Generalized notions of amenability for a class of matrix algebras

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Abstract. We investigate the amenability and its related homological notions for a class of $I \times I$-upper triangular matrix algebra, say $\mathrm{UP}(I, A)$, where A is a Banach algebra equipped with a nonzero character. We show that $\mathrm{UP}(I, A)$ is pseudo-contractible (amenable) if and only if I is singleton and A is pseudo-contractible (amenable), respectively. We also study pseudo-amenability and approximate biprojectivity of $\mathrm{UP}(I, A)$.

Keywords: upper triangular Banach algebra; amenability; left φ-amenability; approximate biprojectivity

Classification: 46M10, 43A07, 43A20

1. Introduction and preliminaries

B. E. Johnson studied the class of amenable Banach algebras. Indeed a Banach algebra A is amenable if every continuous derivation $D\colon A \to X^*$ is inner for every Banach A-bimodule X, that is, there exists $x_0 \in X^*$ such that

$$D(a) = a \cdot x_0 - x_0 \cdot a, \qquad a \in A.$$

B. E. Johnson also showed that A is amenable if and only if there exists a bounded net (m_α) in $A \otimes_p A$ such that

$$a \cdot m_\alpha - m_\alpha \cdot a \to 0, \quad \pi_A(m_\alpha)a \to a, \qquad a \in A,$$

where $\pi_A\colon A \otimes_p A \to A$ is given by $\pi_A(a \otimes b) = ab$ for every $a, b \in A$. About the same time A. Y. Helemskii defined the homological notions of biflatness and biprojectivity for Banach algebras. In fact a Banach algebra A is called biflat (biprojective), if there exists a bounded A-bimodule morphism $\varrho\colon A \to (A \otimes_p A)^{**}$ ($\varrho\colon A \to A \otimes_p A$) such that $\pi_A^{**} \circ \varrho$ is the canonical embedding of A into A^{**} (ϱ is a right inverse for π_A), respectively. Note that a Banach algebra A is amenable if and only if A is biflat and A has a bounded approximate identity. It is known that for a locally compact group G, $L^1(G)$ is biflat (biprojective) if and only if G is amenable (compact), respectively. For more information about amenability and homological properties of Banach algebras, see [17].

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Upper triangular Banach algebras are 2×2-matrix algebras. B. E. Forrest and L. W. Marcoux studied this class of Banach algebras in [8]. Also they investigated some notions of amenability and homological properties of triangular Banach algebras, see [9]. The l^1-Munn algebras are another matrix algebra. G. H. Esslamzadeh studied amenability and some homological properties of these matrix algebras, for more information see [7].

In this paper, we investigate amenability and its related homological notions for a class of matrix algebras which is a generalization for 2×2-upper triangular Banach algebras. We show that for a Banach algebra A with a nonzero character, $I\times I$-upper triangular Banach algebra $\mathrm{UP}(I,A)$ is amenable (pseudo-contractible) if and only if I is singleton and A is amenable (pseudo-contractible), respectively. Also we characterize whether $\mathrm{UP}(I,A)$ is approximate amenable, pseudo-amenable and approximate biprojective. The paper concludes by studying amenability and approximate biprojectivity of some semigroup algebras related to a matrix algebra.

We remark some standard notations and definitions that we shall need in this paper. Let A be a Banach algebra. Throughout this paper the character space of A is denoted by $\Delta(A)$, that is, all nonzero multiplicative linear functionals on A. The projective tensor product $A\otimes_p A$ is a Banach A-bimodule via the following actions

$$a\cdot(b\otimes c)=ab\otimes c,\quad (b\otimes c)\cdot a=b\otimes ca,\qquad a,b,c\in A.$$

Let A be a Banach algebra and I be a nonempty totally ordered set. Let $\mathrm{UP}(I,A)$ be denoted for the set of all $I\times I$ upper triangular matrices which entries come from A and

$$\|(a_{i,j})_{i,j\in I}\|=\sum_{i,j\in I}\|a_{i,j}\|<\infty.$$

With the usual matrix operations and $\|\cdot\|$ as a norm, $\mathrm{UP}(I,A)$ becomes a Banach algebra.

2. A class of matrix algebras and generalized notions of amenability

In this section, we study generalized notions of amenability for upper triangular Banach algebras.

We remind that a Banach algebra A with $\varphi\in\Delta(A)$ is called left (right) φ-contractible, if there exists $m\in A$ such that $am=\varphi(a)m$ ($ma=\varphi(a)m$) and $\varphi(m)=1$ for every $a\in A$, respectively. For more information the reader is referred to [16].

A Banach algebra A is called pseudo-amenable (pseudo-contractible) if there exists a not necessarily bounded net (m_α) in $A \otimes_p A$ such that

$$a \cdot m_\alpha - m_\alpha \cdot a \to 0, \quad (a \cdot m_\alpha = m_\alpha \cdot a), \quad \pi_A(m_\alpha)a \to a, \qquad a \in A.$$

For more information about these new concepts the reader is referred to [12] and [3].

Theorem 2.1. *Let I be a nonempty totally ordered set and A be a unital Banach algebra with $\Delta(A) \neq \emptyset$. Then $\mathrm{UP}(I, A)$ is pseudo-contractible if and only if I is singleton and A is pseudo-contractible.*

Proof: We will prove this theorem in two steps:
Step 1: We show that if $\mathrm{UP}(I, A)$ is pseudo-contractible, then I must be finite.

Let $\mathrm{UP}(I, A)$ be pseudo-contractible. Then $\mathrm{UP}(I, A)$ has a central approximate identity, say (e_α). Put $F_{i,j}$ for a matrix belongs to $\mathrm{UP}(I, A)$ whose (i, j)th entrying is e_A and others are zero, where e_A is the identity of A. Thus $F_{i,j}e_\alpha = e_\alpha F_{i,j}$ for every $i, j \in I$. This equation implies that the entries on main diagonal of e_α is equal. We go towards a contradiction and suppose that I is infinite. Since the entries on main diagonal of e_α are equal, it implies that $\|e_\alpha\| = \infty$ or the main diagonal of e_α is zero. In the case $\|e_\alpha\| = \infty$, e_α does not belong to $\mathrm{UP}(I, A)$ which is impossible. Otherwise if the main diagonal of e_α is zero, then $e_\alpha F_{i,i} = 0$. Thus $0 = e_\alpha F_{i,i} \to F_{i,i}$ which is impossible, hence I must be finite.
Step 2: In this step, we show that if I is a finite subset and $\mathrm{UP}(I, A)$ is pseudo-contractible, then I is singleton and A is pseudo-contractible.

To see this, suppose that $I = \{i_1, i_2, \ldots, i_n\}$ and $\varphi \in \Delta(A)$. Define $\psi \in \Delta(\mathrm{UP}(I, A))$ by $\psi((a_{i,j})_{i,j\in I}) = \varphi(a_{i_n,i_n})$ for every $(a_{i,j}) \in \mathrm{UP}(I, A)$. Since $\mathrm{UP}(I, A)$ is pseudo-contractible, by [2, Theorem 1.1] $\mathrm{UP}(I, A)$ is left and right ψ-contractible. Set

$$J = \{(a_{i,j}) \in \mathrm{UP}(I, A) \colon a_{i,j} = 0 \text{ for all } j \neq i_n\}.$$

It is clear that J is a closed ideal of $\mathrm{UP}(I, A)$ and $\psi|_J \neq 0$, hence by [16, Proposition 3.8] J is left and right ψ-contractible. So there exist $m_1, m_2 \in J$ such that $jm_1 = \psi(j)m_1$ and $m_2 j = \psi(j)m_2$ and also $\psi(m_1) = \psi(m_2) = 1$ for each $j \in J$. Set $m = m_1 m_2 \in J$. Clearly we have

$$jm = mj = \psi(j)m, \quad \psi(m) = \psi(m_1 m_2) = \psi(m_1)\psi(m_2) = 1, \qquad j \in J. \tag{2.1}$$

Proceeding by contradiction, suppose that $|I| > 1$. Since $m \in J$, there exist $x_1, x_2, \ldots, x_n \in A$ such that $m = \begin{pmatrix} 0 & \cdots & x_1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_{n-1} \\ 0 & \cdots & x_n \end{pmatrix}$. Let $a = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & a_n \end{pmatrix}$ where a_n is an arbitrary element of A. Applying (2.1) we have

$$x_1 a_n = x_2 a_n = \cdots = x_{n-1} a_n = 0, \quad \varphi(a_n)x_1 = \varphi(a_n)x_2 = \cdots = \varphi(a_n)x_{n-1} = 0,$$

and also

$$a_n x_n = x_n a_n = \varphi(a_n)x_n, \qquad \varphi(x_n) = 1.$$

Pick an element $a_n \in A$ such that $\varphi(a_n) = 1$. Applying (2.1) it follows that $x_1 = x_2 = \cdots = x_{n-1} = 0$. Then $m = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & x_n \end{pmatrix}$. Put $b = \begin{pmatrix} 0 & \cdots & b_1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_{n-1} \\ 0 & \cdots & b_n \end{pmatrix}$, where $b_2 = \cdots = b_n = 0$ and $\varphi(b_1) = 1$. By (2.1) we have $b_1 x_n = 0$. Applying φ on this equation, we have $0 = \varphi(b_1 x_n) = \varphi(b_1)\varphi(x_n) = 1$ which is a contradiction. Therefore I must be singleton. So A is pseudo-contractible.

Converse is clear. $\square$

A Banach algebra A is said to be approximately amenable, if for every continuous derivation $D\colon A \to X^*$, there exists a net (x_α) in X^* such that

$$D(a) = \lim_\alpha (a \cdot x_\alpha - x_\alpha \cdot a), \qquad a \in A,$$

see [10] and [11].

Suppose that A is a Banach algebra and $\varphi \in \Delta(A)$. Then A is called (approximately) left φ-amenable if there exists (a not necessarily) bounded net (m_α) in A such that

$$am_\alpha - \varphi(a)m_\alpha \to 0, \quad \varphi(m_\alpha) \to 1, \qquad a \in A,$$

respectively. Right case is defined similarly. For more information about these concepts of amenability and its related homological notions see [1], [15], [13] and [20].

Theorem 2.2. *Let I be a nonempty totally ordered set with a smallest element. Also let A be a Banach algebra with a left unit such that $\Delta(A) \neq \emptyset$. Then $\mathrm{UP}(I, A)$ is pseudo-amenable (approximately amenable) if and only if I is singleton and A is pseudo-amenable (approximately amenable), respectively.*

Proof: Here we proof the pseudo-amenable case, the approximate amenability is similar. Suppose that $\mathrm{UP}(I, A)$ is pseudo-amenable. Then there exists a net (m_α) in $\mathrm{UP}(I, A) \otimes_p \mathrm{UP}(I, A)$ such that

$$a \cdot m_\alpha - m_\alpha \cdot a \to 0, \quad \pi_{\mathrm{UP}(I,A)}(m_\alpha)a \to a, \qquad a \in \mathrm{UP}(I, A).$$

Let i_0 be a smallest element of I. It is easy to see that the map ψ, given by $\psi(a) = \varphi(a_{i_0,i_0})$ is a character on $\mathrm{UP}(I, A)$ for each $a = (a_{i,j}) \in \mathrm{UP}(I, A)$. Define

$$T\colon \mathrm{UP}(I, A) \otimes_p \mathrm{UP}(I, A) \to \mathrm{UP}(I, A)$$

by $T(a \otimes b) = \psi(a)b$ for each $a, b \in \mathrm{UP}(I, A)$. It is easy to see that T is a bounded linear map which satisfies the following:

$$T(a \cdot x) = \psi(a)T(x), \quad T(x \cdot a) = T(x)a, \quad \psi \circ T(x) = \psi \circ \pi_{\mathrm{UP}(I,A)}(x)$$

for each $a,b \in \mathrm{UP}(I,A)$ and $x \in \mathrm{UP}(I,A) \otimes_p \mathrm{UP}(I,A)$. Thus we have

$$\psi(a)T(m_\alpha) - T(m_\alpha)a = T(a \cdot m_\alpha - m_\alpha \cdot a) \to 0$$

and $\psi \circ T(m_\alpha) = \psi \circ \pi_{\mathrm{UP}(I,A)}(m_\alpha) \to 1$. Hence $\mathrm{UP}(I,A)$ is approximately right ψ-amenable. Define

$$J = \{(a_{i,j})_{i,j\in I} \in \mathrm{UP}(I,A) \colon a_{i,j} = 0,\ i \neq i_0\}.$$

It is easy to see that J is a closed ideal of $\mathrm{UP}(I,A)$ and $\psi|_J \neq 0$. Then by [19, Proposition 5.1], J is approximately right $\psi-$amenable. Now, if we proceed similar to the arguments as in the proof of [19, Theorem 5.1], we can see that $|I| = 1$. Therefore A is pseudo-amenable (approximately amenable), respectively.

Converse is clear. □

Let A be a Banach algebra and $a \in A$. By $a\varepsilon_{i,j}$ we mean a matrix belonging to $\mathrm{UP}(I,A)$ with (i,j)th entry a and zero elsewhere.

Theorem 2.3. *Let I be a nonempty totally ordered set and let A be a Banach algebra such that $\Delta(A) \neq \emptyset$. Then $\mathrm{UP}(I,A)$ is amenable if and only if I is singleton and A is amenable.*

Proof: Let $\mathrm{UP}(I,A)$ be amenable. Then $\mathrm{UP}(I,A)$ has a bounded approximate identity, say (E^α). Let $M > 0$ be a bound for (E^α). We claim that A has a bounded left approximate identity. To see this, fix $k,l \in I$. Then for each $a \in A$, we have

$$\begin{aligned} 0 &= \lim_\alpha \|E^\alpha a\varepsilon_{k,l} - a\varepsilon_{k,l}\| = \lim_\alpha \left\| \Big(\sum_{i,j} E^\alpha_{i,j}\varepsilon_{i,j}\Big) a\varepsilon_{k,l} - a\varepsilon_{k,l}\right\| \\ &= \lim_\alpha \left\| \sum_i E^\alpha_{i,l} a\varepsilon_{i,l} - a\varepsilon_{k,l}\right\| = \lim_\alpha \left(\left\| \sum_{i\neq k} E^\alpha_{i,l}a \right\| + \|E^\alpha_{k,l}a - a\|\right). \end{aligned} \tag{2.2}$$

Thus $e_\alpha = E^\alpha_{k,l}$ is a left approximate identity of A. It is easy to see that $\|e_\alpha\| \leq \|E^\alpha\| \leq M$. So (e_α) is a bounded left approximate identity for A. We claim that I is finite. Suppose conversely that I is infinite. Pick $a \in A$ such that $\|a\| = 1$. Since (e_α) is a bounded left approximate identity for A, then $\lim_\alpha e_\alpha a = a$ for each $a \in A$. Thus there exists an $\alpha_{l,k}$ such that $\alpha \geq \alpha_{k,l}$ with $1/2 < \|e_\alpha a\|$. Hence for $\alpha \geq \alpha_{k,l}$ we have

$$\frac{1}{2} < \|e_\alpha a\| \leq \|e_\alpha\| = \|E^\alpha_{k,l}\|. \tag{2.3}$$

Since I is infinite we can choose $N \in \mathbb{N}$ such that $N > 2M$. Then choose distinct $k_1, l_1, k_2, l_2, \ldots, k_N, l_N$ in I and $\alpha \geq \alpha_{k_i,l_i}$, $i = 1,2,\ldots,N$. Using (2.3) one can see that

$$M < \frac{1}{2}N = \sum_{i=1}^{N} \|E^{\alpha}_{k_i,l_i}\| \leq \sum_{i,j\in I} \|E^{\alpha}_{i,j}\| \leq M,$$

which is a contradiction. So I is finite.

Applying the same method as in the proof of previous theorem, it is easy to see that I must be singleton, then A is amenable. □

3. A class of matrix algebra and approximate biprojectivity

Recently approximate versions of homological notions of Banach algebras have been under more observations, see [21]. In fact a Banach algebra A is said to be approximately biprojective, if there exists a net of A-bimodule morphisms $\varrho_\alpha\colon A \to A\otimes_p A$ such that

$$\pi_A \circ \varrho_\alpha(a) \to a, \qquad a \in A.$$

Note that A is a pseudo-contractible Banach algebra if and only if A is approximately biprojective and has a central approximate identity.

In this section we study the approximate biprojectivity of some matrix algebras. We also investigate the relation of approximate biprojectivity and discreteness of maximal ideal space of a Banach algebra.

Theorem 3.1. *Let I be a totally ordered set with a smallest element. Also let A be a Banach algebra with a right identity such that $\Delta(A)\neq\emptyset$. Then* $\mathrm{UP}(I,A)$ *is approximately biprojective if and only if I is singleton and A is approximately biprojective.*

Proof: Let i_0 be smallest element of I. Define $\psi\in\Delta(\mathrm{UP}(I,A))$ by $\psi(a)=\varphi(a_{i_0,i_0})$, where $a=(a_{i,j})\in\mathrm{UP}(I,A)$. Suppose that $\mathrm{UP}(I,A)$ is approximately biprojective. Since A has a right identity, by [19, Lemma 5.1], $\mathrm{UP}(I,A)$ has a right approximate identity. Applying [18, Theorem 3.9], $\mathrm{UP}(I,A)$ is right ψ-contractible. Using the same arguments as in the proof of the Theorem 2.2, I is singleton and A is approximately biprojective.

Converse is clear. □

Remark 3.2. Let A be a Banach algebra with a left approximate identity and I be a finite set which has at least two elements. Then $\mathrm{UP}(I,A)$ is never approximately biprojective. To see this, since $I=\{i_1,i_2,\ldots,i_n\}$ is finite then left approximate identity of A gives a left approximate identity for $\mathrm{UP}(I,A)$. Define $\psi\in\Delta(\mathrm{UP}(I,A))$ by $\psi(a)=\varphi(a_{i_n,i_n})$ for every $a=(a_{i,j})\in\mathrm{UP}(I,A)$. By [18, Theorem 3.9] approximate biprojectivity of $\mathrm{UP}(I,A)$ implies that $\mathrm{UP}(I,A)$ is left ψ-contractible, then the rest is similar to the proof of Theorem 2.2.

Proposition 3.3. *Let A be a Banach algebra with a left approximate identity and $\Delta(A)$ be a nonempty set. If A is approximately biprojective, then $\Delta(A)$ is discrete with respect to the w^*-topology.*

PROOF: Since A is an approximately biprojective Banach algebra with a left approximate identity, by [18, Theorem 3.9] A is left φ-contractible for every $\varphi \in \Delta(A)$. Applying [4, Corollary 2.2] one can see that $\Delta(A)$ is discrete. □

Corollary 3.4. *Let A be a Banach algebra with a left identity, $\varphi \in \Delta(A)$ and let I be a totally ordered set. If $\mathrm{UP}(I, A)$ is approximate biprojective, then $\Delta(\mathrm{UP}(I, A))$ is discrete with respect to the w^*-topology.*

PROOF: Note that, since $\varphi \in \Delta(A)$, $\Delta(\mathrm{UP}(I, A))$ is a nonempty set. The existence of left identity for A implies that $\mathrm{UP}(I, A)$ has a left approximate identity. Applying previous proposition one can see that $\Delta(\mathrm{UP}(I, A))$ is discrete with respect to the w^*-topology. □

Let A be a Banach algebra and $\varphi \in \Delta(A)$, A is φ-inner amenable if there exists a bounded net (a_α) in A such that

$$aa_\alpha - a_\alpha a \to 0, \quad \varphi(a_\alpha) \to 1, \qquad a \in A.$$

For more information about φ-inner amenability, see [14].

Lemma 3.5. *Let A be a Banach algebra and $\varphi \in \Delta(A)$. Suppose that A has an approximate identity. Then approximate biprojectivity of A implies that A is φ-inner amenable.*

PROOF: Suppose that A is approximately biprojective. Using [18, Theorem 3.9], the existence of approximate identity implies that A is left and right φ-contractible. Then there exist m_1 and m_2 in A such that

$$am_1 = \varphi(a)m_1 (m_2 a = \varphi(a)m_2), \quad \varphi(m_1) = \varphi(m_2) = 1, \qquad a \in A,$$

respectively. Since

$$m_1 = \varphi(m_2)m_1 = m_2 m_1 = \varphi(m_1)m_2 = m_2,$$

one can see that

$$am_1 = m_1 a = \varphi(a)m_1, \quad \varphi(m_1) = 1, \qquad a \in A.$$

It follows that A is φ-inner amenable. □

Remark 3.6. We can not drop the assumption of the existence of an approximate identity in Lemma 3.5. In fact, there exists a matrix algebra which is approximately biprojective but it is not φ-inner amenable.

To see this, let $A = \begin{pmatrix} 0 & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$ and also let $a_0 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Define $\varrho \colon A \to$

$A \otimes_p A$ by $\varrho(a) = a \otimes a_0$ for every $a \in A$. It is easy to see that ϱ is a bounded A-bimodule morphism and

$$\pi_A \circ \varrho(a) = a, \qquad a \in A.$$

Then A is biprojective and it follows that A is approximately biprojective. Set $\varphi\left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}\right) = b$ for every $a, b \in \mathbb{C}$. It is easy to see that $\varphi \in \Delta(A)$. We claim that A is not φ-inner amenable. Suppose, for contradiction, that A is φ-inner amenable. Then there exists a bounded net (a_α) in A such that

$$aa_\alpha - a_\alpha a \to 0, \quad \varphi(a_\alpha) \to 1, \qquad a \in A.$$

It is easy to see that $ab = \varphi(b)a$ for every $a \in A$. Hence we have

$$0 = \lim_\alpha a_0 a_\alpha - a_\alpha a_0 = \lim \varphi(a_\alpha)a_0 - \varphi(a_0)a_\alpha = \lim a_0 - a_\alpha.$$

It follows that $a_0 = \lim a_\alpha$. Hence for each $a \in A$, we have

$$aa_0 = a_0 a, \qquad \varphi(a_0) = 1.$$

It implies that $a = \varphi(a)a_0$. Thus $\dim A = 1$ which is a contradiction.

4. Examples of semigroup algebras related to the matrix algebras

Example 4.1. Suppose that A is a Banach algebra and I is a nonempty totally ordered set. Put $B = \mathrm{UP}(I, A)$. It is obvious that B with matrix multiplication can be observed as a semigroup. Equip this semigroup with the discrete topology and denote it with S_B. Suppose that A has a nonzero idempotent. We claim that $l^1(S_B)$ is not amenable, whenever I is an infinite set. Suppose, for contradiction, that $l^1(S_B)$ is amenable. Let e be an idempotent for A and let $E_{i,i}$ denotes for a matrix belonging to B whose (i,i)th entry is e, otherwise is 0. It is easy to see that $E_{i,i}$ is an idempotent for the semigroup S_B for every $i \in I$. So the set of idempotents of S_B is infinite, whenever I is infinite. Thus by [6, Theorem 2] $l^1(S_B)$ is not amenable which is a contradiction.

Suppose that A is a nonzero Banach algebra with a left identity, also suppose that I is an infinite totally ordered set with smallest element. We also claim that $l^1(S_B)$ is not approximately biprojective. To see this, proceeding by contradiction, suppose that $l^1(S_B)$ is approximately biprojective. We denote the augmentation character on $l^1(S_B)$ by φ_{S_B}. It is easy to see that $\delta_{\hat{0}} \in S_B$ and $\varphi_{S_B}(\delta_{\hat{0}}) = 1$, where $\hat{0}$ is denoted for the zero matrix belonging to S_B. One can see that the center of S_B, $Z(S_B)$, is nonempty, because $\hat{0}$ belongs to $Z(S_B)$. So we can show that $l^1(S_B)$ is left φ_{S_B}-contractible. Let i_0 be smallest element of I. Define

$$J = \{(a_{i,j}) \in S_B \colon a_{i,j} = 0 \text{ for all } i \neq i_0\}.$$

It is easy to see that J is an infinite ideal of S_B, then by [5, page 50] $l^1(J)$ is a closed ideal of $l^1(S_B)$. Since $\varphi_{S_B}|_{l^1(J)}$ is nonzero, $l^1(J)$ is left φ_{S_B}-contractible. Thus there exists $m \in l^1(J)$ such that $am = \varphi_{S_B}(a)m$ and $\varphi_{S_B}(m) = 1$ for every $a \in l^1(J)$. On the other hand since A has a left identity, then J has a left identity. So we have

$$m(j) = m(e_l j) = \delta_j m(e_l) = \varphi_{S_B}(\delta_j)m(e_l) = m(e_l), \qquad j \in J,$$

where e_l is a left unit for J. It follows that m is a constant function belonging to $l^1(J)$. Since $\varphi_{S_B}(m) = 1$, we have $m \neq 0$. Then J is finite which is impossible.

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