

## A $\mathbb{Q}$ -linear automorphism of the reals with non-measurable graph

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*Abstract.* This note contains a proof of the existence of a one-to-one function  $\Theta$  of  $\mathbb{R}$  onto itself with the following properties:  $\Theta$  is a rational-linear automorphism of  $\mathbb{R}$ , and the graph of  $\Theta$  is a non-measurable subset of the plane.

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The existence of functions with non-measurable graphs was known to Sierpinski a century ago, but he apparently never published the details of a proof. However, see [2] and [1]. What is new in this note is a proof of the existence of an isomorphism  $\Theta$  of  $\mathbb{R}$  onto itself, as a rational vector space, which has the property that the graph of  $\Theta$  has zero inner measure and its complement  $\mathbb{R}^2 \setminus \Theta$  also has zero inner measure. Thus, the graph of  $\Theta$  is not measurable. (The characteristic function of a non-measurable set is not a measurable function, but its graph is measurable.) As in rudimentary set theory, a function is identical with its graph.

In what follows every linear combination has rational coefficients.

For any set  $Z$  let  $|Z|$  be the cardinal of  $Z$ . As usual  $c$  is the cardinal of the continuum. Let  $m_1$  be the Lebesgue measure on  $\mathbb{R}$  and  $m_2$  be the Lebesgue (2-dimensional) measure on  $\mathbb{R}^2$ . Let  $\mathbf{K}$  be the set of compact  $K \subset \mathbb{R}^2$  for which  $m_2(K) > 0$ . For  $x \in \mathbb{R}$  let  $V_x$  be the vertical line  $\{(x, y) : y \in \mathbb{R}\}$ .

**Lemma 1.** *If  $K \in \mathbf{K}$ , then there are  $c$  elements  $x$  of  $\mathbb{R}$  for which  $|V_x \cap K| = c$ .*

PROOF: Let  $0 < m_2(K) = \int m_1(V_x \cap K) dm_1(x)$  by Fubini's theorem. So  $m_1(V_x \cap K) > 0$  for a set  $X$  (of such  $x$ 's) of positive measure. Therefore  $X$  has  $c$  points, and for each such point  $V_x \cap K$  must contain  $c$  points, since its measure is positive. Lemma 1 is proved, along with the statement that the graph of any function has zero inner measure. (We remark that  $K$  may not contain a set  $I \times J$  of positive measure.)

Now we proceed with the proof of the main statement (the title and the abstract). The set  $\mathbf{K}$  has  $c$  members: Let us well order  $\mathbf{K} = \{K_\alpha : \alpha < c\}$ . We shall find a one-one map  $F$  of  $2 \times c$  into  $\mathbb{R}$ , with these properties: the image of  $F$  is linearly independent, and defining  $x_\alpha = F(0, \alpha)$  and  $y_\alpha = F(1, \alpha)$ , we have  $(x_\alpha, y_\alpha) \in K_\alpha$ .

Once we have the above, let  $A = \{x_\alpha : \alpha < c\}$  and  $B = \{y_\alpha : \alpha < c\}$ , two sets whose union is linearly independent. Clearly, the linear span of  $A$  is of dimension  $c$  and of co-dimension  $c$ ; the same is true for  $B$ . Extend  $A$  to a (Hamel) basis  $A'$  for  $\mathbb{R}$ , and extend  $B$  to  $B'$ , similarly. Since  $|A' \setminus A| = |B' \setminus B| = c$ , the one-one map  $\Theta : A \rightarrow B$  defined by  $\Theta(x_\alpha) = y_\alpha$  can be extended to a one-one map, also called  $\Theta$  from  $A'$  onto  $B'$ . Of course, this  $\Theta$  extends naturally to an isomorphism  $\Theta$  from  $\mathbb{R}$  onto itself. The set  $\Theta$  meets every  $K \in \mathbf{K}$ , since  $\Theta(x_\alpha) = y_\alpha$  means exactly that  $(x_\alpha, y_\alpha) \in \Theta$ .

It follows that  $\mathbb{R}^2 \setminus \Theta$  cannot contain a set of positive measure; that is, the inner measure of  $\mathbb{R}^2 \setminus \Theta$  is zero. The inner measure of  $\Theta$  is also zero, by Lemma 1.

In order to prove the existence of a function  $F$  with the desired properties, we shall proceed by transfinite induction. The set  $\mathbf{K}$  is already well ordered. Well order  $\mathbb{R} = \{z_\lambda : \lambda < c\}$ .

By transfinite induction we shall produce a collection  $\{F_\alpha : \alpha < c\}$  with these properties:

- (1) each  $F_\alpha : 2 \times \alpha \rightarrow \mathbb{R}$  is one-to-one;
- (2) the image of each  $F_\alpha$  is linearly independent;
- (3) for  $\beta < \alpha$   $F_\beta \subset F_\alpha$ ; and
- (4) for  $\beta < \alpha$   $(x_\beta, y_\beta) \in K_\beta$ , where  $x_\beta = F(0, \beta)$  and  $y_\beta = F(1, \beta)$ .

Then our desired  $F = \bigcup_{\alpha < c} F_\alpha$ .

Suppose we have a collection as above for all  $\alpha < \gamma < c$ . We are to find  $F_\gamma$  to satisfy the four properties. Note that for  $\gamma = 0$ , the properties hold vacuously. If  $\gamma$  is not a successor, simply take  $F_\gamma = \bigcup_{\alpha < \gamma} F_\alpha$ .

For  $\gamma = \beta + 1$  we proceed as follows. Put  $L$  to be the linear span of the image of  $F_\beta$ . The span  $L$  has fewer than  $c$  points. Lemma 1 shows that there are  $c$  points  $z$  not in  $L$  for which  $V_z \cap K_\beta$  contains  $c$  points. Let  $x_\beta$  be one of those  $z$ 's, say  $z_\eta$ , where  $\eta$  is the smallest possible for definiteness. Let  $L'$  be the linear span of  $L \cup \{x_\beta\}$ . In a manner similar to the selection of  $x_\beta$  choose  $y_\beta$  so that  $y_\beta$  is not in  $L'$  but  $(x_\beta, y_\beta) \in K_\beta$ . Put  $F(0, \beta) = x_\beta$  and  $F(1, \beta) = y_\beta$ , and extend  $F_\beta$  to  $F_{\beta+1}$  accordingly. Properties (1)–(4) are evident, and the proof is complete.  $\square$

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#### REFERENCES

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