof: Work in the $Q$-extension. Now $\Xi$ is a partial FAM. Let $J$ enumerate all suitable sequences $\bar{q} \in V$ with $\lim_\Xi(\bar{q}) \in G$, and for such a sequence $\bar{q}^j$ set $a^j_k = |\{l \in I_k \colon q^j_l \in G\}|/|I_k|$, and $b^j = 1-\text{loss}^*$. Using that $\Xi$ satisfies Definition 1.7, we can apply Fact 1.3 (b), we can extend $\Xi$ to some FAM $\Xi^+$ such that $\text{Av}_{\Xi^+}(\bar{a}^j) \geq 1 - \text{loss}^*$ for $j < j^*$. So $\Xi^+(A_{\bar{q}^j}) + (1 - \Xi^+(A_{\bar{q}^j}))(1 - \sqrt{\text{loss}^*}) \geq \text{Av}_{\Xi^+}(a^j_k) \geq 1 - \text{loss}^*$, and thus $\Xi^+(A_{\bar{q}^j}) \geq 1 - \sqrt{\text{loss}^*}$. □

**Definition 1.10.** $(Q, \text{trunk}, \text{loss})$ as in Assumption 1.5 "has strong FAM limits for intervals", if for every FAM $\Xi$ there is a function $\lim_\Xi$ that is a strong FAM limit for intervals.

**Lemma 1.11** ([16])**.** *Random forcing has strong FAM-limits for intervals.*

Proof: $\lim_\Xi$ is implicitly defined in [16, 2.18], in the following way: Given a sequence $r_l$ with $(\text{trunk}(p_l), \text{loss}(p_l)) = (\text{trunk}^*, \text{loss}^*)$, we can set $r^* = [\text{trunk}^*]$ and $b = 1 - \text{loss}^*$; and we set $n^*_k$ such that $I_k = [n^*_k, n^*_{k+1} - 1]$. We now use these objects to apply [16, 2.18] (note that (c)($*$) is satisfied). This gives $r^\otimes$, and we define $\lim_\Xi(\bar{r})$ to be $r^\otimes$.

In [16, 2.17], it is shown that this $r^\otimes$ satisfies Definition 1.7, i.e., is a limit: If $r$ is stronger than all limits $r^{\otimes i}$, then $r$ satisfies [16, 2.17 ($*$)]. □

**1.2 The forcing $\widetilde{\mathbb{E}}$.** We now define $\widetilde{\mathbb{E}}$, a variant of the forcing notion $Q^2$ defined in [9]:

**Definition 1.12.** By induction on the height $h \geq 0$, we define a compact homogeneous tree $T^* \subset \omega^{<\omega}$, and set

(1.13) $\varrho(h) := \max(|T^* \cap \omega^h|, h+2)$ and $\pi(h) := ((h+1)^2 \varrho(h)^{h+1})^{\varrho(h)^h}$,

we set $\Omega_s$ to be the set $\{t \rhd s \colon t \in T^*\}$, i.e., the set of immediate successors of $s$, and define for each $s$ a norm $\mu_s$ on the subsets of $\Omega_s$. In more detail:

- The unique element of $T^*$ of height 0 is $\langle\rangle$, i.e., $T^* \cap \omega^0 = \{\langle\rangle\}$.
- We set

$$a(h) = \pi(h)^{h+2}, \qquad M(h) = a(h)^2, \qquad \text{and} \qquad \mu_h(n) = \log_{a(h)}\Big(\frac{M(h)}{M(h)-n}\Big)$$

  for natural numbers $0 \leq n < M(h)$, and we set $\mu_h(M(h)) = \infty$.
- For any $s \in T^* \cap \omega^h$, we set $\Omega_s = \{s^\frown l \colon l \in M(h)\}$ (which defines $T^* \cap \omega^{h+1}$). For $A \subset \Omega_s$, we set $\mu_s(A) := \mu_h(|A|)$. So $|\Omega_s| = M(h)$, $\mu_s(\emptyset) = 0$ and $\mu_s(\Omega_s) = \infty$. Note that $|A| = |\Omega_s|(1 - a(h)^{-\mu_s(A)})$.

We can now define $\widetilde{\mathbb{E}}$:

**Definition 1.14.**
- For a subtree $p \subseteq T^*$, the stem of $p$ is the smallest splitting node. For $s \in p$, we set $\mu_s(p) = \mu_s(\{t \in p \colon t \rhd s\})$.
  The set $\widetilde{\mathbb{E}}$ consists of subtrees $p$ with some stem $s^*$ of height $h^*$ such that $\mu_t(p) \geq 1 + 1/h^*$ for all $t \in p$ above the stem. (So the only condition with $h^* = 0$ is the full condition, where all norms are $\infty$.)
  The set $\widetilde{\mathbb{E}}$ is ordered by inclusion.
- trunk$(p)$ is the stem of $p$.
  loss$(p)$ is defined if there is an $m \geq 2$ satisfying the following, and in that case loss$(p) = 1/m$ for the maximal such $m$:
  - $p$ has stem $s^*$ of height $h^* > 3m$,
  - $\mu_s(p) \geq 1 + 1/m$ for all $s \in p$ of height greater than or equal to $h^*$.

  We set $Q' = \mathrm{dom}(\mathrm{loss})$.

By simply extending the stem, we can find for any $p \in \widetilde{\mathbb{E}}$ and $\varepsilon > 0$ some $q \leq p$ in $Q'$ with loss$(q) < \varepsilon$; i.e., one of Assumptions 1.5 is satisfied. (The other one is dealt with in Lemma 1.19 (a).) In particular $Q' \subseteq \widetilde{\mathbb{E}}$ is dense.

We list a few trivial properties of the loss function:

**Facts 1.15.** Assume $p \in Q'$ with $s = \mathrm{trunk}(p)$ of height $h$.

(a) loss$(p) < 1$, $\mu_s(p) \geq 1 + \mathrm{loss}(p)$ for any $s$ above the stem, and loss$(p) > 3/h$.

(b) If $q$ is a subtree of $p$ such that all norms above the stem are greater than or equal to $1 + \mathrm{loss}(p) - 2/h$, then $q$ is a valid $\widetilde{\mathbb{E}}$-condition.

(c) $\prod_{l=h}^{\infty}(1 - 1/l^2) = 1 - 1/h > 1 - \mathrm{loss}(p)/3$.

**Lemma 1.16.** *Let $s \in T^*$ be of height $h$ and $A \subset \Omega_s$.*

(a) *If $\mu_s(A) \geq 1$, then $|A| \geq |\Omega_s|(1 - 1/h^2)$.*

(b) *If $A \subsetneq \Omega_s$, i.e., $A$ is a proper subset, then $\mu_s(A \setminus \{t\}) > \mu_s(A) - 1/h$ for $t \in A$.*

(c) *For $i < \pi(h)$, assume that $A_i \subseteq \Omega_s$ satisfies $\mu_s(A_i) \geq x$. Consequently $\mu_s\big(\bigcap_{i\in\pi(h)} A_i\big) > x - 1/h$.*

(d) *For $i < I$ (an arbitrary finite index set) pick proper subsets $A_i \subsetneq \Omega_s$ such that $\mu_s(A_i) \geq x$, and assign weighs $a_i$ to $A_i$ such that $\sum_{i\in I} a_i = 1$. Then*

$$\mu_s(B) > x - \frac{1}{h} \qquad \text{for } B := \Big\{t \in \Omega_s\colon \sum_{t\in A_i} a_i > 1 - \frac{1}{h^2}\Big\}. \tag{1.17}$$

Proof: (a) Trivial, as $a(h)^{-\mu_s(A)} \leq 1/a(h) < 1/h^2$.

(b)
$$\begin{aligned}\mu_s(A \setminus \{t\}) &= \log_{a(h)}(|\Omega_s|) - \log_{a(h)}(|\Omega_s| - |A| + 1)\\ &\geq \log_{a(h)}(|\Omega_s|) - \log_{a(h)}(2(|\Omega_s| - |A|))\\ &\geq \mu_s(A) - \log_{a(h)}(2) > \mu_s(A) - 1/h.\end{aligned}$$

(c)
$$\begin{aligned}\mu_s\big(\textstyle\bigcap_{i\in\pi(h)} A_i\big) &= \log_{a(h)}(|\Omega_s|) - \log_{a(h)}\big(|\Omega_s| - \big|\textstyle\bigcap_{i\in\pi(h)} A_i\big|\big)\\ &= \log_{a(h)}(|\Omega_s|) - \log_{a(h)}\big(\big|\textstyle\bigcup_{i\in\pi(h)}(\Omega_s - A_i)\big|\big)\\ &\geq \log_{a(h)}(|\Omega_s|) - \log_{a(h)}\big(\pi(h)\cdot \max_{i\in\pi(h)} |\Omega_s - A_i|\big)\\ &\geq x - \log_{a(h)}(\pi(h)) > x - 1/h.\end{aligned}$$

(d) Set $y = \sum_{i\in I} a_i|A_i|$. On the one hand, $y \geq |\Omega_s|(1 - a(h)^{-x})$. On the other hand, $y = \sum_{t\in\Omega_s}\sum_{t\in A_i} a_i \leq |B| + (|\Omega_s \setminus B|)(1 - 1/h^2)$.
So $|B| \geq |\Omega_s|(1 - h^2 a(h)^{-x}) > |\Omega_s|(1 - a(h)^{-(x-1/h)})$, as $a(h)^{1/h} > \pi(h) > h^2$.

□

The set $\widetilde{\mathbb{E}}$ is not $\sigma$-centered, but it satisfies a property, first defined in [15], which is between $\sigma$-centered and $\sigma$-linked:

**Definition 1.18.** Fix $f, g$ functions from $\omega$ to $\omega$ converging to infinity. Set $Q$ is $(f, g)$-linked if there are $g(i)$-linked $Q^i_j \subseteq Q$ for $i < \omega$, $j < f(i)$ such that each $q \in Q$ is in every $\bigcup_{j<f(i)} Q^i_j$ for sufficiently large $i$.

Recall that we have defined $\varrho$ and $\pi$ in (1.13).

**Lemma 1.19.** (a) *If $\pi(h)$ many conditions $(p_i)_{i\in\pi(h)}$ have a common node $s$ above their stems, $|s| = h$, then there is a $q$ stronger than each $p_i$.*

(b) *The set $\widetilde{\mathbb{E}}$ is $(\varrho, \pi)$-linked (in particular it is countable chain condition (ccc)).*

(c) *The $\widetilde{\mathbb{E}}$-generic real $\eta$ is eventually different (from every real in $\lim(T^*)$, and therefore from every real in $\omega^\omega$ as well).*

(d) $\mathrm{Leb}(p) \geq \mathrm{Leb}([\mathrm{trunk}(p)])(1 - \mathrm{loss}(p)/2)$; *more explicitly: for any $h > |\operatorname{trunk}(p)|$,*

$$\frac{|p \cap \omega^h|}{|T^* \cap \omega^h \cap [\mathrm{trunk}(p)]|} \geq 1 - \frac{1}{2}\,\mathrm{loss}(p).$$

(e) *The set $Q'$ (which is a dense subset of $\widetilde{\mathbb{E}}$) is an incompatibility-preserving subforcing of random forcing, where we use the variant*[5] *of random forcing on* $\lim(T^*)$ *instead of* $2^\omega$. *Let $B'$ be the sub-Boolean-algebra of* Borel/Null *generated by* $\{\lim(q)\colon q \in Q'\}$. *Then $Q'$ is dense in $B'$.*

(Here, Borel refers to the set of Borel subsets of $\lim(T^*)$. In the following proof, we will denote the equivalence class of a Borel set $A$ by $[A]_\mathcal{N}$.)

Proof: (a) Set $S = [s] \cap \bigcap_{i<\pi(h)} p_i$. According to 1.16 (c), for each $t \in S$ of height $h' \geq h$, the successor set has norm bigger than $1 + 1/h - 1/h' > 1$, so in particular there is a branch $x \in S$, and $S \cap [x \restriction 2h]$ is a valid condition stronger than all $p_i$.

(b) For each $h \in \omega$, enumerate $T^* \cap \omega^h$ as $\{s^h_1, \dots, s^h_{\varrho(h)}\}$, and set $Q^h_i = \{p \in \widetilde{\mathbb{E}}\colon s^h_i \in p \text{ and } |\operatorname{trunk}(p)| \leq h\}$. So for all $h$, $Q^h_i$ is $\pi(h)$-linked, and $p \in \bigcup_{i<\varrho(h)} Q^h_i$ for all $p \in Q$ with $|\operatorname{trunk}(p)| \leq h$.

(c) Use 1.16 (b).

(d) Use 1.16 (a) and the definition of loss.

(e) As in the previous item, we get that $\operatorname{Leb}(p \cap [t]) > 0$ whenever $p \in Q'$ and $t \in p$. So $Q'$ is a subset of random forcing. As both sets are ordered by inclusion, $Q'$ is a subforcing. If $q_1, q_2 \in Q'$ and $q_1, q_2$ are compatible as random conditions, then $q_1 \cap q_2$ has arbitrary high nodes, in particular a node above both stems, which implies that $q_1$ is compatible with $q_2$ in $\widetilde{\mathbb{E}}$ and therefore in $Q'$. It remains to show that $Q'$ is dense in $B'$. It is enough to show: If $x \neq 0$ in $B'$ has the form $x = \bigwedge_{i<i^*} [\lim(q_i)]_\mathcal{N} \wedge \bigwedge_{j<j^*} [\lim(T^*) \setminus \lim(q_j)]_\mathcal{N}$ then there is some $q \in Q'$ with $[\lim(q)]_\mathcal{N} < x$. Note that $0 \neq x = [A]_\mathcal{N}$ for $A = \lim\big(\bigcap_{i<i^*} q_i\big) \setminus \bigcup_{j<j^*} \lim(q_j)$, so pick some $r \in A$ and pick $h > i^*$ large enough such that $s = r \restriction h$ is not in any $q_j$. Then any $q \in Q'$ stronger than all $q_i \cap [s]$ for $i < i^*$ is as required. □

**Lemma 1.20.** *The set $\widetilde{\mathbb{E}}$ has strong FAM-limits for intervals.*

Proof: Let $(p_l)_{l\in\omega}$ be a $(s^*, \text{loss}^*)$-sequence, $s^*$ of height $h^*$. Set $\widetilde{\zeta}^{h^*} = 0$ and

$$\widetilde{\zeta}^{h} := 1 - \prod_{m=h^*}^{h-1} \Big(1 - \frac{1}{m^2}\Big) \qquad \text{for } h > h^*.$$

This is a strictly increasing sequence below $\text{loss}^*/3$, cf. Fact 1.15 (c). Also, all norms in all conditions of the sequence are at least $1 + \text{loss}^*$, cf. Fact 1.15 (a).

We will first construct $(q_k)_{k\in\omega}$ with stem $s^*$ and all norms greater than $1 + \text{loss}^* - 1/h^*$ such that $q_k$ forces $|\{l \in I_k : p_l \in G\}|/|I_k| > 1 - \text{loss}^*/3$. We will then use $\bar{q}$ to define $\lim_\Xi(\bar{p})$, and in the third step show that it is as required.

Step 1: So let us define $q_k$. Fix $k \in \omega$.

[5] We can use Definition 1.6, replacing $2^\omega$ with $\lim(T^*)$.

- Set $X_t = \{l \in I_k : t \in p_l\}$ and $Y_h = \{t \in [s^*] \cap \omega^h : |X_t| \geq |I_k|(1 - \widetilde{\zeta}^{\,h})\}$.
- We define $q_k$ by induction on the level, such that $q_k \cap \omega^h \subseteq Y_h$. The stem is $s^*$. (Note that $X_{s^*} = I_k$ and so $s^* \in Y_{h^*}$.) For $s \in q_k \cap \omega^h$ (and thus, by induction hypothesis, in $Y_h$), we set $q_k \cap [s] \cap \omega^{h+1} = [s] \cap Y_{h+1}$, i.e., a successor $t$ of $s$ is in $q_k$ if and only if it is $Y_{h+1}$. Then $\mu_s(q_k) > 1 + \text{loss}^* - 1/h$.

  Proof: Set $I = X_s$. By induction, $|X_s| \geq |I_k|(1 - \widetilde{\zeta}^{\,h})$. For $l \in I$, set $A_l = p_l \cap [s] \cap \omega^{h+1}$, i.e., the immediate successors of $s$ in $p_l$. Obviously $\mu_s(A_l) \geq 1 + \text{loss}^*$. We give each $A_l$ equal weight $a_l = 1/|I|$. According to (1.17), the set $B = \{t \rhd s : |\{l \in X_s : t \in A_l\}| \geq |I|(1 - 1/h^2)\}$ has norm greater than $1 + \text{loss}^* - 1/h$. □

- The condition $q_k$ forces that $p_l \in G$ for $\geq |I_k|(1 - \text{loss}^*/2)$ many $l \in I_k$.

  Proof: Let $r < q_k$ have stem $s'$ of length $h'$, without loss of generality $h' > |I_k| + 1$. As $s' \in Y_{h'}$, there are greater than $|I_k|(1 - \text{loss}^*/3)$ many $l \in I_k$ such that $s' \in p_l$. So we can find a condition $r'$ stronger than $r$ and all these $p_l$ (as these are at most $|I_k| + 1 \leq h'$ many conditions all containing $s'$ above the stem). □

Step 2: Now we use $(q_k)_{k\in\omega}$ to construct by induction on the height $q^* = \lim_\Xi(\bar{p})$, a condition with stem $s^*$ and all norms greater than or equal to $1 + \text{loss}^* - 2/h$ such that for all $s \in q^*$ of height $h \geq h^*$,

$$(*)\quad \Xi(Z_s) \geq 1 - \tilde{\zeta}^h \qquad \text{for } Z_s := \{k \in \omega : s \in q_k\}. \qquad \text{So } \ \Xi(Z_s) > 1 - \frac{1}{3}\,\text{loss}^*.$$

Note that $Z_{s^*} = \omega$, so $(*)$ is satisfied for $s^*$. Fix an $s \geq s^*$ satisfying $(*)$. Set $A(k)$ to be the $s$-successors in $q_k$ for each $k \in Z_s$. Enumerate the (finitely many) $A(k)$ as $(A_i)_{i\in I}$. Clearly $\mu_s(A_i) > 1 + \text{loss}^* - 1/h$. Assign to $A_i$ the weight $a_i = (1/\Xi(Z_s))\Xi(\{k \in Z_s : A(k) = A_i\})$. Again using (1.17), $\mu_s(B) \geq 1 + \text{loss}^* - 2/h$, where $B$ consists of those successors $t$ of $s$ such that

$$1 - \frac{1}{h^2} < \sum_{t\in A_i} a_i = \frac{1}{\Xi(Z_s)}\Xi(\{k \in Z_s : t \in q_k\}) \leq \frac{1}{\Xi(Z_s)}\Xi(Z_t).$$

So every $t \in B$ satisfies $\Xi(Z_t) > \Xi(Z_s)(1 - 1/h^2) \geq \widetilde{\zeta}^{\,h+1}$, i.e., satisfies $(*)$. So we can use $B$ as the set of $s$-successors in $q^*$.

This defines $q^*$, which is a valid condition by Fact 1.15 (b).

Step 3: We now show that this limit works: As in Definition 1.7, fix $m^*$, $(B_m)_{m<m^*}$, $\varepsilon$, $k^*$, $i^*$ and sequences $(p^i_l)_{l<\omega}$ for $i < i^*$, such that $(\text{trunk}(p^i_l), \text{loss}(p^i_l)) = (\text{trunk}^*, \text{loss}^*)$.

For each $i < i^*$, $\bar{q}^{\,i} = (q^i_k)_{k\in\omega}$ is defined from $\bar{p}^{\,i} = (p^i_l)_{l\in\omega}$, and in turn defines the limit $\lim_\Xi(\bar{p}^{\,i})$. Let $q$ be stronger than all $\lim_\Xi(\bar{p}^{\,i})$.

Let $M$ be as in Lemma 1.2 for $N = m^* + i^*$. So for any $N$ many sets there is a $u$ of size at most $M$ (above $k^*$) which approximates the measure well. We use the following $N$ many sets:

- $B_m$ for $m < m^*$.
- Fix an $s \in q$ of height $h > M \cdot i^*$, and use the $i^*$ many sets $Z^i_s \subseteq \omega$ defined in $(*)$.

Accordingly, there is a $u$ (starting above $k^*$) of size less than or equal to $M$ with

- $\Xi(B_m) - \varepsilon \leq |B_m \cap u|/|u| \leq \Xi(B_m) + \varepsilon$ for each $m < m^*$, and
- $|Z^i_s \cap u|/|u| \geq 1 - \text{loss}^*/3 - \varepsilon$ for each $i < i^*$.

So for each $i \in i^*$ there are at least $|u|(1 - \text{loss}^*/2 - \varepsilon)$ many $k \in u$ with $s \in q^i_k$. There is a condition $r$ stronger than $q$ and all those $q^i_k$ (as less than or equal to $Mi^* + 1$ many conditions of height $h > M \cdot i^*$ with common node $s$ above their stems are compatible). So $r$ forces for all $i < i^*$ and $k \in u \cap Z^i_s$ that $q^i_k \in G$ and therefore that $|\{l \in I_k : p^i_l \in G\}| \geq |I_k|(1 - \text{loss}^*/3)$. By increasing $r$ to some $q'$, we can assume that $r$ decides which $p^i_l$ are in $G$ and that $r$ is actually stronger than each $p^i_l$ decided to be in $G$. So all in all we get $q' \leq q$ such that

$$\frac{1}{|u|} \sum_{k \in u} \frac{|\{l \in I_k : q' \leq p^j_l\}|}{|I_k|} \;\geq\; \frac{1}{|u|} |\{k \in u : k \in Z^j_s\}| \Big(1 - \frac{1}{3}\,\text{loss}^*\Big) \;>\; 1 - \text{loss}^* - \varepsilon,$$

as required. □

## 2. The left hand side of Cichoń's diagram

We write $\mathfrak{x}_1$ for $\text{add}(\mathcal{N})$, $\mathfrak{x}_2$ for $\mathfrak{b}$ (which will also be $\text{add}(\mathcal{M})$), $\mathfrak{x}_3$ for $\text{cov}(\mathcal{N})$ and $\mathfrak{x}_4$ for $\text{non}(\mathcal{M})$.

**2.1 Good iterations and the LCU property.** We want to show that some forcing $\mathbb{P}^5$ results in $\mathfrak{x}_i = \lambda_i$ for $i = 1, \dots, 4$. So we have to show two "directions", $\mathfrak{x}_i \leq \lambda_i$ and $\mathfrak{x}_i \geq \lambda_i$.

For $i = 1, 3, 4$ (i.e., for all the characteristics on the left hand side apart from $\mathfrak{b} = \text{add}(\mathcal{M})$), the direction $\mathfrak{x}_i \leq \lambda_i$ will be given by the fact that $\mathbb{P}^5$ is $(R_i, \lambda_i)$-good for a suitable relation $R_i$. (For $i = 2$, i.e., the unbounding number, we will have to work more.)

We will use the following relations:

**Definition 2.1.** 1. Let $\mathcal{C}$ be the set of strictly positive rational sequences $(q_n)_{n\in\omega}$ such that $\sum_{n\in\omega} q_n \leq 1$.[6] Let $\mathrm{R}_1 \subseteq \mathcal{C}^2$ be defined by: $f \,\mathrm{R}_1\, g$ if $(\forall^* n \in \omega)\, f(n) \leq g(n)$.

2. $\mathrm{R}_2 \subseteq (\omega^\omega)^2$ is defined by: $f \,\mathrm{R}_2\, g$ if $(\forall^* n \in \omega)\, f(n) \leq g(n)$.

4. $\mathrm{R}_4 \subseteq (\omega^\omega)^2$ is defined by: $f \,\mathrm{R}_4\, g$ if $(\forall^* n \in \omega)\, f(n) \neq g(n)$.

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[6] It is easy to see that $\mathcal{C}$ is homeomorphic to $\omega^\omega$, when we equip the rationals with the discrete topology and use the product topology.

So far, these relations fit the usual framework of goodness, as introduced in [10] and [3] and summarized, e.g., in [2, 6.4] or [8, Section 3] or [13, Section 2]. For $\mathfrak{x}_3$, i.e., $\mathrm{cov}(\mathcal{N})$, we will use a relation $\mathrm{R}_3$ that does not fit this framework (as the range of the relation is not a Polish space). Nevertheless, the property "$(\mathrm{R}_3,\lambda)$-good" behaves just as in the usual framework (e.g., finite support limits of good forcings are good, etc.). The relation $\mathrm{R}_3$ was implicitly used by S. Kamo and N. Osuga in [15], who investigated $(\mathrm{R}_3,\lambda)$-goodness.[7] It was also used in [4]; a unifying notation for goodness (which works for the usual cases as well as relations such as $\mathrm{R}_3$) is given in [5, Section 4].

**Definition 2.2.** We call a set $\mathcal{E}\subset\omega^\omega$ an $\mathrm{R}_3$-parameter, if for all $e\in\mathcal{E}$

- $\lim e(n)=\infty$, $e(n)\leq n$, $\lim(n-e(n))=\infty$,
- there is some $e'\in\mathcal{E}$ such that $(\forall^* n)\, e(n)+1\leq e'(n)$, and
- for all countable $\mathcal{E}'\subseteq\mathcal{E}$ there is some $e\in\mathcal{E}$ such that for all $e'\in\mathcal{E}'$ $(\forall^* n)\, e(n)\geq e'(n)$.

Note that such an $\mathrm{R}_3$-parameter of size $\aleph_1$ exists. This is trivial if we assume continuum hypothesis (CH), which we could in this paper, but also true without this assumption, see [5, 4.20]. Recall that $\varrho$ and $\pi$ were defined in equation (1.13).

**Definition 2.3.** We fix for the rest of the paper, an $\mathrm{R}_3$-parameter $\mathcal{E}$ of size $\aleph_1$, and set

$$b(h)=(h+1)^2\varrho(h)^{h+1},\quad \mathcal{S}=\Big\{\psi\in\prod\nolimits_{h\in\omega}P(b(h))\colon (\forall\, h\in\omega)\,|\psi(h)|\leq\varrho(h)^h\Big\},$$

$$\mathcal{S}_e=\Big\{\varphi\in\prod\nolimits_{h\in\omega}P(b(h))\colon (\forall\, h\in\omega)\,|\varphi(h)|\leq\varrho(h)^{e(h)}\Big\}\qquad\text{and}\qquad \widehat{\mathcal{S}}=\bigcup_{e\in\mathcal{E}}\mathcal{S}_e.$$

We can now define the relation for $\mathrm{cov}(\mathcal{N})$:

3. $\mathrm{R}_3\subseteq\mathcal{S}\times\widehat{\mathcal{S}}$ is defined by: $\psi\,\mathrm{R}_3\,\varphi$ if and only if $(\forall^* n\in\omega)\,\varphi(n)\not\subseteq\psi(n)$.

Note that $\mathcal{S}_e\subset\widehat{\mathcal{S}}\subset\mathcal{S}$ and that $\mathcal{S}_e$ and $\mathcal{S}$ are Polish spaces. Assume that $M$ is a forcing extension of $V$ by either a ccc forcing (or by a $\sigma$-closed forcing). Then $\mathcal{E}$ is an "$\mathrm{R}_3$-parameter" in $M$ as well, and we can evaluate in $M$ for each $e\in\mathcal{E}$ the sets $\mathcal{S}_e^M$ and $\mathcal{S}^M$, as well as $\widehat{\mathcal{S}}^M=\bigcup_{e\in\mathcal{E}}\mathcal{S}_e^M$. Absoluteness gives $\mathcal{S}_e^V=\mathcal{S}_e^M\cap V$ and $\widehat{\mathcal{S}}^V=\widehat{\mathcal{S}}^M\cap V$.

**Definition 2.4.** Fix one of these relations $R\subseteq X\times Y$.

- We say "$f$ is bounded by $g$" if $f\,\mathrm{R}\,g$, and for $\mathcal{Y}\subseteq\omega^\omega$ "$f$ is bounded by $\mathcal{Y}$" if $(\exists\, y\in\mathcal{Y})\,f\,\mathrm{R}\,y$. We say "unbounded" for "not bounded". (I.e., $f$ is unbounded by $\mathcal{Y}$ if $(\forall\, y\in\mathcal{Y})\,\neg f\,\mathrm{R}\,y$.)
- We call $\mathcal{X}$ an R-unbounded family, if $\neg(\exists\, g)\,(\forall\, x\in\mathcal{X})x\,\mathrm{R}\,g$, and an R-dominating family if $(\forall\, f)\,(\exists\, x\in\mathcal{X})\,f\,\mathrm{R}\,x$.

[7] They use the notation $(*^{<\lambda}_{c,h})$, cf. [15, Definition 6].

- Let $\mathfrak{b}_i$ be the minimal size of an $\mathrm{R}_i$-unbounded family,
- and let $\mathfrak{d}_i$ be the minimal size of an $\mathrm{R}_i$-dominating family.

We only need the following connection between $\mathrm{R}_i$ and the cardinal characteristics:

**Lemma 2.5.** (1) $\mathrm{add}(\mathcal{N}) = \mathfrak{b}_1$ *and* $\mathrm{cof}(\mathcal{N}) = \mathfrak{d}_1$.
(2) $\mathfrak{b} = \mathfrak{b}_2$ *and* $\mathfrak{d} = \mathfrak{d}_2$.
(3) $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}_3$ *and* $\mathrm{non}(\mathcal{N}) \geq \mathfrak{d}_3$.
(4) $\mathrm{non}(\mathcal{M}) = \mathfrak{b}_4$ *and* $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}_4$.

Proof: (2) holds by definition. (1) can be found in [2, 6.5.B]. (4) is a result of [14] and [1], cf. [2, 2.4.1 and 2.4.7].

To see (3), we work in the space $\Omega = \prod_{h\in\omega} b(h)$, with the $b$ defined in Definition 2.3 and the usual (uniform) measure. It is well known that we get the same values for the characteristics $\mathrm{cov}(\mathcal{N})$ and $\mathrm{non}(\mathcal{N})$ whether we define them using $\Omega$ or, as usual, $2^\omega$ (or $[0,1]$ for that matter, etc). Given $\psi \in \mathcal{S}$, note that

$$N_\psi = \{\eta \in \Omega\colon (\exists^\infty h)\ \eta(h) \in \psi(h)\}$$

is a Null set, as $\{\eta \in \Omega\colon (\forall h > k)\ \eta(h) \notin \psi(h)\}$ has measure $\prod_{h>k}(1 - |\psi(h)|/b(h)) \geq \prod_{h>k}(1-(h+1)^{-3})$, which converges to 1 for $k \to \infty$.

Let $\mathcal{A} \subseteq \mathcal{S}$ be an $\mathrm{R}_3$-unbounded family. So for every $\varphi \in \widehat{\mathcal{S}}$ there is some $\psi \in A$ such that $(\exists^\infty h)\ \psi(h) \supseteq \varphi(h)$. In particular, for each $\eta \in \Omega$ there is a $\psi \in A$ with $\eta \in N_\psi$; i.e., $\mathrm{cov}(\mathcal{N}) \leq |\mathcal{A}|$.

Analogously, let $X$ be a non-null set (in $\Omega$). For each $\psi$ there is an $x \in X \setminus N_\psi$, so $\varphi_x(n) = \{x(n)\}$ satisfies $\psi\,\mathrm{R}_3\,\varphi_x$. □

**Remark 2.6.** As shown implicitly in [15], and explicitly in [5, 4.22], we actually get $\mathrm{cov}(\mathcal{N}) \leq c^{\exists}_{b,\varrho^{\mathrm{Id}}} \leq \mathfrak{b}_3$.

**Definition 2.7.** Let $P$ be a ccc forcing, $\lambda$ an uncountable regular cardinal, and $\mathrm{R}_i \subseteq X \times Y$ one of the relations above (so for $i = 1, 2, 4$, $Y = X$, and for $i = 3$ $Y = \widehat{\mathcal{S}}_e$). The forcing $P$ is $(\mathrm{R}_i, \lambda)$-good, if for each $P$-name $r$ for an element of $Y$ there is (in $V$) a nonempty set $\mathcal{Y} \subseteq Y$ of size less than $\lambda$ such that every $f \in X$ (in $V$) that is $\mathrm{R}_i$-unbounded by $\mathcal{Y}$ is forced to be $\mathrm{R}_i$-unbounded by $r$ as well.

Note that $\lambda$-good trivially implies $\mu$-good if $\mu \geq \lambda$ are regular.

**Lemma 2.8.** *Let $\lambda$ be uncountable regular.*

(a) *Forcings of size less than $\lambda$ are $(\mathrm{R}_i, \lambda)$-good. In particular, Cohen forcing is $(\mathrm{R}_i, \aleph_1)$-good.*
(b) *A FS ccc iteration of $(\mathrm{R}_i, \lambda)$-good forcings (and in particular, a composition of two such forcings) is $(\mathrm{R}_i, \lambda)$-good.*
(1) *A sub-Boolean-algebra of the random algebra is $(\mathrm{R}_1, \aleph_1)$-good. Any $\sigma$-centered forcing notion is $(\mathrm{R}_1, \aleph_1)$-good.*
(3) *A $(\varrho, \pi)$-linked forcing is $(\mathrm{R}_3, \aleph_1)$-good (for the $\varrho, \pi$ of Definition 1.12).*

Proof: (a) & (b) For $i = 1, 2, 4$ this is proven in [10], cf. [2, 6.4]. The same proof works for $i = 3$, as shown in [15, Lemmas 12, 13]. The proof for the uniform framework can be found in [5, 4.10, 4.14].

(1) follows from [10] and [11], cf. [2, 6.5.17–18].

(3) is shown in [15, Lemma 10], cf. [5, Lemma 4.24]; as our choice of $\pi$, $\varrho$ and $b$ (see Definition 2.3) satisfies $\pi(h) \geq b(h)^{\varrho(h)^h} = ((h+1)^2 \varrho(h)^{h+1})^{\varrho(h)^h}$. □

Each relation $\mathrm{R}_i$ is a subset of some $X \times Y$, where $X$ is either $2^\omega$, $\omega^\omega$ (or homeomorphic to it) or $\mathcal{S}$, and $Y$ is the range of $\mathrm{R}_i$.

**Lemma 2.9.** *For each $i$ and each $g \in Y$, the set $\{f \in X \colon f\,\mathrm{R}_i\, g\} \subseteq X$ is meager.*

Proof: We have explicitly defined each $f\,\mathrm{R}_i\, g$ as $\forall^* n\ \ R^n_i(f,g)$ for some $R^n_i$. The lemma follows easily from the fact that for each $n \in \omega$, the set $\{f \in X \colon R^n_i(f,g)\}$ is closed nowhere dense. □

**Lemma 2.10.** *Let $\lambda \leq \kappa \leq \mu$ be uncountable regular cardinals. Force with $\mu$ many Cohen reals $(c_\alpha)_{\alpha\in\mu}$, followed by an $(\mathrm{R}_i, \lambda)$-good forcing. Note that each Cohen real $c_\beta$ can be interpreted as element of the Polish space $X$ where $\mathrm{R}_i \subseteq X \times Y$. Then we get: For every real $r$ in the final extension $Y$, the set $\{\alpha \in \kappa \colon c_\alpha$ is $\mathrm{R}_i$-unbounded by $r\}$ is cobounded in $\kappa$. I.e., $(\exists\, \alpha \in \kappa)\,(\forall\, \beta \in \kappa \setminus \alpha)$ $\neg c_\alpha\, \mathrm{R}_i\, r$.*

Proof: Work in the intermediate extension after $\kappa$ many Cohen reals, let us call it $V_\kappa$. The remaining forcing (i.e., $\mu \setminus \kappa$ many Cohens composed with the good forcing) is good; so applying the definition we get (in $V_\kappa$) a set $\mathcal{Y} \subseteq Y$ of size less than $\lambda$.

As the initial Cohen extension is ccc, and $\kappa \geq \lambda$ is regular, we get some $\alpha \in \kappa$ such that each element $y$ of $\mathcal{Y}$ already exists in the extension by the first $\alpha$ many Cohens, call it $V_\alpha$.

Fix some $\beta \in \kappa \setminus \alpha$ and $y \in Y$. As $\{x \in X \colon x\,\mathrm{R}_i\, y\}$ is a meager set already defined in $V_\alpha$, we get $\neg c_\beta\, \mathrm{R}_i\, y$. Accordingly, $c_\beta$ is unbounded by $\mathcal{Y}$; and, by the definition of good, unbounded by $r$ as well. □

In the light of this result, let us revisit Lemma 2.5 with some new notation, the "linearly cofinally unbounded" property $\mathsf{LCU}$:

**Definition 2.11.** For $i = 1, 2, 3, 4$, $\gamma$ a limit ordinal, and $P$ a ccc forcing notion, let $\mathsf{LCU}_i(P,\gamma)$ stand for:

> There is a sequence $(x_\alpha)_{\alpha\in\gamma}$ of $P$-names such that for every $P$-name $y$ $\ (\exists\, \alpha \in \gamma)\,(\forall\, \beta \in \gamma \setminus \alpha)\ P \Vdash \neg x_\beta\, \mathrm{R}_i\, y)$.

**Lemma 2.12.** ∘ *The $\mathsf{LCU}_i(P,\delta)$ property is equivalent to $\mathsf{LCU}_i(P, \mathrm{cf}(\delta))$.*

∘ *If $\lambda$ is regular, then $\mathsf{LCU}_i(P,\lambda)$ implies $\mathfrak{b}_i \leq \lambda$ and $\mathfrak{d}_i \geq \lambda$.*

*In particular:*

1. *The $\mathsf{LCU}_1(P,\lambda)$ property implies $P \Vdash (\,\mathrm{add}(\mathcal{N}) \leq \lambda\ \&\ \mathrm{cof}(\mathcal{N}) \geq \lambda\,)$.*
2. *The $\mathsf{LCU}_2(P,\lambda)$ property implies $P \Vdash (\,\mathfrak{b} \leq \lambda\ \&\ \mathfrak{d} \geq \lambda\,)$.*

(3) *The* $\mathsf{LCU}_3(P,\lambda)$ *property implies* $P \Vdash (\operatorname{cov}(\mathcal{N}) \leq \lambda \ \&\ \operatorname{non}(\mathcal{N}) \geq \lambda)$.
(4) *The* $\mathsf{LCU}_4(P,\lambda)$ *property implies* $P \Vdash (\operatorname{non}(\mathcal{M}) \leq \lambda \ \&\ \operatorname{cov}(\mathcal{M}) \geq \lambda)$.

PROOF: Assume that $(\alpha_\beta)_{\beta\in\mathrm{cf}(\delta)}$ is increasing continuous and cofinal in $\delta$. If $(x_\alpha)_{\alpha\in\delta}$ witnesses $\mathsf{LCU}_i(P,\delta)$, then $(x_{\alpha_\beta})_{\beta\in\mathrm{cf}(\delta)}$ witnesses $\mathsf{LCU}_i(P,\mathrm{cf}(\delta))$. And if $(x_\beta)_{\beta\in\mathrm{cf}(\delta)}$ witnesses $\mathsf{LCU}_i(P,\mathrm{cf}(\delta))$, then $(y_\alpha)_{\alpha\in\delta}$ witnesses $\mathsf{LCU}_i(P,\mathrm{cf}(\delta))$, where $y_\alpha := x_\beta$ for $\alpha\in[\alpha_\beta,\alpha_{\beta+1})$.

The set $\{x_\alpha\colon \alpha\in\lambda\}$ is certainly forced to be $\mathrm{R}_i$-unbounded; and given a set $Y=\{y_j\colon j<\theta\}$ of $\theta<\lambda$ many $P$-names, each has a bound $\alpha_j\in\lambda$ so that $(\forall\beta\in\lambda\setminus\alpha_j)\ P\Vdash\neg x_\beta\,\mathrm{R}_i\,y_j)$, so for any $\beta\in\lambda$ above all $\alpha_j$ we get $P\Vdash\neg x_\beta\,\mathrm{R}_i\,y_j$ for all $j$; i.e., $Y$ cannot be dominating. □

**2.2 The initial forcing $\mathbb{P}^5$ and the $\mathsf{COB}$ property.** We will assume the following throughout the paper:

**Assumption 2.13.**
- $\lambda_1<\lambda_2<\lambda_3<\lambda_4<\lambda_5$ are regular uncountable cardinals such that $\mu<\lambda_i$ implies $\mu^{\aleph_0}<\lambda_i$.
- We set $\delta_5=\lambda_5+\lambda_5$, and partition $\delta_5\setminus\lambda_5$ into unbounded sets $S^i$ for $i=1,\dots,4$. Fix for each $\alpha\in\delta_5\setminus\lambda_5$ a $w_\alpha\subseteq\alpha$ such that $\{w_\alpha\colon\alpha\in S^i\}$ is cofinal[8] in $[\delta_5]^{<\lambda_i}$ for each $i=1,\dots,4$.

The reader can assume that $(\lambda_i)_{i=1,\dots,5}$ and $(S^i)_{i=1,\dots,4}$ have been fixed once and for all (let us call them "fixed parameters"), whereas we will investigate various possibilities for $\overline{w}=(w_\alpha)_{\alpha\in\delta_5\setminus\lambda_5}$ in the following. (We will call a $\overline{w}$ which satisfies the assumption a "cofinal parameter".)

We define by induction:

**Definition 2.14.** We define the FS iteration $(P_\alpha,Q_\alpha)_{\alpha\in\delta_5}$ and for $\alpha>\lambda_5$, $P'_\alpha$ as follows: If $\alpha\in\lambda_5$, then $Q_\alpha$ is Cohen forcing. In particular, the generic at $\alpha$ is determined by the Cohen real $\eta_\alpha$. For $\alpha\in\delta_5\setminus\lambda_5$:

(1) $$Q^{\mathrm{full}}_\alpha := \left\{\begin{array}{c}\text{Amoeba}\\ \text{Hechler}\\ \text{Random}\\ \widetilde{\mathbb{E}}\end{array}\right\} \qquad \text{for } \alpha \text{ in} \left\{\begin{array}{c} S^1\\ S^2\\ S^3\\ S^4\end{array}\right. .$$

So $Q^{\mathrm{full}}_\alpha$ is a Borel definable subset of the reals, and the $Q^{\mathrm{full}}_\alpha$-generic is determined, in a Borel way, by the canonical generic real $\eta_\alpha$.

(2) The set $P'_\alpha$ is the set of conditions $p\in P_\alpha$ satisfying the following for each $\beta\in\mathrm{supp}(p)$: $\beta\in w_\alpha$ and there is (in the ground model) a countable $u\subseteq w_\alpha\cap\beta$ and a Borel function $B\colon(\omega^\omega)^u\to Q^{\mathrm{full}}_\beta$ such that $p\restriction\beta$ forces that $p(\beta)=B((\eta_\gamma)_{\gamma\in u})$. We *assume* that

$$P'_\alpha \text{ is a complete subforcing of } P_\alpha. \tag{2.15}$$

(3) In the $P_\alpha$-extension, let $M_\alpha$ be the induced $P'_\alpha$-extension of $V$. Then $Q_\alpha$ is the $M_\alpha$-evaluation of $Q^{\mathrm{full}}_\alpha$. Or equivalently (by absoluteness): $Q_\alpha=$

[8] i.e., if $\alpha\in S^i$ then $|w_\alpha|<\lambda_i$, and for all $u\subseteq\delta_5$, $|u|<\lambda_i$ there is some $\alpha\in S^i$ with $w_\alpha\supseteq u$.

$Q^{\mathrm{full}}_\alpha \cap M_\alpha$. We call $Q_\alpha$ a "partial $Q^{\mathrm{full}}_\alpha$ forcing" (e.g.: a "partial random forcing").

Some notes:

- For item (3) of Definition 2.14 to make sense, (2.15) is required.
- We do not require any "transitivity" of the $w_\alpha$, i.e., $\beta \in w_\alpha$ does generally not imply $w_\beta \subseteq w_\alpha$.
- We do not require (and it will generally not be true) that $P_\alpha$ forces that $Q_\alpha$ is a *complete* subforcing of $Q^{\mathrm{full}}_\alpha$.

A simple absoluteness argument (between $M_\alpha$ and $V[G_\alpha]$) shows:

**Lemma 2.16.** *$P_\alpha$ forces:*

(a) *The forcing $Q_\alpha$ is an incompatibility preserving subforcing of $Q^{\mathrm{full}}_\alpha$ and in particular ccc. (And so, $P_\alpha$ itself is ccc for all $\alpha$.)*
(b) *For $\alpha \in S^i$, $|Q_\alpha| < \lambda_i$.*
(c) *The forcing $Q_\alpha$ forces that its generic filter $G(\alpha)$ is also generic over $M_\alpha$. So from the point of view of $M_\alpha$, $M_\alpha[G(\alpha)]$ is a $Q^{\mathrm{full}}_\alpha$-extention.*
(2) *For $\alpha \in S^2$, the partial Hechler forcing $Q_\alpha$ is $\sigma$-centered.*
(3) *For $\alpha \in S^3$, the partial random forcing $Q_\alpha$ is equivalent to a subalgebra of the random algebra.*
(4) *For $\alpha \in S^4$, a partial $\widetilde{\mathbb{E}}$ forcing is $(\varrho,\pi)$-linked and basically equivalent to a subalgebra of the random algebra (as in Lemma* 1.19 (e)*).*

PROOF: (b) $|P'_\alpha| \leq |w_\alpha|^{\aleph_0} \times 2^{\aleph_0} < \lambda_i$ by Assumption 2.13. There is a set of nice $P'_\alpha$-names of size less than $\lambda_i$ such that every $P'_\alpha$-name for a real has an equivalent name in this set. Accordingly, the size of the reals in $M_\alpha$ is forced to be less than $\lambda_i$.

(c) is trivial, as $Q_\alpha$ is element of the transitive class $M_\alpha$.

(4) By Lemma 1.19 (b) we know that $M_\alpha$ thinks that $\widetilde{\mathbb{E}}$ is $(\varrho,\pi)$-linked; i.e., that there is a family[9] $Q^i_j$ as in Definition 1.18. Being $l$-linked is obviously absolute between $M_\alpha$ and $V[G_\alpha]$ for any $l<\omega$, and $M_\alpha \vDash \bigcup_{h\in\omega, i<\varrho(h)} Q^h_i = Q^{\mathrm{full}}_\alpha$ translates to $V[G_\alpha] \vDash \bigcup_{h\in\omega, i<\varrho(h)} Q^h_i = Q_\alpha$.

Similarly, $M_\alpha$ thinks that $\widetilde{\mathbb{E}}$ satisfies 1.19 (e), i.e., that there is some dense $Q' \subseteq \widetilde{\mathbb{E}}$ and a dense embedding from $Q'$ to a subalgebra $B'$ of the random algebra.

So from the point of view of $V[G_\alpha]$, there is a $Q'$ dense in $\widetilde{\mathbb{E}} \cap M_\alpha$ and a dense embedding of $Q'$ into some $B'$, which is a subalgebra of the random algebra in $M_\alpha$ and therefore of the random algebra in $V[G_\alpha]$. □

It is easy to see that (2.15) is a "closure property" of $w_\alpha$:

**Lemma 2.17.** *Assume we have constructed (in the ground model) $(P_\beta, Q_\beta)_{\beta<\alpha}$ and $w_\alpha$ according to Definition* 2.14 *for some $\alpha \in S^i$, $i = 1,\dots,4$. This determines the (limit or composition) $P_\alpha$.*

[9] Actually there is even a Borel definable family $Q^i_j$, see the proof of Lemma 1.19 (a), but this is not required here.

(a) *For every $P_\alpha$-name $\tau$ of a real, there is (in $V$) a countable $u \subseteq \alpha$ and a Borel function $B\colon (\omega^\omega)^u \to \omega^\omega$ such that $P_\alpha$ forces $\tau = B((\eta_\gamma)_{\gamma\in u})$. (So if $w_\alpha \supseteq u$ satisfies* (2.15)*, then $P_\alpha$ forces that $\tau \in M_\alpha$.)*

(b) *The set of $w_\alpha$ satisfying* (2.15) *is an $\omega_1$-club in $[\alpha]^{<\lambda_i}$ (in the ground model).*

(A set $A \subseteq [\alpha]^{<\lambda_i}$ is an $\omega_1$-club, if for each $a \in [\alpha]^{<\lambda_i}$ there is a $b \supseteq a$ in $A$, and if $(a^i)_{i\in\omega_1}$ is an increasing sequence of sets in $A$, then the limit $b := \bigcup_{i\in\omega_1} a^i$ is in $A$ as well.)

Proof: The first item follows easily from the fact that we are dealing with a forsing set (FS) ccc iteration where the generics of all iterands $Q_\beta$ are Borel-determined by some generic real $\eta_\beta$. (See, e.g., [12, 1.2] for more details.)

Any $w \in [\alpha]^{<\lambda_i}$ defines some $P^w_\alpha$. We first define $w'$ for such a $w$:

Set $X = [P^w_\alpha]^{\leq\aleph_0}$, as set of size at most $(2^{\aleph_0} \times |w|^{\aleph_0})^{\aleph_0} < \lambda_i$. For $x \in X$, pick some $p \in P_\alpha$ stronger than all conditions in $x$ (if such a condition exists), and some $q \in P_\alpha$ incompatible to each element of $x$ (again, if possible). There is a countable $w_x \subseteq \alpha$ such that $p, q \in P^{w_x}$. Set $w' := w \cup \bigcup_{x\in X} w_x$.

Start with any $w_0 \in [\alpha]^{<\lambda_i}$. Construct an increasing continuous chain in $[\alpha]^{<\lambda_i}$ with $w^{k+1} = (w^k)'$. Then $w^{\omega_1} \supseteq w_0$ is in the set of $w$ satisfying (2.15); which shows that this set is unbounded. It is equally easy to see that it is closed under increasing sequences of length $\omega_1$. □

For later reference, we explicitly state the assumption we used (for every $\alpha \in \delta_5 \setminus \lambda_5$):

**Assumption 2.18.** The set $w_\alpha$ is sufficiently closed so that (2.15) is satisfied.

Let us also restate Lemma 2.17 (a):

**Lemma 2.19.** *For each $\mathbb{P}^5$-name $f$ of a real, there is a countable set $u \subseteq \delta_5$ such that $w_\alpha \supseteq u$ implies that ($\mathbb{P}^5$ forces that) $f \in M_\alpha$.*

**Lemma 2.20.** *The $\mathsf{LCU}_i(\mathbb{P}^5, \kappa)$ property holds for $i = 1, 3, 4$ and each regular cardinal $\kappa$ in $[\lambda_i, \lambda_5]$.*

Proof: This follows from Lemma 2.16:

For $i = 1$, partial random and partial $\widetilde{\mathbb{E}}$ forcings are basically equivalent to a sub-Boolean-algebra of the random algebra; and partial Hechler forcings are $\sigma$-centered. The partial amoeba forcings are small, i.e., have size less than $\lambda_1$. So according to Lemma 2.8, all iterands $Q_\alpha$ (and therefore the limits as well) are $(\mathrm{R}_1, \lambda_1)$-good.

For $i = 3$, note that partial $\widetilde{\mathbb{E}}$ forcings are $(\varrho, \pi)$-linked. All other iterands have size less than $\lambda_3$, so the forcing is $(\mathrm{R}_3, \lambda_3)$-good.

For $i = 4$ it is enough to note that *all* iterands are small, i.e., of size less than $\lambda_4$.

We can now apply Lemma 2.10. □

So in particular, $\mathbb{P}^5$ forces $\mathrm{add}(\mathcal{N}) \leq \lambda_1$, $\mathrm{cov}(\mathcal{N}) \leq \lambda_3$, $\mathrm{non}(\mathcal{M}) \leq \lambda_4$ and $\mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \lambda_5 = 2^{\aleph_0}$; i.e., the respective left hand characteristics are small. We now show that they are also large, using the "cone of bounds" property $\mathsf{COB}$:

**Definition 2.21.** For a ccc forcing notion $P$, regular uncountable cardinals $\lambda, \mu$ and $i = 1, 2, 4$, let $\mathsf{COB}_i(P, \lambda, \mu)$ stand for:

> There is a $<\lambda$-directed partial order $(S, \prec)$ of size $\mu$ and a sequence $(g_s)_{s\in S}$ of $P$-names for reals such that for each $P$-name $f$ of a real $(\exists\, s \in S)\,(\forall\, t \succ s)\, P \Vdash f\, \mathrm{R}_i\, g_t$.

For $i = 3$, let $\mathsf{COB}_3(P, \lambda, \mu)$ stand for:

> There is a $<\lambda$-directed partial order $(S, \prec)$ of size $\mu$ and a sequence $(g_s)_{s\in S}$ of $P$-names for reals such that for each $P$-name $f$ of a null-set $(\exists\, s \in S)\,(\forall\, t \succ s)\, P \Vdash g_t \notin f$.

So $s$ is the tip of a cone that consists of elements bounding $f$, where in case $i = 3$ we implicitly use an additional relation $N\, \mathrm{R}_3{}'\, r$ expressing that the null-set $N$ does not contain the real $r$. Note that $\mathrm{cov}(\mathcal{N})$ is the bounding number $\mathfrak{b}'_3$ of $\mathrm{R}_3{}'$, and $\mathrm{non}(\mathcal{N})$ the dominating number $\mathfrak{d}'_3$. So $\mathrm{add}(\mathcal{N}) = \mathfrak{b}'_3 \leq \mathfrak{b}_3$ and $\mathrm{non}(\mathcal{N}) = \mathfrak{d}'_3 \geq \mathfrak{d}_3$ (as defined in Lemma 2.5).

The $\mathsf{COB}_i(P, \lambda, \mu)$ property implies that $P$ forces that $\mathfrak{b}_i \geq \lambda$ and that $\mathfrak{d}_i \leq \mu$ for $i = 1, 2, 4$, and the same for $i = 3$ and $\mathfrak{b}'_3$, $\mathfrak{d}'_3$: Clearly $P$ forces that $\{g_s\colon s \in \mathcal{S}\}$ is dominating. And if $A$ is set of names of size $\kappa < \lambda$, then for each $f \in A$ the definition gives a bound $s(f)$ and directedness some $t \succ s(f)$ for all $f$, i.e., $g_t$ bounds all elements of $A$. So we get:

**Lemma 2.22.** (1) *The* $\mathsf{COB}_1(P, \lambda, \mu)$ *property implies* $P \Vdash (\mathrm{add}(\mathcal{N}) \geq \lambda\ \&\ \mathrm{cof}(\mathcal{N}) \leq \mu)$.
(2) *The* $\mathsf{COB}_2(P, \lambda, \mu)$ *property implies* $P \Vdash (\mathfrak{b} \geq \lambda\ \&\ \mathfrak{d} \leq \mu)$.
(3) *The* $\mathsf{COB}_3(P, \lambda, \mu)$ *property implies* $P \Vdash (\mathrm{cov}(\mathcal{N}) \geq \lambda\ \&\ \mathrm{non}(\mathcal{N}) \leq \mu)$.
(4) *The* $\mathsf{COB}_4(P, \lambda, \mu)$ *property implies* $P \Vdash (\mathrm{non}(\mathcal{M}) \geq \lambda\ \&\ \mathrm{cov}(\mathcal{M}) \leq \mu)$.

**Lemma 2.23.** *The* $\mathsf{COB}_i(\mathbb{P}^5, \lambda_i, \lambda_5)$ *property holds for* $i = 1, 2, 3, 4$.

Proof: We use the following facts (provable in ZFC, or true in the $P_\alpha$-extention, respectively):

(1) Amoeba forcing adds a sequence $\bar{b}$ which $\mathrm{R}_1$-dominates the old elements of $\mathcal{C}$.
(The simple proof can be found in [7, Lemma 1.4], a slight variation in [2].) Accordingly (by absoluteness), the generic real $\eta_\alpha$ for partial amoeba forcing $Q_\alpha$ $\mathrm{R}_1$-dominates $\mathcal{C} \cap M_\alpha$.
(2) Hechler forcing adds a real which $\mathrm{R}_2$-dominates all old reals.
Accordingly, the generic real $\eta_\alpha$ for partial Hechler forcing $Q_\alpha$ $\mathrm{R}_2$-dominates all reals in $M_\alpha$.
(3) Random forcing adds a random real.

Accordingly, the generic real $\eta_\alpha$ for partial random forcing $Q_\alpha$ is not in any null set whose Borel-code is in $M_\alpha$.

(4) The generic branch $\eta \in \lim(T^*)$ added by $\widetilde{\mathbb{E}}$ is eventually different to each old real, i.e., $\mathrm{R}_4$-dominates the old reals.
(This was shown in Lemma 1.19 (c).)
Accordingly, the generic branch $\eta_\alpha$ for partial $\widetilde{\mathbb{E}}$ forcing $Q_\alpha$ $\mathrm{R}_4$-dominates the reals in $M_\alpha$.

Fix $i \in \{1,2,3,4\}$, and set $S = S^i$ and $s \prec t$ if $w_s \subsetneq w_t$, and let $g_s$ be $\eta_s$, i.e., the generic added at $s$ (e.g., the partial random real in case of $i = 3$, etc.).

Fix a $\mathbb{P}^5$-name $f$ for a real. It depends (in a Borel way) on a countable index set $w^* \subseteq \delta_5$. Fix some $s \in S^i$ such that $w_s \supseteq w^*$. Pick any $t \succ s$. Then $w_t \supseteq w_s \supseteq w^*$, so ($\mathbb{P}^5$ forces that) $f \in M_t$, so, as just argued, $\mathbb{P}^5 \Vdash f\,\mathrm{R}_i\,g_t$ (or: $\mathbb{P}^5 \Vdash f\,\mathrm{R}'_3\,g_t$ for $i = 3$). □

So to summarize what we know so far about $\mathbb{P}^5$: Whenever we choose (in addition to the "fixed" $\lambda_i$, $S^i$) a cofinal parameter $\overline{w}$ satisfying Assumptions 2.13 and 2.18, we get

**Fact 2.24.**
- ◦ The $\mathsf{COB}_i$ property holds for $i = 1,2,3,4$. So the left hand side characteristics are large.
- ◦ The $\mathsf{LCU}_i$ property holds for $i = 1,3,4$. So the left hand side characteristics other than $\mathfrak{b}$ are small.

What is missing is "$\mathfrak{b}$ small". We do not claim that this will be forced for every $\overline{w}$ as above; but we will show in the rest of Section 2 that we can choose such a $\overline{w}$.

### 2.3 FAMs in the $P_\alpha$-extension compatible with $M_\alpha$, explicit conditions.

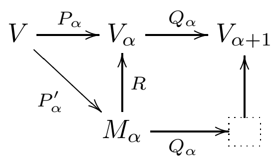
We first investigate sequences $\bar{q} = (q_l)_{l\in\omega}$ of $Q_\alpha$-conditions that are in $M_\alpha$, i.e., the (evaluations of) $P'_\alpha$-names for $\omega$-sequences in $Q_\alpha^{\mathrm{full}}$. For $\alpha \in S^3 \cup S^4$, $M_\alpha$ thinks that $Q_\alpha$ (i.e., $Q_\alpha^{\mathrm{full}}$) has FAM-limits. So if $M_\alpha$ thinks that $\Xi_0$ is a FAM, then for any sequence $\bar{q}$ in $M_\alpha$ there is a condition $\lim_{\Xi_0}(\bar{q})$ in $M_\alpha$ (and thus in $Q_\alpha$). We can relativize Lemma 1.8 to sequences in $M_\alpha$:

**Lemma 2.25.** *Assume that $\alpha \in S^3 \cup S^4$, that $\Xi$ is a $P_\alpha$-name for a FAM and that $\Xi_0$, the restriction of $\Xi$ to $M_\alpha$, is forced to be in $M_\alpha$. Then there is a $P_{\alpha+1}$-name $\Xi^+$ for a FAM such that for all* $(\mathrm{trunk}^*, \mathrm{loss}^*)$*-sequences $\bar{q}$ in $M_\alpha$,*

$$\lim_{\Xi_0}(\bar{q}) \in G(\alpha) \ \ \textit{implies} \ \ \Xi^+(A_{\bar{q}}) \geq 1 - \sqrt{\mathrm{loss}^*}.$$

$A_{\bar{q}}$ was defined in (1.9) (here we use $G(\alpha)$ instead of $G$, of course).

Proof: This Lemma is implicitly used in [16]. Note that $P'_\alpha$ is a complete subforcing of $P_\alpha$, and so there is a quotient $R$ such that $P_\alpha = P'_\alpha * R$. We consider the following (commuting) diagram:




Note that ($P'_\alpha$ forces that) $R * Q_\alpha = R \times Q_\alpha$. So from the point of view of $M_\alpha$:

- $Q_\alpha = Q_\alpha^{\text{full}}$ has FAM limits, and $\Xi_0$ is a FAM. So there is a $Q_\alpha$-name for a FAM $\Xi_0^+$ satisfying Lemma 1.8.
- $R$ is a ccc forcing, and there is an $R$-name[10] $\Xi$ for a FAM extending $\Xi_0$.
- So there is $R \times Q_\alpha$-name $\Xi^+$ for a FAM extending both $\Xi_0^+$ and $\Xi$ (cf. [16, Claim 1.6]).

Back in $V$, this defines the $P_{\alpha+1}$-name $\Xi^+$. Let $\bar{q} = (q_l)_{l \in \omega}$ be a sequence in $M_\alpha$. Then $M_\alpha[G(\alpha)]$ thinks: If $\lim_{\Xi_0}(\bar{q}) \in G(\alpha)$, then $\Xi_0^+(A_{\bar{q}})$ is large enough. This is upwards absolute to $V[G_{\alpha+1}]$ (as $A_{\bar{q}}$ is absolute). $\square$

For later reference, we will reformulate the lemma for a specific instance of "sequence in $M_\alpha$". Recall that a sequence in $M_\alpha$ corresponds to a "$P'_\alpha$-name of a sequence in $Q_\alpha^{\text{full}}$". This is not equivalent to a "$P_\alpha$-name for a sequence in $Q_\alpha$", which would correspond to an arbitrary sequence in $Q_\alpha$ (of which there are $|\alpha + \aleph_0|^{\aleph_0}$ many, while there are only less than $\lambda_i$ many sequences in $M_\alpha$). However, we can define the following:

**Definition 2.26.**
- An explicit $Q_\alpha$-condition (in $V$) is a $P'_\alpha$-name for a $Q_\alpha^{\text{full}}$ condition.
- A condition $p \in \mathbb{P}^5$ is explicit, if for all $\alpha \in \operatorname{supp}(p) \cap (S^4 \cup S^5)$, $p(\alpha)$ is an explicit $Q_\alpha$-condition.

Here we mean that for $p(\alpha)$ there is a $P'_\alpha$-name $q_\alpha$ such that $p \restriction \alpha \Vdash p(\alpha) = q_\alpha$ (and the map $\alpha \mapsto q_\alpha$ exists in the ground model, i.e., we do not just have a $P_\alpha$-name for a $P'_\alpha$-condition $q_\alpha$).

**Lemma 2.27.** *The set of explicit conditions is dense.*

Proof: We show by induction that the set $D_\alpha$ of explicit conditions in $P_\alpha$ is dense in $P_\alpha$. As we are dealing with FS iterations, limits are clear. Assume that $(p, q) \in P_{\alpha+1}$. Then $p$ forces that there is a $P'_\alpha$-name $q'$ such that $q' = q$. Strengthen $p$ to some $p' \in D_\alpha$ deciding $q'$. Then $(p', q') \leq (p, q)$ is explicit. $\square$

Note that any sequence in $V$ of explicit $Q_\alpha$-conditions defines a sequence of conditions in $M_\alpha$ (as $V \subseteq M_\alpha$). So we get:

**Lemma 2.28.** *Let $\alpha$, $\Xi$, and $\Xi^+$ be as in Lemma 2.25, and let $(p_l)_{l \in \omega}$ be (in $V$) a sequence of explicit conditions in $\mathbb{P}^5$ such that $\alpha \in \operatorname{supp}(p_l)$ for all $l \in \omega$. Set*

---

[10] We identify the $P_\alpha$-name $\Xi$ in $V$ and the induced $R$-name in $M_\alpha = V[G'_\alpha]$.

*$q_l := p_l(\alpha)$ and $\bar{q} := (q_l)_{l\in\omega}$, and assume that* $(\mathrm{trunk}(q_l), \mathrm{loss}(q_l))$ *is forced to be equal to some constant* $(\mathrm{trunk}^*, \mathrm{loss}^*)$.

*Then there is a $P'_\alpha$-name for a $Q^{\mathrm{full}}_\alpha$-condition (and thus a $P_\alpha$-name for a $Q_\alpha$-condition)* $\lim_{\Xi_0}(\bar{q})$ *such that* $\lim_{\Xi_0}(\bar{q})$ *forces that* $\Xi^+(A_{\bar{q}}) \le 1 - \sqrt{\mathrm{loss}^*}$.

**2.4 Dealing with $\mathfrak{b}$ (without generalized continuum hypothesis (GCH)).** In this section, we follow [7, 1.3], additionally using techniques inspired by [16].

We assume the following (in addition to Assumption 2.13):

**Assumption 2.29.** (This section only.) Let $\chi < \lambda_3$ is regular such that $\chi^{\aleph_0} = \chi$, $\chi^+ \ge \lambda_2$ and $2^\chi = |\delta_5| = \lambda_5$.

Set $S^0 = \lambda_5 \cup S^1 \cup S^2$. So $\delta_5 = S^0 \cup S^3 \cup S^4$, and $\mathbb{P}^5$ is a FS ccc iteration along $\delta_5$ such that $\alpha \in S^0$ implies $|Q_\alpha| < \lambda_2$, i.e., $|Q_\alpha| \le \chi$ (and $Q_\alpha$ is a partial random forcing for $\alpha \in S^3$ and a partial $\widetilde{\mathbb{E}}$-forcing for $\alpha \in S^4$).

Let us fix for each $\alpha \in S^0$ a $P_\alpha$-name

$$i_\alpha \colon Q_\alpha \to \chi \ \text{ injective.} \tag{2.30}$$

**Definition 2.31.**
- A "partial guardrail" is a function $h$ defined on a subset of $\delta_5$ such that for $\alpha \in \mathrm{dom}(h)$: $h(\alpha) \in \chi$ if $\alpha \in S^0$; and $h(\alpha)$ is a pair $(x, y)$ with $x \in H(\aleph_0)$ and $y$ a rational number otherwise. (Any $(\mathrm{trunk}, \mathrm{loss})$-pair is of this form.)
- A "countable guardrail" is a partial guardrail with countable domain. A "full guardrail" is a partial guardrail with domain $\delta_5$.

We will use the following lemma, which is a consequence of the Engelking–Karlowicz theorem, see [6], on the density of box products (cf. [8, 5.1]):

**Lemma 2.32** (as $|\delta_5| \le 2^\chi$)**.** *There is a family $H^*$ of full guardrails of cardinality $\chi$ such that each countable guardrail is extended by some $h \in H^*$. We will fix such an $H^*$.*

Note that the notion of guardrail (and the density property required in Lemma 2.32) only depends on the "fixed" parameters $\chi$, $\delta_5$, $S^0$, $S^3$ and $S^4$; so we can fix an $H^*$ that will work for all these fixed parameters and all choices of the cofinal parameter $\overline{w}$.

Once we have decided on $\overline{w}$, and thus have defined $\mathbb{P}^5$, we can define the following:

**Definition 2.33.** The set $D^* \subseteq \mathbb{P}^5$ consists of $p$ such that there is a partial guardrail $h$ (and we say: "$p$ follows $h$") with $\mathrm{dom}(h) \supseteq \mathrm{supp}(p)$ and for all $\alpha \in \mathrm{supp}(p)$ applies:
- If $\alpha \in S^0$, then $p \restriction \alpha \Vdash i_\alpha(p(\alpha)) = h_\alpha$.
- If $\alpha \in S^3 \cup S^4$, the empty condition of $P_\alpha$ forces

$$p(\alpha) \in Q_\alpha \qquad \text{and} \qquad (\mathrm{trunk}(p(\alpha)), \mathrm{loss}(p(\alpha))) = h(\alpha).$$

- Furthermore, $\sum_{\alpha\in\mathrm{supp}(p)\cap(S^3\cup S^4)}\sqrt{\mathrm{loss}(p(\alpha))} < 1/2$.
- A condition $p$ is explicit (as in Definition 2.26).

**Lemma 2.34.** *The set $D^* \subseteq \mathbb{P}^5$ is dense.*

PROOF: By induction we show that for any sequence $(\varepsilon_i)_{i\in\omega}$ of positive numbers the following set of $p$ is dense: If $\mathrm{supp}(p) = \{\alpha_0,\dots,\alpha_m\}$, where $\alpha_0 > \alpha_1 > \dots$ (i.e., we enumerate downwards), $\mathrm{loss}^p_{\alpha_n} < \varepsilon_n$ whenever $\alpha_n \in S^3 \cup S^4$. For the successor step, we use that the set of $q \in Q_\alpha$ such that $\mathrm{loss}(q) < \varepsilon_0$ is forced to be dense. □

**Remark 2.35.** So the set of conditions following *some* guardrail is dense. For each *fixed* guardrail $h$, the set of all conditions $p$ following $h$ is $n$-linked, provided that each loss in the domain of $h$ is less than $1/n$ (cf. Assumption 1.5).

**Definition 2.36.** A "$\Delta$-system with heart $\nabla$ following the guardrail $h$" is a family $\bar p = (p_i)_{i\in I}$ of conditions such that:

- all $p_i$ are in $D^*$ and follow $h$;
- $(\mathrm{supp}(p_i))_{i\in I}$ is a $\Delta$ system with heart $\nabla$ in the usual sense (so $\nabla \subseteq \delta_5$ is finite);
- the following is independent of $i \in I$:
  - $|\mathrm{supp}(p_i)|$, which we call $m^{\bar p}$.
    Let $(\alpha_i^{\bar p,n})_{n<m^{\bar p}}$ increasingly enumerate $\mathrm{supp}(p_i)$.
  - Whether $\alpha_i^{\bar p,n}$ is less than, equal to or bigger than the $k$th element of $\nabla$.
    In particular it is independent of $i$ whether $\alpha_i^{\bar p,n} \in \nabla$, in which case we call $n$ a "heart position".
  - Whether $\alpha_i^{\bar p,n}$ is in $S^0$, in $S^3$ or in $S^4$.
    If $\alpha_i^{\bar p,n} \in S^j$, we call $n$ an "$S^j$-position".
  - If $n$ is not an $S^0$-position,[11] the value of $h(\alpha_i^{\bar p,n}) =: (\mathrm{trunk}^{\bar p,n}, \mathrm{loss}^{\bar p,n})$.
    If $n$ is an $S^0$-position, we set $\mathrm{loss}^{\bar p,n} := 0$.

A "countable $\Delta$-system" $\bar p = (p_l : l \in \omega)$ is a $\Delta$ system that additionally satisfies:

- For each non-heart position[12] $n < m^{\bar p}$, the sequence $(\alpha_l^{\bar p,n})_{l\in\omega}$ is strictly increasing.

**Fact 2.37.**
- Each infinite $\Delta$-system $(p_i)_{i\in I}$ contains a countable $\Delta$-system. I.e., there is a sequence $i_l$ in $I$ such that $(p_{i_l})_{l\in\omega}$ is a countable $\Delta$-system.
- If $\bar p$ is a $\Delta$-system (or: a countable $\Delta$-system) following $h$ with heart $\nabla$, and $\beta \in \nabla \cup (\max(\nabla + 1))$, then $\bar p \restriction \beta := (p_i \restriction \beta)_{i\in I}$ is again a $\Delta$-system (or: a countable $\Delta$-system, respectively) following $h$, now with heart $\nabla \cap \beta$.

[11] If $n$ is a $S^0$-position, $h(\alpha_i^{\bar p,n})$ will generally not be independent of $i$; unless of course $n$ is a heart position.

[12] For a heart position $n$, $(\alpha_l^{\bar p,n})_{l\in\omega}$ is of course constant.

**Definition 2.38.** Let $\bar{p}$ be a countable $\Delta$-system, and assume that a sequence $\bar{\Xi} = (\Xi_\alpha)_{\alpha \in \nabla \cap (S^3 \cup S^4)}$ is such that each $\Xi_\alpha$ is a $P_\alpha$-name for a FAM and $P_\alpha$ forces that $\Xi_\alpha$ restricted to $M_\alpha$ is in $M_\alpha$. Then we can define $q = \lim_{\bar{\Xi}}(\bar{p})$ to be the following $\mathbb{P}^5$-condition with support $\nabla$:

- If $\alpha \in \nabla \cap S^0$, then $q(\alpha)$ is the common value of all $p_n(\alpha)$. (Recall that this value is already determined by the guardrail $h$.)
- If $\alpha \in \nabla \cap (S^3 \cup S^4)$, then $q(\alpha)$ is (forced by $\mathbb{P}^5_\alpha$ to be) $\lim_{\Xi_\alpha}(p_l(\alpha))_{l \in \omega}$, see Lemma 2.28.

We now give a specific way to construct such $\overline{w}$, which allows to keep $\mathfrak{b}$ small.

**Lemma/Construction 2.39.** *We can construct by induction on $\alpha \in \delta_5$ for each $h \in H^*$ some $\Xi^h_\alpha$, and if $\alpha > \kappa_5$, also $w_\alpha$, such that:*

(a) *Each $\Xi^h_\alpha$ is a $P_\alpha$-name of a FAM extending $\bigcup_{\beta<\alpha} \Xi^h_\beta$.*

(b) *Let $\alpha$ be a limit of countable cofinality: Assume $\bar{p}$ is a countable $\Delta$-system in $P_\alpha$ following $h$, and $n < m^{\bar{p}}$ such that $(\alpha^{\bar{p},n}_l)_{l\in\omega}$ has supremum $\alpha$. Then $A_{\bar{p},n}$ is forced to have $\Xi^h_\alpha$-measure 1, where*

$$A_{\bar{p},n} := \{k \in \omega \colon |\{l \in I_k : p_l(\alpha^{\bar{p},n}_l) \in G(\alpha^{\bar{p},n}_l)\}| \geq |I_k|(1 - \sqrt{\mathrm{loss}^{\bar{p},n}}\,)\}.$$

(c) *For each countable $\Delta$-system $\bar{p}$ in $P_\alpha$ following $h$, the $P_\alpha$-condition $\lim_{(\Xi^h_\beta)_{\beta<\alpha}}(\bar{p})$ is well-defined and forces*

$$\Xi^h_\alpha(A_{\bar{p}}) \geq 1 - \sum_{n<m^{\bar{p}}} \sqrt{\mathrm{loss}^{\bar{p},n}}, \text{ where}$$

$$A_{\bar{p}} := \left\{k \in \omega \colon |\{l \in I_k : p_l \in G_\alpha\}| \geq |I_k|\left(1 - \sum_{n<m^{\bar{p}}} \sqrt{\mathrm{loss}^{\bar{p},n}}\right)\right\}.$$

(d) *For $\alpha > \kappa_5$, $w_\alpha$ is "sufficiently closed". More specifically: It satisfies Assumptions 2.13 and 2.18, and if $\alpha \in S^3 \cup S^4$ then $P_\alpha$ forces that $\Xi^h_\alpha$ restricted to $M_\alpha$ is in $M_\alpha$.*

*Actually, the set of $w_\alpha$ satisfying this is an $\omega_1$-club set.*

Proof: (a&c) for $\mathrm{cf}(\alpha) > \omega$: We set $\Xi^h_\alpha = \bigcup_{\beta<\alpha} \Xi^h_\beta$. As there are no new reals at uncountable confinalities, this is a FAM. Each countable $\Delta$-system is bounded by some $\beta < \alpha$, and, by induction, (c) holds for $\beta$; so (c) holds for $\alpha$ as well.

(a&b) for $\mathrm{cf}(\alpha) = \omega$: Fix $h$. We will show that $P_\alpha$ forces $A \cap \bigcap_{j<j^*} A_{\bar{p}^j,n^j} \neq \emptyset$, where $A$ is a $\Xi^h_\beta$-positive set for some $\beta < \alpha$, and each $(\bar{p}^j, n^j)$ is as in (b).

Then we can work in the $P_\alpha$-extension and apply Fact 1.3 (a), using $\bigcup_{\beta<\alpha} \Xi^h_\beta$ as the partial FAM $\Xi'$. This gives an extension of $\Xi'$ to a FAM $\Xi^h_\alpha$ that assigns measure one to all $A_{\bar{p},n}$, showing that (a) and (b) are satisfied.

So assume towards a contradiction that some $p \in P_\alpha$ forces

$$A \cap \bigcap_{j<j^*} A_{\bar{p}^j,n^j} = \emptyset.$$

We can assume that $p$ decides the $\beta$ such that $A \in V_\beta$, that $\beta$ is above the hearts of all $\Delta$-sequences $\bar{p}^j$ involved, and that $\mathrm{supp}(p) \subseteq \beta$. We can extend $p$ to some $p^* \in P_\beta$ to decide $k \in A$ for some "large" $k$: By large, we mean:

- Let $F(l; n, p)$ (the cumulative binomial probability distribution) be the probability that $n$ independent experiments, each with success probability $p$, will have at most $l$ successful outcomes. As $\lim_{n\to\infty} F(np'; n, p) = 0$ for all $p' < p$, and as $\lim_{k\to\infty} |I_k| = \infty$, we can find some $k$ such that

$$F(|I_k| p'_j; |I_k|, p_j) < \frac{1}{2j^*} \tag{2.40}$$

  for all $j < j^*$, where we set $p'_j := 1 - \sqrt{\mathrm{loss}^{\bar{p}^j, n^j}}$ and $p_j := 1 - (1 + \sqrt{2}/2) \times \mathrm{loss}^{\bar{p}^j, n^j}$. (Note that $p'_j < p_j$, as $\mathrm{loss}^{\bar{p}^j, n^j} \leq 1/2$.)
- All elements of $Y = \{\alpha_l^{\bar{p}^j, n^j} : j < j^* \text{ and } l \in I_k\}$ are larger than $\beta$. (This is possible as each sequence $(\alpha_l^{\bar{p}^j, n^j})_{l<\omega}$ has supremum $\alpha$.) We enumerate $Y$ by the increasing sequence $(\beta_i)_{i\in M}$, and set $\beta_{-1} = \beta$.

We will find $q \leq p^*$ forcing that $k \in \bigcap_{j<j^*} A_{\bar{p}^j, n^j}$.

To this end, we define a finite tree $\mathcal{T}$ of height $M$, and assign to each $s \in \mathcal{T}$ of height $i$ a condition $q_s \in P_{\beta_{i-1}+1}$ (decreasing along each branch) and a probability $\mathrm{pr}_s \in [0, 1]$, such that $\sum_{t \rhd s} \mathrm{pr}_t = 1$ for all non-terminal nodes $s \in \mathcal{T}$. For $s$ the root of $\mathcal{T}$, i.e., for the unique $s$ of height 0, we set $q_s = p^* \in P_{\beta_{-1}}$ and $\mathrm{pr}_s = 1$.

So assume we have already constructed $q_s \in P_{\beta_{i-1}+1}$ for some $s$ of height $i < M$. We will now take care of index $\beta_i$ and construct the set of successors of $s$, and for each successor $t$, a $q_t \leq q_s$ in $P_{\beta_i+1}$.

- If $\beta_i \in S^0$, the guardrail guarantees that $\beta_i \in \mathrm{supp}(p_l^j)$ implies $p_l^j \restriction \beta_i \Vdash i_{\beta_i}(p_l^j(\beta_i)) = h(\beta_i)$. In that case we use a unique $\mathcal{T}$-successor $t$ of $s$, and we set $q_t = q_s^\frown(\beta_i, i_{\beta_i}^{-1} h(\beta_i))$, and $\mathrm{pr}_t = 1$.
  In the following we assume $\beta_i \notin S^0$.
- Let $J_i$ be the set of $j < j^*$ such that there is an $l \in I_k$ with $\alpha_l^{\bar{p}^j, n^j} = \beta_i$ (there is at most one such $l$). For $j \in J_i$ set $r_i^j = p_l^j(\beta_i)$ for the according $l$. So each $r_i^j$ is a $P_{\beta_i}$-name for an element of $Q_{\beta_i}$.
  The guardrail gives us the constant value $(\mathrm{trunk}_i^*, \mathrm{loss}_i^*) := h(\beta_i)$ (which is equal to $(\mathrm{trunk}^{\bar{p}^j, n^j}, \mathrm{loss}^{\bar{p}^j, n^j})$ for all $j \in J_i$).
- The case $\beta_i \in S^3$, i.e., the case of random forcing, is basically [16, 2.14]: For $x \subseteq [\mathrm{trunk}_i^*]$, set $\mathrm{Leb}^{\mathrm{rel}}(x) = \mathrm{Leb}(x)/\mathrm{Leb}([\mathrm{trunk}_i^*])$. Note that the $r_i^j$ are closed subsets of $[\mathrm{trunk}_i^*]$ and $\mathrm{Leb}^{\mathrm{rel}}(r_i^j) \geq 1 - \mathrm{loss}_i^*$.

  Let $\mathcal{B}^*$ be the power set of $[\mathrm{trunk}_i^*]$; and let $\mathcal{B}$ be the sub-Boolean-algebra generated by $r_i^j$, $j \in J_i$, let $\mathcal{X}$ be the set of atoms and $\mathcal{X}' = \{x \in \mathcal{X} \colon \mathrm{Leb}^{\mathrm{rel}}(x) > 0\}$. So $|\mathcal{X}'| \leq 2^{J_i} \leq 2^{j^*}$, $\sum_{x\in\mathcal{X}'} \mathrm{Leb}^{\mathrm{rel}}(x) = 1$, and $\sum_{x\in\mathcal{X}', x\subseteq r_i^j} \mathrm{Leb}^{\mathrm{rel}}(x) = \mathrm{Leb}^{\mathrm{rel}}(r_i^j)$.

So far, $\mathcal{X}'$ is a $P_{\beta_i}$-name. Now we increase $q_s$ inside $P_{\beta_i}$ to some $q^+$ deciding which of the (finitely many) Boolean combinations result in elements of $\mathcal{X}'$, and also deciding rational numbers $y_x$, $x \in \mathcal{X}'$, with sum 1 such that $|\operatorname{Leb}^{\mathrm{rel}}(x) - y_x| < ((\sqrt{2}-1)/2)\operatorname{loss}_i^* \cdot 2^{-j^*}$.

We can now define the immediate successors of $s$ in $\mathcal{T}$: For each $x \in \mathcal{X}'$, add an immediate successor $t_x$ and assign to it the probability $\mathrm{pr}_{t_x} = y_x$ and the condition $q_{t_x} = q^{+\frown}(\beta_i, r_x)$, where $r_x$ is a (name for a) partial random condition below $x$ (such a condition exists, as the Lebesgue positive intersection of finitely many partial random condition contains a partial random condition).

Note that when we choose a successor $t$ randomly (according to the assigned probabilities $\mathrm{pr}_t$), then for each $j \in J$ the probability of $q^+ \Vdash q_t(\beta_i) \leq r_i^j$ is at least

$$\begin{aligned}\sum_{x\in\mathcal{X}', x\subseteq r_i^j} \mathrm{pr}_x &\geq \sum_{x\in\mathcal{X}', x\subseteq r_i^j} \Big(\operatorname{Leb}^{\mathrm{rel}}(x) - \frac{\sqrt{2}-1}{2}\operatorname{loss}_i^* \cdot 2^{-j^*}\Big)\\ &\geq \Big(\sum_{x\in\mathcal{X}', x\subseteq r_i^j} \operatorname{Leb}^{\mathrm{rel}}(x)\Big) - \frac{\sqrt{2}-1}{2}\operatorname{loss}_i^*\\ &= \operatorname{Leb}^{\mathrm{rel}}(r_i^j) - \frac{\sqrt{2}-1}{2}\operatorname{loss}_i^*\\ &\geq 1 - \operatorname{loss}_i^* - \frac{\sqrt{2}-1}{2}\operatorname{loss}_i^*\\ &= 1 - \frac{1+\sqrt{2}}{2}\operatorname{loss}_i^*.\end{aligned}$$

- The case $\beta_i \in S^4$, i.e., the case of $\widetilde{\mathbb{E}}$:

  Recall that $\widetilde{\mathbb{E}}$-conditions are subtrees of some basic compact tree $T^*$, and there is a $h$ such that: if $\max\{|I_k|, j^*\}$ many conditions share a common node (above their stems) at height $h$, then they are compatible.

  All conditions $r_i^j$ have the same stem $s^* = \operatorname{trunk}_i^*$. For each $j \in J_i$, set $d(j) = r_i^j \cap \omega^h$. Note that ($P_{\beta_i}$ forces that) $d(j)$ is a subset of $T^* \cap [s^*] \cap \omega^h$ of relative size greater than or equal to $1 - \operatorname{loss}_i^*/2$ (according to Lemma 1.19 (d)). First find $q^+ \leq q_s$ in $P_{\beta_i}$ deciding all $d(j)$.

  We can now define the immediate successors of $s$ in $\mathcal{T}$: For each $x \in T^* \cap [s^*] \cap \omega^h$ add an immediate successor $t_x$, and assign to it the uniform probability (i.e., $\mathrm{pr}_{t_x} = |T^* \cap [s^*] \cap \omega^h|^{-1}$) and the condition $q_{t_x} = q^{+\frown}(\beta_i, r_x)$, where $r_x$ is a partial $\widetilde{\mathbb{E}}$-condition stronger than all $r_i^j$ that satisfy $x \in d(j)$. (Such a condition exists, as we can intersect less than or equal to $j^*$ many conditions of height $h$.)

  If we choose $t$ randomly, then for each $j \in J$ the probability of $q^+ \Vdash q_t \leq r_i^j$ is at least $1 - \operatorname{loss}_i^*/2 \geq 1 - ((1+\sqrt{2})/2)\operatorname{loss}_i^*$.

In the end, we get a tree $\mathcal{T}$ of height $M$, and we can choose a random branch through $\mathcal{T}$, according to the assigned probabilities. We can identify the branch with its terminal node $t^*$, so in this notation the branch $t^*$ has probability $\prod_{n\le M} \mathrm{pr}_{t^*\restriction n}$.

Fix $j < j^*$. There are $|I_k|$ many levels $i < M$ such that at $\beta_i$ we deal with the $(\bar p^j, n^j)$-case. Let $M^j$ be the set of these levels. For each $i \in M^j$, we perform an experiment, by asking whether the next step $t \in \mathcal{T}$ (from the current $s$ at level $i$) will satisfy $q_t \restriction \beta_i \Vdash q_t(\beta_i) \le r^j_i$. While the exact probability for success will depend on which $s$ at level $i$ we start from, a lower bound is given by $1 - ((1+\sqrt{2})/2)\,\mathrm{loss}^*_i$. Recall that $\mathrm{loss}^*_i = \mathrm{loss}^{\bar p^j, n^j}$, and that we set $p_j := 1 - (1+\sqrt{2}/2)\,\mathrm{loss}^*_i$ and $p'_j := 1 - \sqrt{\mathrm{loss}^{\bar p^j, n^j}}$ in (2.40). So the chance of our branch $t^*$ having success fewer than $|I_k|(1-\sqrt{\mathrm{loss}^{\bar p^j, n^j}})$ many times, out of the $|I_k|$ many tries, (let us call such a $t^*$ "bad for $j$") is at most $F(|I_k|p'; |I_k|, p) \le 1/(2j^*)$.

Accordingly, the measure of branches that are not bad for *any* $j < j^*$ is at least $1/2$. Fix such a branch $t^*$. Then for each $j < j^*$,

$$|\{i \in M^j : q_{t^*} \restriction \beta_i \Vdash q_{t^*}(\beta_i) \le r^j_i\}| \ge |I_k|\big(1 - \sqrt{\mathrm{loss}^{\bar p^j, n^j}}\,\big),$$

and thus $q_{t^*}$ forces that

$$|\{l \in I_k : p_l(\alpha_l^{\bar p^j, n^j}) \in G(\alpha_l^{\bar p^j, n^j})\}| \ge |I_k|\big(1 - \sqrt{\mathrm{loss}^{\bar p^j, n^j}}\,\big).$$

(c) for $\mathrm{cf}(\alpha) = \omega$: Fix $\bar p$ as in the assumption of (c). To simplify notation, let us assume that $\nabla \ne \emptyset$ and that $\sup(\nabla) < \sup(\mathrm{supp}(p_l))$ (for some, or equivalently: all $l \in \omega$). Let $0 < n_0 < m^{\bar p}$ be such that $\sup(\nabla)$ is at position $n_0 - 1$ in $\mathrm{supp}(p_l)$, i.e., $\sup(\nabla) = \alpha_l^{\bar p, n_0 - 1}$ (independent of $l$), and set $\beta := \sup(\nabla) + 1$.

The system $\bar p \restriction \beta$ is again a countable $\Delta$-system following the same $h$, and $\lim_{(\Xi^h_\gamma)_{\gamma<\alpha}}(\bar p)$ is by definition identical to $\lim_{(\Xi^h_\gamma)_{\gamma<\beta}}(\bar p \restriction \beta)$, which by induction is a valid condition and forces (c) for $\bar p \restriction \beta$. This gives us the set $A_{\bar p \restriction \beta}$ of measure at least $1 - \sum_{n<n_0} \sqrt{\mathrm{loss}^{\bar p, n}}$.

For the positions $n_0 \le n < m^{\bar p}$, all $(\alpha_l^{\bar p, n})_{l\in\omega}$ are strictly increasing sequences above $\beta$ with some limit $\alpha_n \le \alpha$. Then (b) (applied to $\alpha_n$) gives us an according measure-1-set $A_{\bar p, n}$.

So $\lim_{(\Xi^h_\gamma)_{\gamma<\alpha}}(\bar p)$ forces that $A' = A_{\bar p\restriction\beta} \cap \bigcap_{n_0 \le n < m^{\bar p}} A_{\bar p, n}$ has measure $\Xi^h_\alpha(A') \ge 1 - \sum_{n<n_0} \sqrt{\mathrm{loss}^{\bar p, n}} \ge 1 - \sum_{n<m^{\bar p}} \sqrt{\mathrm{loss}^{\bar p, n}}$.

Note that $p_l \in G$ if and only if $p_l \restriction \beta \in G_\beta$ and $p_l(\alpha^{\bar p, n}) \in G(\alpha^{\bar p, n})$ for all $n_0 \le n < m^{\bar p}$.

Fix $k \in A'$. As $k \in A_{\bar p \restriction \beta}$, the relative frequency for $l \in I_k$ *not* to satisfy $p_l \restriction \beta \in G_\beta$ is at most $\sum_{n<n_0} \sqrt{\mathrm{loss}^{\bar p, n}}$. For any $n_0 \le n < m^{\bar p}$, as $k \in A_{\bar p, n}$, the relative frequency for *not* $p_l(\alpha^{\bar p, n}) \in G(\alpha^{\bar p, n})$ is at most $\sqrt{\mathrm{loss}^{\bar p, n}}$. So the relative frequency for $p_l \in G$ to fail is at most $\sum_{n<n_0} \sqrt{\text{loss}^{\bar{p},n}} + \sum_{n_0 \le n < m^{\bar{p}}} \sqrt{\text{loss}^{\bar{p},n}}$, as required.

(a&c) for $\alpha = \gamma + 1$ successor: For $\gamma \in S^0$ this is clear: Let $\Xi^h_\alpha$ be the name of some FAM extending $\Xi^h_\gamma$. Let $\bar{p}$ be as in (c), without loss of generality $\gamma \in \nabla$. Then $q^+ := \lim_{(\Xi^h_\beta)_{\beta<\alpha}}(\bar{p}) = q^\frown(\gamma, r)$, where $q := \lim_{(\Xi^h_\beta)_{\beta<\gamma}}(\bar{p} \restriction \gamma)$ and $r$ is the condition determined by $h(\gamma)$, i.e., each $p_l \restriction \gamma$ forces $p_l(\gamma) = r$. In particular, $q^+$ forces that $p_l \in G_\alpha$ if and only if $p_l \restriction \gamma \in G_\alpha$. By induction, (c) holds for $\gamma$, and therefore we get (c) for $\alpha$.

Assume $\gamma \in S^3 \cup S^4$. By induction we know that (d) holds for $\gamma$, i.e., that $\Xi^h_\gamma$ restricted to $M_\gamma$ (call it $\Xi_0$) is in $M_\gamma$. So the requirement in the Definition 2.38 of the limit is satisfied, and thus the limit $q^+ := \lim_{\Xi^h}(\bar{p})$ is well defined for any countable $\Delta$-system $\bar{p}$ as in (c): $q^+$ has the form $q^\frown(\gamma, r)$ with $q$ and $r$ such that $q = \lim_{(\Xi^h_\beta)_{\beta<\gamma}}(\bar{p} \restriction \gamma)$ and $r = \lim_{\Xi_0}((p_l(\gamma))_{l\in\omega})$. Now Lemma 2.28 gives us the $P_\alpha$-name $\Xi^+$, which will be our new $\Xi^h_\alpha$.

This works as required: Again without loss of generality we can assume $\gamma \in \nabla$. By induction, $q$ forces that $\Xi^h_\gamma(A_{\bar{p}\restriction\gamma}) \ge 1 - \sum_{n<m^p-1} \sqrt{\text{loss}^{\bar{p},n}}$. According to Lemma 2.28 $r$ forces that $\Xi^+(A_{(p_l(\gamma))_{l\in\omega}}) \ge 1 - \sqrt{\text{loss}^{\bar{p},m^p-1}}$. So $q^+ = q^\frown r$ forces that $\Xi^h_\alpha(A_{\bar{p}}) \ge 1 - \sum_{n<m^p} \sqrt{\text{loss}^{\bar{p},n}}$.

(d): So we have (in $V$) the $P_\alpha$-name $\Xi^h_\alpha$. We already know that there is (in $V$) an $\omega_1$-club set $X_0$ in $[\alpha]^{<\lambda_i}$ (for the appropriate $i \in \{3, 4\}$) such that $w \in X_0$ implies that $w$ satisfies Assumptions 2.13 and 2.18. So each such $w \in X_0$ defines a complete subforcing $P_w$ of $P_\alpha$ and the $P_\alpha$-name for the according $P_w$-extention $M_w$.

Fix some $w \in X_0$. We will define $w' \supseteq w$ as follows: For a $P_w$-name (and thus a $P_\alpha$-name) $r \in 2^\omega$, let $s$ be the name of $\Xi_\alpha(r) \in [0, 1]$. As in Lemma 2.17 (a), we can find a countable $w_r$ determining $s$. (I.e., there is a Borel function that calculates the real $s$ from the generics at $w_r$; moreover we know this Borel function in the ground model.) Let $w' \supseteq w$ be in $X_0$ and contain all these $w_r$, for a (small representative set of) all $P_w$-names for reals.

Iterating this construction $\omega_1$ many steps gives us a suitable $w_\alpha$: Note that the assignment of a name $r$ to the $\Xi_\alpha$-value $s$ can be done in $V$, and thus is known to $M_\alpha$. In addition, $M_\alpha$ sees that for each "actual real" (i.e., element of $M_\alpha$), the value $s$ is already determined (by $P'_\alpha$). So the assignment $r \mapsto s$, which is $\Xi_\alpha$ restricted to $M_\alpha$, is in $M_\alpha$. □

Note that in (c), when we deal with a countable $\Delta$-system $\bar{p}$ following the guardrail $h \in H^*$, the condition $\lim_{\Xi^h} \bar{p}$ forces in particular that infinitely many $p_l$ are in $G$. So after carrying out the construction as above, we get a forcing notion $\mathbb{P}^5$ satisfying the following (which is actually the only thing we need from the previous construction, in addition to the fact that we can choose each $w_\alpha$ in an $\omega_1$-club):

**Lemma 2.41.** *For every countable $\Delta$-system $\bar{p}$ there is some $q$ forcing that infinitely many $p_l$ are in the generic filter.*

PROOF: According to Lemma 2.32, $\bar{p}$ follows some $h \in H^*$; so $q = \lim_{\Xi^h}(\bar{p})$ will work. □

**Lemma 2.42.** *The property $\mathsf{LCU}_2(\mathbb{P}^5, \kappa)$ is for $\kappa \in [\lambda_2, \lambda_5]$ regular, witnessed by the sequence $(c_\alpha)_{\alpha<\kappa}$ of the first $\kappa$ many Cohen reals.*

PROOF: Fix a $\mathbb{P}^5$-name $y \in \omega^\omega$. We have to show that $(\exists\, \alpha \in \kappa)\, (\forall\, \beta \in \kappa \setminus \alpha)\, \mathbb{P}^5 \Vdash \neg c_\beta \leq^* y)$.

Assume towards a contradiction that $p^*$ forces that there are unboundedly many $\alpha \in \kappa$ with $c_\alpha \leq^* y$, and enumerate them as $(\alpha_i)_{i\in\kappa}$. Pick $p^i \leq p^*$ deciding $\alpha_i$ to be some $\beta^i$, and also deciding $n_i$ such that $(\forall\, m \geq n_i)\, c_{\alpha_i}(m) \leq y(m)$. We can assume that $\beta^i \in \mathrm{supp}(p^i)$. Note that $\beta^i$ is a Cohen position (as $\beta^i < \kappa \leq \lambda_5$), and we can assume that $p^i(\beta^i)$ is a Cohen condition in $V$ (and not just a $P_{\beta_i}$-name for such a condition). By strengthening and thinning out, we may assume:

- The sequence $(p^i)_{i\in\kappa}$ forms a $\Delta$ system with heart $\nabla$.
- All $n_i$ are equal to some $n^*$.
- The condition $p^i(\beta_i)$ is always the same Cohen condition $s \in \omega^{<\omega}$, without loss of generality of length $|s| = n^{**} \geq n^*$.
- For some position $n < m^{\bar{p}}$, $\beta^i$ is the $n$th element of $\mathrm{supp}(p^i)$.

Note that this $n$ cannot be a heart condition: For any $\beta \in \kappa$, at most $|\beta|$ many $p^i$ can force $\alpha_i = \beta$, as $p^i$ forces that $\alpha_i \geq i$ for all $i$.

Pick a countable subset of this $\Delta$-system which forms a countable $\Delta$-system $\bar{p} := (p_l)_{l\in\omega}$. So $p_l = p^{i_l}$ for some $i_l \in \kappa$, and we set $\beta_l = \beta^{i_l}$. In particular all $\beta_l$ are distinct. Now extend each $p_l$ to $p'_l$ by extending the Cohen condition $p_l(\beta_l) = s$ to $s^\frown l$ (i.e., forcing $c_{\beta_l}(n^{**}) = l$). Note that $\bar{p}' := (p'_i)_{i\in\omega}$ is still a countable $\Delta$-system[13], and by Lemma 2.41 some $q$ forces that infinitely many of the $p'_l$ are in the generic filter. But each such $p'_l$ forces that $c_{\beta_l}(n^{**}) = l \leq y(n^{**})$, a contradiction. □

**2.5 The left hand side.** We have now finished the consistency proof for the left hand side:

**Theorem 2.43.** *Assume GCH and let $\lambda_i$ be an increasing sequence of regular cardinals, none of which is a successor of a cardinal of countable cofinality for $i = 1, \ldots, 5$. Then there is a cofinalities-preserving forcing $P$ resulting in*

$$\mathrm{add}(\mathcal{N}) = \lambda_1 < \mathrm{add}(\mathcal{M}) = \mathfrak{b} = \lambda_2 < \mathrm{cov}(\mathcal{N}) = \lambda_3$$
$$< \mathrm{non}(\mathcal{M}) = \lambda_4 < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0} = \lambda_5.$$

PROOF: Set $\chi = \lambda_2$, and let $R$ be the set of partial functions $f\colon \chi \times \lambda_5 \to 2$ with $|\,\mathrm{dom}(f)| < \chi$ (ordered by inclusion). The set $R$ is $<\chi$-closed, $\chi^+$-cc, and adds $\lambda_5$ many new elements to $2^\chi$. So in the $R$-extension, Assumption 2.29 is satisfied,

[13] Note that $\bar{p}'$ will not follow the same guardrail as $\bar{p}$.

and we can construct $\mathbb{P}^5$ according to Assumption 2.13 and Construction 2.39. Fact 2.24 gives us all inequalities for the left hand side, apart from $\mathfrak{b} \le \lambda_2$, which we get from 2.42.

In the $R$-extension, CH holds and $P$ is a FS ccc iteration of length $\delta_5$, $|\delta_5| = \lambda_5$, and each iterated is a set of reals; so $2^{\aleph_0} \le \lambda_5$ is forced. Also, any FS ccc iteration of length $\delta$ (of nontrivial iterands) forces $\mathrm{cov}(\mathcal{M}) \ge \mathrm{cf}(\delta)$: Without loss of generality $\mathrm{cf}(\delta) = \lambda$ is uncountable. Any set $A$ of (Borel codes for) meager sets that has size less than $\lambda$ already appears at some stage $\alpha < \delta$, and the iteration at state $\alpha + \omega$ adds a Cohen real over the $V_\alpha$, so $A$ will not cover all reals. □

**Remark 2.44.** So this consistency result is reasonably general, we can, e.g., use the values $\lambda_i = \aleph_{i+1}$. This is in contrast to the result for the whole diagram, where in particular the small $\lambda_i$ have to be separated by strongly compact cardinals.

## 3. Ten different values in Cichoń's diagram

We can now apply, with hardly any change, the technique of [7] to get the following:

**Theorem 3.1.** *Assume GCH and that $\aleph_1 < \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 < \kappa_6 < \lambda_4 < \lambda_5 < \lambda_6 < \lambda_7 < \lambda_8 < \lambda_9$ are regular, $\lambda_i$ is not a successor of a cardinal of countable cofinality for $i = 1, \dots, 5$, $\lambda_2 = \chi^+$, with $\chi$ regular, and $\kappa_i$ strongly compact for $i = 6, 7, 8, 9$. Then there is a ccc forcing notion $\mathbb{P}^9$ resulting in:*

$$\begin{aligned}\mathrm{add}(\mathcal{N}) = \lambda_1 < \mathfrak{b} = \mathrm{add}(\mathcal{M}) = \lambda_2 < \mathrm{cov}(\mathcal{N}) = \lambda_3 < \mathrm{non}(\mathcal{M}) = \lambda_4 < \mathrm{cov}(\mathcal{M})\\ = \lambda_5 < \mathrm{non}(\mathcal{N}) = \lambda_6 < \mathfrak{d} = \mathrm{cof}(\mathcal{M}) = \lambda_7 < \mathrm{cof}(\mathcal{N}) = \lambda_8 < 2^{\aleph_0} = \lambda_9.\end{aligned}$$

To do this, we first have to show that we can achieve the order for the left hand side, i.e., Theorem 2.43, starting with GCH and using a FS ccc iteration $\mathbb{P}^5$ alone (instead of using $P = R * \mathbb{P}^5$, where $R$ is not ccc). This is the only argument that requires $\lambda_2 = \chi^+$. We will just briefly sketch it here, as it can be found with all details in [7, 1.4]:

- We already know that in the $R$-extension, (where $R$ is $<\chi$-closed, $\chi^+$-cc and forces $2^\chi = \lambda_5$) we can find by the inductive Construction 2.39 suitable $w_\alpha$ such that $R * \mathbb{P}^5$ works.
- We now perform a similar inductive construction in the ground model: At stage $\alpha$, we know that there is an $R$-name for a suitable $w^1_\alpha$ of size less than $\lambda_i$ (where $i$ is 3 in the random and 4 in the $\widetilde{\mathbb{E}}$-case). This name can be covered by some set $\widetilde{w}^1_\alpha$ in $V$, still of size less than $\lambda_i$, as $R$ is $\chi^+$-cc. Moreover, in the $R$-extension, the suitable parameters form an $\omega_1$-club; so there is a suitable $w^2_\alpha \supseteq \widetilde{w}^1_\alpha$, etc. Iterating $\omega_1$ many times and taking the union at the end leads to $w_\alpha$ in $V$ which is forced by $R$ to be suitable.
- Not only $w_\alpha$ is in $V$, but the construction for $w_\alpha$ is performed in $V$, so we can construct the whole sequence $\overline{w} = (w_\alpha)_{\alpha \in \delta_5}$ in $V$.

- We now know that in the $R$-extension, the forcing $\mathbb{P}^5$ defined from $\overline{w}$ will satisfy $\mathsf{LCU}_2(\mathbb{P}^5, \kappa)$ in the form of Lemma 2.42.
- By an absoluteness argument, we can show that actually in $V$ the forcing $\mathbb{P}^5$ defined form $\overline{w}$ will satisfy Lemma 2.42 as well.

The rest of the proof is the same as in [7, Section 2], where we interchange $\mathfrak{b}$ and $\mathrm{cov}(\mathcal{N})$ as well as $\mathfrak{d}$ and $\mathrm{non}(\mathcal{N})$.

We cite the following facts from [7, 2.2–2.5]:

**Facts 3.2.** (a) If $\kappa$ is a strongly compact cardinal and $\theta > \kappa$ regular, then there is an elementary embedding $j_{\kappa,\theta}\colon V \to M$ (in the following just called $j$) such that
- the critical point of $j$ is $\kappa$, $\mathrm{cf}(j(\kappa)) = |j(\kappa)| = \theta$,
- $\max(\theta, \lambda) \leq j(\lambda) < \max(\theta, \lambda)^+$ for all $\lambda \geq \kappa$ regular, and
- $\mathrm{cf}(j(\lambda)) = \lambda$ for $\lambda \neq \kappa$ regular,

and such that the following is satisfied:

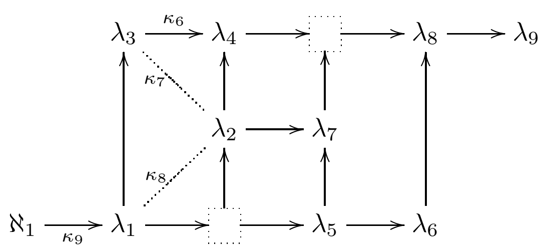
(b) If $P$ is a FS ccc iteration along $\delta$, then $j(P)$ is a FS ccc iteration along $j(\delta)$.

(c) The $\mathsf{LCU}_i(P, \lambda)$ property implies the $\mathsf{LCU}_i(j(P), \mathrm{cf}(j(\lambda)))$ property, and thus $\mathsf{LCU}_i(j(P), \lambda)$ if $\lambda \neq \kappa$ regular.[14]

(d) If $\mathsf{COB}_i(P, \lambda, \mu)$, then $\mathsf{COB}_i(j(P), \lambda, \mu')$ for $\mu' = \begin{cases} |j(\mu)| & \text{if } \kappa > \lambda, \\ \mu & \text{if } \kappa < \lambda. \end{cases}$

Using these facts, it is easy to finish the proof[15]:

Proof of Theorem 3.1: Recall that we want to force the following values to the characteristics of Figure 2 (where we indicate the positions of the $\kappa_i$ as well):



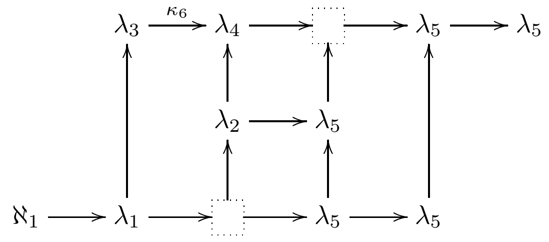

Step 5: Our first step, called "Step 5" for notational reasons, just uses $\mathbb{P}^5$. This is an iteration of length $\delta_5$ with $\mathrm{cf}(\delta_5) = |\delta_5| = \lambda_5$, satisfying:

(3.3) For all $i$: $\mathsf{LCU}_i(\mathbb{P}^5, \mu)$ for all $\mu \in [\lambda_i, \lambda_5]$ regular, and $\mathsf{COB}_i(\mathbb{P}^5, \lambda_i, \lambda_5)$.

[14] In [7], we only used "classical" relations $\mathrm{R}_3$ that are defined on a Polish space in an absolute way. In this paper, we use the relation $\mathrm{R}_3$ which is not of this kind. However, the proof still works without any change: The parameter $\mathcal{E}$ used to define the relation $\mathrm{R}_3$, cf. Definition 2.2, is a set of reals. So $j(\mathcal{E}) = \mathcal{E}$, and we can still use the usual absoluteness arguments between $M$ and $V$. (A parameter not element of $H(\kappa_9)$ might be a problem.)

[15] This is identical to the argument in [7], with the roles of $\mathfrak{b}$ and $\mathrm{cov}(\mathcal{N})$, as well as their duals, switched.

As a consequence, the characteristics are forced by $\mathbb{P}^5$ to have the following values[16] (we also mark the position of $\kappa_6$, which we are going to use in the following step):




Step 6: Consider the embedding $j_6 := j_{\kappa_6,\lambda_6}$. According to Fact 3.2 (b), $\mathbb{P}^6 := j_6(\mathbb{P}^5)$ is a FS ccc iteration of length $\delta_6 := j_6(\delta_5)$. As $|\delta_6| = \lambda_6$, the continuum is forced to have size $\lambda_6$.

For $i = 1$, we have $\mathsf{LCU}_1(\mathbb{P}^5, \mu)$ for all regular $\mu \in [\lambda_1, \lambda_5]$, so using Fact 3.2 (c) we get $\mathsf{LCU}_1(\mathbb{P}^6, \mu)$ for all regular size $\mu \in [\lambda_1, \lambda_5]$ different to $\kappa_6$; as well as $\mathsf{LCU}_1(\mathbb{P}^6, \lambda_6)$ (as $\mathrm{cf}(j(\kappa_6)) = \lambda_6$). For $\mu = \lambda_1$ the former implies for the iteration $\mathbb{P}^6 \Vdash \mathrm{add}(\mathcal{N}) \leq \lambda_1$, and the latter $\mathbb{P}^6 \Vdash \mathrm{cof}(\mathcal{N}) \geq \lambda_6 = 2^{\aleph_0}$.

More generally, we get from (3.3) and Fact 3.2 (c):

$$\text{(3.4)}\qquad \begin{aligned} &\text{For all } i\colon\ \mathsf{LCU}_i(\mathbb{P}^6, \mu) \text{ for all regular } \mu \in [\lambda_i, \lambda_5] \setminus \{\kappa_6\}.\\ &\text{For } i < 4\colon\ \mathsf{LCU}_i(\mathbb{P}^6, \lambda_6). \end{aligned}$$

So in particular for $\mu = \lambda_i$, we see that the characteristics on the left do not increase; for $\mu = \lambda_5$ that the ones on the right are still at least $\lambda_5$; and for $i < 4$ and $\mu = \lambda_6$ that the according characteristics on the right will have size continuum. (But not for $i = 4$, as $\kappa_4 < \lambda_4$. And we will see that $\mathrm{cov}(\mathcal{M})$ is at most $\lambda_5$.)

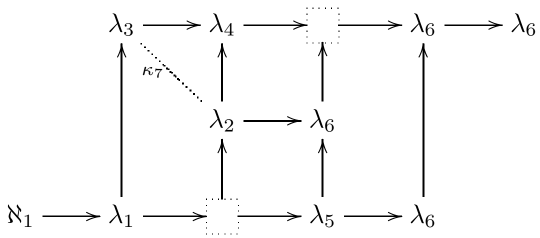
Dually, because $\lambda_3 < \kappa_6 < \lambda_4$, we get from (3.3) and Fact 3.2 (d):

$$\text{(3.5)}\qquad \begin{aligned} &\text{For } i < 4\colon\ \mathsf{COB}_i(\mathbb{P}^6, \lambda_i, \lambda_6).\\ &\text{For } i = 4\colon\ \mathsf{COB}_4(\mathbb{P}^6, \lambda_4, \lambda_5). \end{aligned}$$

(The former because $|j_6(\lambda_5)| = \max(\lambda_6, \lambda_5) = \lambda_6$.) So the characteristics on the left do not decrease, and $\mathbb{P}^6 \Vdash \mathrm{cov}(\mathcal{M}) \leq \lambda_5$.

[16] These values, and the ones forced by the "intermediate forcings" $\mathbb{P}^6$ to $\mathbb{P}^8$, are not required for the argument; they should just illustrate what is going on.

Accordingly, $\mathbb{P}^6$ forces the following values:




Step 7: We now apply a new embedding, $j_7 := j_{\kappa_7,\lambda_7}$, to the forcing $\mathbb{P}^6$ that we just constructed. (We always work in $V$, not in any inner model $M$ or any forcing extention.) As before, set $\mathbb{P}^7 := j_7(\mathbb{P}^6)$, a FS ccc iteration of length $\delta_7 = j_7(\delta_6)$, forcing the continuum to have size $\lambda_7$.

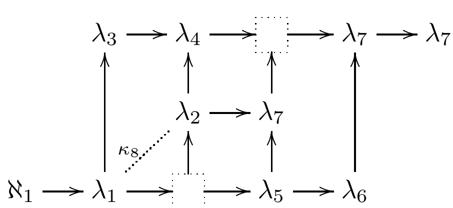
Now $\kappa_7 \in (\lambda_2, \lambda_3)$, so arguing as before, we get from (3.4):

(3.6)
For all $i$: $\mathsf{LCU}_i(\mathbb{P}^7, \mu)$ for all regular $\mu \in [\lambda_i, \lambda_5] \setminus \{\kappa_6, \kappa_7\}$.
For $i < 4$: $\mathsf{LCU}_i(\mathbb{P}^7, \lambda_6)$.
For $i < 3$: $\mathsf{LCU}_i(\mathbb{P}^7, \lambda_7)$.

And from (3.5):

(3.7)
For $i < 3$: $\mathsf{COB}_i(\mathbb{P}^7, \lambda_i, \lambda_7)$.
For $i = 3$: $\mathsf{COB}_3(\mathbb{P}^7, \lambda_3, \lambda_6)$.
For $i = 4$: $\mathsf{COB}_4(\mathbb{P}^7, \lambda_4, \lambda_5)$.

Accordingly, $\mathbb{P}^7$ forces the following values:



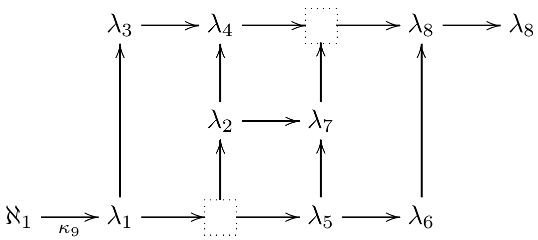

Step 8: Now we set $\mathbb{P}^8 := j_{\kappa_8,\lambda_8}(\mathbb{P}^7)$, a FS ccc iteration of length $\delta_8$. Now $\kappa_8 \in (\lambda_1, \lambda_2)$, and as before, we get from (3.6):

(3.8)
For all $i$: $\mathsf{LCU}_i(\mathbb{P}^8, \mu)$ for all regular $\mu \in [\lambda_i, \lambda_5] \setminus \{\kappa_6, \kappa_7, \kappa_8\}$.
For $i < 4$: $\mathsf{LCU}_i(\mathbb{P}^8, \lambda_6)$.
For $i < 3$: $\mathsf{LCU}_i(\mathbb{P}^8, \lambda_7)$.
For $i < 2$ (i.e., $i = 1$): $\mathsf{LCU}_1(\mathbb{P}^8, \lambda_8)$.

And from (3.7):

$$
\text{(3.9)}\qquad
\begin{aligned}
&\text{For } i=1\colon\ \mathsf{COB}_1(\mathbb{P}^8,\lambda_1,\lambda_8).\\
&\text{For } i=2\colon\ \mathsf{COB}_2(\mathbb{P}^8,\lambda_2,\lambda_7).\\
&\text{For } i=3\colon\ \mathsf{COB}_3(\mathbb{P}^8,\lambda_3,\lambda_6).\\
&\text{For } i=4\colon\ \mathsf{COB}_4(\mathbb{P}^8,\lambda_4,\lambda_5).
\end{aligned}
$$

Accordingly, $\mathbb{P}^8$ forces the following values:




Step 9: Finally we set $\mathbb{P}^9 := j_{\kappa_9,\lambda_9}(\mathbb{P}^8)$, a FS ccc iteration of length $\delta_9$ with $|\delta_9| = \lambda_9$, i.e., the continuum will have size $\lambda_9$. As $\kappa_9 < \lambda_1$, (3.8) and (3.9) also hold for $\mathbb{P}^9$ instead of $\mathbb{P}^8$. Accordingly, we get the same values for the diagram as for $\mathbb{P}^8$, apart from the value for the continuum, $\lambda_9$. □

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J. Kellner:
Technische Universität Wien,
Institute of Discrete Mathematics and Geometry,
Wiedner Hauptstrasse 8-10/104, 1040 Wien, Austria

*E-mail:* jakob.kellner@tuwien.ac.at

S. Shelah:
The Hebrew University of Jerusalem, Einstein Institute of Mathematics,
Edmond J. Safra Campus, Givat Ram. Jerusalem, 9190401, Israel
and
Rutgers University, Department of Mathematics,
Hill Center - Busch Campus, 110 Frelinghuysen Road, Piscataway,
New Jersey, NJ 08854-8019, U.S.A.

*E-mail:* shlhetal@mat.huji.ac.il

A. R. Tănasie:
Technische Universität Wien,
Institute of Discrete Mathematics and Geometry,
Wiedner Hauptstrasse 8-10/104, 1040 Wien, Austria

*E-mail:* anda-ramona.tanasie@tuwien.ac.at

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