Some results on G_C-flat dimension of modules

Ramalingam Udhayakumar,
Intan Muchtadi-Alamsyah, Chelliah Selvaraj

Abstract. In this paper, we study some properties of G_C-flat R-modules, where C is a semidualizing module over a commutative ring R and we investigate the relation between the G_C-yoke with the C-yoke of a module as well as the relation between the G_C-flat resolution and the flat resolution of a module over GF-closed rings. We also obtain a criterion for computing the G_C-flat dimension of modules.

Keywords: GF-closed ring; G_C-flat module; G_C-flat dimension; semidualizing module

Classification: 18G20, 18G25

1. Introduction

In basic homological algebra, projective, injective and flat modules play an important and fundamental role. Homological properties of the Gorenstein projective, injective and flat modules have been studied by many authors, some references are [2], [3], [5], [8], [15]. The study of semidualizing modules over commutative Noetherian rings was initiated independently (with different names) by H.-B. Foxby in [6], E. S. Golod in [7], and W. V. Vasconcelos in [16]. Over a commutative Noetherian ring, E. S. Holm and P. Jørgensen in [9] introduced the C-Gorenstein projective, C-Gorenstein injective and C-Gorenstein flat modules using semidualizing modules and their associated projective, injective and flat classes which are also called G_C-projective, G_C-injective and G_C-flat module, respectively. D. White introduced in [17] the G_C-projective modules and gave a functorial description of the G_C-projective dimension of modules with respect to a semidualizing module C over a commutative ring; and in particular, many classical results about the Gorenstein projectivity of modules were generalized in [17]. Being motivated from [17], in this paper, we give equivalent conditions for G_C-flat dimension of modules with respect to a semidualizing module C.

This paper is organized as follows. In Section 2, we recall some notions and definitions which will be needed in the later sections. In Section 3, we establish the relation between the G_C-yoke with the C-yoke of a module as well as the relation between the G_C-flat resolution and the flat resolution of a module over a GF-closed ring.

DOI 10.14712/1213-7243.2019.007

In Section 4, we get some properties of G_C-flat dimension of modules. In particular, as an application of the results obtained in Section 3, we get a criterion for computing such a dimension. Let R be a GF-closed ring and let M be an R-module and $n \geq 0$. We prove that the G_C-flat dimension of M is at most n if and only if for every nonnegative integer t such that $0 \leq t \leq n$, there exists an exact sequence of R-modules $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$ such that X_t is G_C-flat and X_i are flat for $i \neq t$.

2. Preliminaries

Throughout this paper, R is a commutative ring with identity and all modules are unitary modules. Let M be an R-module. We denote $\mathrm{Add}_R M$ (or $\mathrm{Prod}_R M$) the subclass of R-modules consisting of all modules isomorphic to direct summands of direct sums (direct products, respectively) of copies of M. At the beginning of this section, we recall some notions from [10], [17].

Definition 2.1 ([17])**.** A degreewise finite projective (or free) resolution of an R-module M is a *projective (or free) resolution* P *of* M such that each P_i is finitely generated projective (free, respectively).

Remark 2.2. Note that M admits a degreewise finite projective resolution if and only if it admits a degreewise finite free resolution. However, it is possible for a module to admit a bounded degreewise finite projective resolution but not to admit a bounded degreewise finite free resolution. For example, if $R = k_1 \oplus k_2$, where k_1 and k_2 are fields, then $M = k_1 \oplus 0$ is a projective R-module, but it does not admit a bounded free resolution.

Definition 2.3 ([17])**.** An R-module C is *semidualizing* if it satisfies the following conditions:

(1) C admits a degreewise finite projective resolution;
(2) the natural homothety morphism $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism; and
(3) $\mathrm{Ext}^i_R(C, C) = 0$ for any $i \geq 1$.

Remark 2.4. A free R-module of rank one is semidualizing. If R is Noetherian and admits a dualizing module D, then D is a semidualizing.

Definition 2.5 ([10])**.** Let C be a semidualizing module for a ring R. An R-module is *C-projective* if it has the form $C \otimes_R P$ for some projective module P. An R-module is called C-injective if it has the form $\mathrm{Hom}_R(C, I)$ for some injective module I. Set

$$\mathcal{P}_C(R) = \{C \otimes_R P \colon P \text{ is } R\text{-projective}\},$$

and

$$\mathcal{I}_C(R) = \{\mathrm{Hom}_R(C, I) \colon I \text{ is } R\text{-injective}\}.$$

Definition 2.6 ([10])**.** An R-module is called *C-flat* if it has the form $C \otimes_R F$ for some flat module F. Set $\mathcal{F}_C(R) = \{C \otimes_R F \colon F \text{ is } R\text{-flat}\}$.

Definition 2.7. Let R be a ring and let $\mathfrak{X}$ be a class of R-modules.

(1) A class $\mathfrak{X}$ is closed under extensions if for every short exact sequence of R-modules $0 \to A \to B \to C \to 0$, the conditions A and C are in $\mathfrak{X}$ imply B is in $\mathfrak{X}$.

(2) A class $\mathfrak{X}$ is closed under kernels of epimorphisms if for every short exact sequence of R-modules $0 \to A \to B \to C \to 0$, the conditions B and C are in $\mathfrak{X}$ imply A is in $\mathfrak{X}$.

(3) A class $\mathfrak{X}$ is projectively resolving if it contains all projective modules and it is closed under both extensions and kernels of epimorphisms, i.e., for every short exact sequence of R-modules $0 \to A \to B \to C \to 0$ with $C \in \mathfrak{X}$, the conditions $A \in \mathfrak{X}$ and $B \in \mathfrak{X}$ are equivalent.

Definition 2.8 ([5]). An R-module M is said to be *Gorenstein flat*, if there exists an exact sequence of flat R-modules,

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that $M \cong \mathrm{Im}(F_0 \to F^0)$ and such that $B \otimes_R -$ leaves the sequence exact whenever B is an injective R-module.

Definition 2.9 ([1]). Let R be a ring. We call R *GF-closed* if the class of Gorenstein flat R-modules is closed under extensions.

3. G_C-flat modules

We start with the following definition.

Definition 3.1 ([9]). A complete $\mathcal{F}\mathcal{F}_C$-resolution is a $\mathcal{I}_C(R)\otimes_R$-exact sequence:

$$\mathscr{X}\colon\ \cdots \to F_1 \to F_0 \to C \otimes_R F^0 \to C \otimes_R F^1 \to \cdots \tag{1}$$

in R-Mod with all F_i and F^i flat. An R-module M is called G_C-flat if there exists a complete $\mathcal{F}\mathcal{F}_C$-resolution as in (1) with $M = \mathrm{Coker}(F_1 \to F_0)$. Set $\mathcal{G}\mathcal{F}_C(R)$ to be the class of G_C-flat R-modules.

It is trivial that in case $C = R$, the G_C-flat modules are just the usual Gorenstein flat modules.

Using the definition, we immediately get the following results.

Proposition 3.2. *If $(F_i)_{i\in I}$ is a family of G_C-flat R-modules, then $\bigoplus F_i$ is G_C-flat.*

Proposition 3.3. *An R-module M is G_C-flat if and only if*

$$\mathrm{Tor}^R_{\geq 1}(\mathrm{Hom}_R(C, I), M) = 0$$

and M admits a $\mathcal{F}_C$-resolution Y with $\mathrm{Hom}_R(C, I) \otimes_R Y$ exact for any injective I.

Proposition 3.4. *Let R be a commutative Noetherian ring and F a flat R-module. If M is an G_C-flat R-module, then $M \otimes_R F$ is a G_C-flat R-module.*

PROOF: There is an exact sequence

$$\cdots \to F_1 \to F_0 \to C \otimes_R F^0 \to C \otimes_R F^1 \to \cdots$$

with F_i and F^i flat and $M = \mathrm{Coker}(F_1 \to F_0)$. Then the sequence

$$\cdots \to F_1 \otimes F \to F_0 \otimes F \to C \otimes_R F^0 \otimes F \to C \otimes_R F^1 \otimes F \to \cdots$$

is exact with $F_i \otimes F$, $F^i \otimes F$ flat by [12, Proposition 2.11]. Let I be any injective R-module and $\mathcal{F} = \mathrm{Hom}(C, I)$. Then

$$\begin{aligned} \mathrm{Tor}_1^R(M \otimes_R F, \mathrm{Hom}(C, I)) &= H_i((M \otimes_R F) \otimes \mathcal{F}) \\ &\cong H_i(M \otimes_R (F \otimes \mathcal{F})) \\ &\cong \mathrm{Tor}_1^R(M, F \otimes_R \mathrm{Hom}(C, I)) = 0 \end{aligned}$$

by [13, page 258, 9.20] for all $i \geq 1$, since $F \otimes_R \mathrm{Hom}(C, I) \cong \mathrm{Hom}(C, F \otimes_R I)$ is a C-injective module by [4, Theorem 3.2.16] and [10, (1.10)]. Hence $M \otimes_R F$ is a G_C-flat R-module. □

The following result is due to [14, Proposition 5.3].

Proposition 3.5. *Let C be a semidualizing R-module. Then the class $\mathcal{GF}_C(R)$ is closed under kernels of epimorphisms and extensions.*

Proposition 3.6. *Let C be a semidualizing R-module. If F is flat R-module, then F and $C \otimes_R F$ are G_C-flat. Thus, every R-module admits a G_C-flat resolution.*

PROOF: Follows from [9, Example 2.8 (a) and (c)] and since the class of G_C-flat modules contains the class of flat modules, every R-module admits a G_C-flat resolution. □

Theorem 3.7. *Let C be a semidualizing module, then the class $\mathcal{GF}_C(R)$ of G_C-flat R-modules is projectively resolving and closed under direct summands.*

PROOF: Using the dual of Theorem 2.8 in [17] and together with the [14, Lemma 5.2], we see that $\mathcal{GF}_C(R)$ is projectively resolving. Now, comparing Proposition 3.5 with Proposition 1.4 in [8], we get $\mathcal{GF}_C(R)$ is closed under direct summands. □

Proposition 3.8. *Let R be a GF-closed ring. Then every cokernel in a complete $\mathcal{FF}_C$-resolution is G_C-flat.*

PROOF: Follows from Proposition 3.3, Theorem 3.7 and [14, Lemma 5.4]. □

Lemma 3.9. *Let R be a GF-closed ring and let M be G_C-flat R-module. Then there exists $\mathcal{I}_C(R)\otimes$-exact sequences of R-modules:*

$$0 \to M \to G \to N \to 0$$

and

$$0 \to K \to F \to M \to 0$$

with N, K G_C-flat, G, F flat.

Proof: By the definition of G_C-flat modules and Proposition 3.8 . □

The following result plays a crucial role in this section and it follows from [11, Proposition 2.2].

Lemma 3.10. *Let R be a GF-closed ring and suppose that*

$$0 \to A \to G_1 \xrightarrow{f} G_0 \to M \to 0 \tag{2}$$

is an exact sequence of R-modules with G_0, G_1 G_C-flat. Then we have the following exact sequences:

$$0 \to A \to C_1 \to G \to M \to 0, \tag{3}$$

and

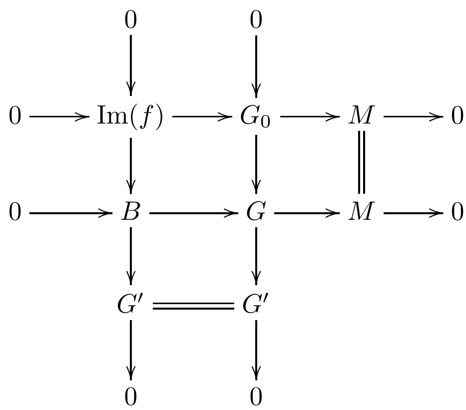
$$0 \to A \to H \to F \to M \to 0 \tag{4}$$

with C_1, F flat, and G, H G_C-flat.

Proof: Since G_1 is G_C-flat, there exists a short exact sequence $0 \to G_1 \to C_1 \to G' \to 0$ with C_1 is flat and G' G_C-flat by Lemma 3.9. Then we have the following pushout diagram:

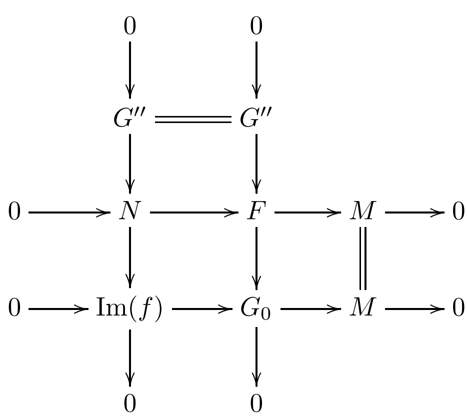
$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & G_1 & \longrightarrow & \mathrm{Im}(f) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & C_1 & \longrightarrow & B & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Consider the following pushout diagram:



Since G_0 and G' are G_C-flat, G is also G_C-flat by Theorem 3.7. Connecting the middle rows in the above two diagrams, we get the first desired exact sequence (3).

Since G_0 is G_C-flat, there exists an exact sequence $0 \to G'' \to F \to G_0 \to 0$ with F flat and G'' G_C-flat by Lemma 3.9. Then we have the following pullback diagram:



Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G'' & = & G'' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & N & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & G_1 & \longrightarrow & \mathrm{Im}(f) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since G_1 and G'' are G_C-flat, H is also G_C-flat by Theorem 3.7. Connecting the middle rows in the above two diagrams, we get the second desired exact sequence (4). □

Definition 3.11. Let n be a positive integer. An R-module A is called an *C-yoke module (of M)* if there exists an exact sequence of R-modules

$$0 \to A \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

with all F_i C-flat.

Definition 3.12. Let n be a positive integer, a module A is called an *G_C-yoke module (of M)* if there exists an exact sequence of R-modules

$$0 \to A \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

with all G_i G_C-flat.

The following result establishes the relation between the G_C-yoke with the C-yoke of a module as well as the relation between the G_C-flat resolution and the flat resolution of a module.

Lemma 3.13. *Let R be a GF-closed ring and let $n \geq 1$ and*

(5) $$0 \to A \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

be an exact sequence of R-modules with all G_i G_C-flat. Then we have the following:

(i) *There exists exact sequences of R-modules:*

(6) $$0 \to A \to C_{n-1} \to \cdots \to C_1 \to C_0 \to N \to 0$$

and

$$0 \to M \to N \to G \to 0$$

with all C_i flat and G G_C-flat.

(ii) *There exist exact sequences of R-modules*

$$0 \to B \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0 \tag{7}$$

and

$$0 \to H \to B \to A \to 0$$

with all F_i flat and H G_C-flat.

Proof: We proceed by induction on n.

(i) When $n = 1$, we have an exact sequence of R-modules $0 \to A \to G_0 \to M \to 0$. Since we have a $\mathcal{I}_C(R)\otimes_R$-exact sequence of R-modules $0 \to G_0 \to C_0 \to G \to 0$ with C_0 is flat and G G_C-flat by Lemma 3.9, we have the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & C_0 & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G & = & G & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The middle row and the last column in the above diagram are the desired two exact sequences.

Now assume that $n \geq 2$ and we have an exact sequence of R-modules $0 \to A \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ with all G_i G_C-flat. Put $K = \mathrm{Coker}(G_{n-1} \to G_{n-2})$. By Lemma 3.10, we get an exact sequence of R-modules

$$0 \to A \to C_{n-1} \to G'_{n-2} \to K \to 0 \tag{8}$$

with C_{n-1} flat and G'_{n-2} G_C-flat. Put $A' = \mathrm{Im}(C_{n-1} \to G'_{n-2})$. Then, we get an exact sequence of R-modules $0 \to A' \to G'_{n-2} \to G_{n-3} \to \cdots \to G_1 \to G_0 \to M \to 0$. So, by the induction hypothesis, we get the assertion.

(ii) When $n = 1$, we have an exact sequence of R-modules $0 \to A \to G_0 \to M \to 0$. Since we have a $\mathcal{I}_C(R)\otimes_R$-exact sequence of R-modules $0 \to H \to F_0 \to G_0 \to 0$ with F_0 flat and H G_C-flat by Lemma 3.9, we have the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H & = & H & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & B & \to & F_0 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & A & \to & G_0 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The middle row and the first column in the above diagram are the desired two exact sequences.

Now assume that $n \geq 2$ and we have an exact sequence of R-modules $0 \to A \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ with all G_i G_C-flat. Put $K = \mathrm{Ker}(G_1 \to G_0)$. By Lemma 3.10, we get an exact sequence of R-modules

$$0 \to K \to G'_1 \to F_0 \to M \to 0 \tag{9}$$

with F_0 flat and G'_1 G_C-flat. Put $M' = \mathrm{Im}(G'_1 \to P_0)$. Then we get an exact sequence of R-modules $0 \to A \to G_{n-1} \to \cdots \to G_2 \to G'_1 \to G_0 \to M \to 0$. So, by the induction hypothesis, we get the assertion. □

4. G_C-flat dimensions of modules

The class of G_C-flat modules can be used to define the G_C-flat dimension.

Definition 4.1. For an R-module M, the G_C-flat dimension of M, denoted by $G_C - fd_R(M)$, is defined as $\inf\{n\colon$ there exists an exact sequence of R-modules $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ with all G_i G_C-flat$\}$. We have $G_C - fd_R(M) \geq 0$, and we set $G_C - fd_R(M) = \infty$ if no such integer exists.

We start with the following standard result.

Lemma 4.2. *Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules.*

(i) $G_C - fd_R(N) \leq \max\{G_C - fd_R(M), G_C - fd_R(L) + 1\}$, *and the equality holds if* $G_C - fd_R(M) \neq G_C - fd_R(L)$.

(ii) $G_C - fd_R(L) \leq \max\{G_C - fd_R(M), G_C - fd_R(N) - 1\}$, *and the equality holds if* $G_C - fd_R(M) \neq G_C - fd_R(N)$.

(iii) $G_C - fd_R(M) \leq \max\{G_C - fd_R(L), G_C - fd_R(N)\}$, *and the equality holds if* $G_C - fd_R(N) \neq G_C - fd_R(L) + 1$.

Proof: It is easy. □

The proof of the following theorem is similar to [8, Theorem 3.15].

Theorem 4.3. *Assume that R is GF-closed and C is a semidualizing module. If any two of the modules M, M' or M'' in a short exact sequence $0 \to M' \to M \to M''$ have finite G_C-flat dimension, then so has the third.*

Next result is a G_C-flat version of the corresponding result about flat dimension of modules.

Proposition 4.4. *Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. If $L \neq 0$ and N is G_C-flat, then $G_C - fd_R(L) = G_C - fd_R(M)$.*

Proof: It follows by Lemma 4.2 (3). □

We give a criterion for computing the G_C-flat dimension of modules as follows. It generalizes [8, Theorem 3.14].

Theorem 4.5. *Let R be a GF-closed ring. The following statements are equivalent for any R-module M and $n \geq 0$.*

(i) *$G_C - fd_R(M) \leq n$.*
(ii) *For every nonnegative integer t such that $0 \leq t \leq n$, there exists an exact sequence of R-modules $0 \to X_n \to \cdots \to X_t \to \cdots \to X_1 \to X_0 \to M \to 0$ such that X_t is G_C-flat and X_i are flat for $i \neq t$.*

Proof: (ii) ⇒ (i). It is trivial.

(i) ⇒ (ii). We proceed by induction on n. Suppose $G_C - fd_R(M) \leq 1$. Then there exists an exact sequence of R-modules $0 \to G_1 \to G_0 \to M \to 0$ with G_0 and G_1 G_C-flat. By Lemma 3.10 with $A = 0$, we get the exact sequences of R-modules $0 \to C_1 \to G'_0 \to M \to 0$ and $0 \to G'_1 \to F_0 \to M \to 0$ with C_1 and F_0 flat, and G'_0, G'_1 G_C-flat.

Now suppose $G_C - fd_R(M) = n \geq 2$. Then there exists an exact sequence of R-modules $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ with G_i G_C-flat for any $0 \leq i \leq n$. Set $A = \mathrm{Coker}(G_3 \to G_2)$. By applying Lemma 3.10 to the exact sequence $0 \to A \to G_1 \to G_0 \to M \to 0$, we get an exact sequence of R-modules $0 \to G_n \to \cdots \to G_2 \to G'_1 \to F_0 \to M \to 0$ with G'_1 G_C-flat and F_0 flat. Set $N = \mathrm{Coker}(G_2 \to G'_1)$. Then we have $G_C - fd_R(N) \leq n-1$. By the induction hypothesis, there exists an exact sequence of R-modules

$$0 \to X_n \to \cdots \to X_t \to \cdots \to X_1 \to F_0 \to M \to 0$$

such that F_0 is flat and X_t is G_C-flat and X_i are flat for $i \neq t$ and $1 \leq t \leq n$.

Now we need only to prove (ii) for $t = 0$. Set $B = \mathrm{Coker}(G_2 \to G_1)$. By the induction hypothesis, we get an exact sequence of R-modules $0 \to X_n \to \cdots \to X_3 \to X_2 \to G'_1 \to B \to 0$ with G'_1 G_C-flat and X_i being flat for any $2 \leq i \leq n$. Set $D = \mathrm{Coker}(X_3 \to X_2)$. Then by applying Lemma 3.10 to the exact sequence $0 \to D \to G'_1 \to G_0 \to M \to 0$, we get the exact sequence of

R-modules $0 \to D \to C_1 \to G'_0 \to M \to 0$ with C_1 flat and G'_0 G_C-flat. Thus we obtain the desired exact sequence of R-modules

$$0 \to X_n \to \cdots \to X_2 \to X_1 \to G'_0 \to M \to 0$$

with all X_i flat and G'_0 G_C-flat. □

References

[1] Bennis D., *Rings over which the class of Gorenstein flat modules is closed under extentions*, Comm. Algebra **37** (2009), no. 3, 855–868.

[2] Christensen L. W., *Gorenstein Dimensions*, Lecture Notes in Mathematics, 1747, Springer, Berlin, 2000.

[3] Christensen L. W., Frankild A., Holm H., *On Gorenstein projective, injective and flat dimensions a functorial description with applications*, J. Algebra **302** (2006), no. 1, 231–279.

[4] Enochs E., Jenda O. M. G., *Relative Homological Algebra*, De Gruyter Expositions in Mathematics, 30, Walter de Gruyter, Berlin, 2000.

[5] Enochs E., Jenda O. M. G., Torrecillas B., *Gorenstein flat modules*, Nanjing Daxue Xuebao Shuxue Bannian Kan **10** (1993), no. 1, 1–9.

[6] Foxby H.-B., *Gorenstein modules and related modules*, Math. Scand. **31** (1972), 267–284.

[7] Golod E. S., *G-dimension and generalized perfect ideals*, Algebraic geometry and its applications, Trudy Mat. Inst. Steklov. **165** (1984), 62–66 (Russian).

[8] Holm H., *Gorenstein homological dimensions*, J. Pure Appl. Algebra **189** (2004), no. 1–3, 167–193.

[9] Holm H., Jørgensen P., *Semidualizing modules and related Gorenstein homological dimensions*, J. Pure. Appl. Algebra **205** (2006), no. 2, 423–445.

[10] Holm H., White D., *Foxby equivalence over associative rings*, J. Math. Kyoto Univ. **47** (2007), no. 4, 781–808.

[11] Huang C., Huang Z., *Gorenstein syzygy modules* J. Algebra **324** (2010), no. 12, 3408–3419.

[12] Liu Z., Yang X., *Gorenstein projective, injective and flat modules*, J. Aust. Math. Soc. **87** (2009), no. 3, 395–407.

[13] Rotman J. J., *An Introduction to Homological Algebra*, Pure and Applied Mathematics, 85, Academic Press, New York, 1979.

[14] Sather-Wagstaff S., Sharif T., White D., *AB-contexts and stability for Gorenstein flat modules with respect to semidualizing modules*, Algebr. Represent. Theory **14** (2011), no. 3, 403–428.

[15] Selvaraj C., Udhayakumar R., Umamaheswaran A., *Gorenstein n-flat modules and their covers*, Asian-Eur. J. Math. **7** (2014), no. 3, 1450051, 13 pages.

[16] Vasconcelos W. V., *Divisor Theory in Module Categories*, North-Holland Mathematics Studies, 14, North-Holland Publishing Co., Amsterdam, American Elsevier Publishing Co., New York, 1974.

[17] White D., *Gorenstein projective dimension with respect to a semidualizing module*, J. Commut. Algebra **2** (2010), no. 1, 111–137.

R. Udhayakumar:
Department of Mathematics, School of Advanced Sciences,
Vellore Institute of Technology, Vellore – 632014, Tamil Nadu, India

E-mail: udhayaram_v@yahoo.co.in

I. Muchtadi-Alamsyah:
Algebra Research Division, Faculty of Mathematics and Natural Sciences,
Institut Teknologi Bandung, Jalan Ganesha 10, Bandung 40132, Indonesia

E-mail: ntan@math.itb.ac.id

C. Selvaraj:
Department of Mathematics, Periyar University, Periyar Palkalai Nagar,
Salem – 636 011, Tamil Nadu, India

E-mail: selvavlr@yahoo.com

(*Received* March 16, 2018, *revised* December 29, 2018)