The reciprocal Dunford–Pettis property of order p in projective tensor products

Ioana Ghenciu

Abstract. We investigate whether the projective tensor product of two Banach spaces X and Y has the reciprocal Dunford–Pettis property of order p, $1 \leq p < \infty$, when X and Y have the respective property.

Keywords: reciprocal Dunford–Pettis property; spaces of compact operators

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1. Introduction

The set of all continuous linear transformations from X to Y will be denoted by $L(X,Y)$, and the compact operators will be denoted by $K(X,Y)$.

In [18] we introduced the reciprocal Dunford–Pettis property of order p (RDP_p) for $1 \leq p < \infty$, a property which is intermediate between property (V) and the reciprocal Dunford–Pettis property (RDP). In [14] and [12] it was studied whether $X \otimes_\pi Y$ has property (V) or the reciprocal Dunford–Pettis property (RDP), when X and Y have the respective property. In this note we use results about relative weak compactness in spaces of compact operators to study whether property RDP_p lifts from the Banach spaces X and Y to the projective tensor product space $X \otimes_\pi Y$. We prove that in some cases, if $X \otimes_\pi Y$ has property RDP_p, then $L(X,Y^*) = K(X,Y^*)$.

2. Definitions and notation

Throughout this paper, X and Y will denote Banach spaces. The unit ball of X will be denoted by B_X and X^* will denote the continuous linear dual of X. The space X embeds in Y (in symbols $X \hookrightarrow Y$) if X is isomorphic to a closed subspace of Y. An operator $T \colon X \to Y$ will be a continuous and linear function. The set of all operators, weakly compact operators, and compact operators from X to Y will be denoted by $L(X,Y)$, $W(X,Y)$, and $K(X,Y)$.

A subset S of X is said to be *weakly precompact* provided that every sequence from S has a weakly Cauchy subsequence. An operator $T \colon X \to Y$ is called *weakly precompact* (or *almost weakly compact*) if $T(B_X)$ is weakly precompact.

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An operator $T\colon X \to Y$ is called *completely continuous* (or *Dunford–Pettis*) if T maps weakly convergent sequences to norm convergent sequences. The set of all completely continuous operators from X to Y is denoted by $\mathrm{CC}(X, Y)$.

For $1 \le p < \infty$, p^* denotes the conjugate of p. If $p = 1$, l_{p^*} plays the role of c_0. The unit vector basis of l_p will be denoted by (e_n). Let $1 \le p < \infty$. A sequence (x_n) in X is called (*strongly*) *p-summable* if $(\|x_n\|) \in l_p$, see [8, page 32], [7, page 59]. Let $l_p(X)^{\text{strong}}$ denote the set of all p-summable sequences in X with the norm

$$\|(x_n)\|_p^{\text{strong}} = \left(\sum_{n=1}^{\infty} \|x_n\|^p\right)^{1/p}.$$

Let $1 \le p \le \infty$. A sequence (x_n) in X is called *weakly p-summable* if $(x^*(x_n)) \in l_p$ for each $x^* \in X^*$ [8, page 32]. Let $l_p^w(X)$ denote the set of all weakly p-summable sequences in X. The space $l_p^w(X)$ is a Banach space with the norm

$$\|(x_n)\|_p^w = \sup\left\{\left(\sum_{n=1}^{\infty} |\langle x^*, x_n\rangle|^p\right)^{1/p} : x^* \in B_{X^*}\right\}.$$

We recall the following isometries: $L(l_{p^*}, X) \simeq l_p^w(X)$ for $1 < p < \infty$; $L(c_0, X) \simeq l_p^w(X)$ for $p = 1$; $T \to (T(e_n))$, see [8, Proposition 2.2, page 36].

A series $\sum x_n$ in X is said to be *weakly unconditionally convergent* (wuc) if for every $x^* \in X^*$, the series $\sum |x^*(x_n)|$ is convergent. An operator $T\colon X \to Y$ is *unconditionally converging* if it maps weakly unconditionally convergent series to unconditionally convergent ones.

Let $1 \le p \le \infty$. An operator $T\colon X \to Y$ is called *p-convergent* if T maps weakly p-summable sequences into norm null sequences, see [5]. The set of all p-convergent operators is denoted by $C_p(X, Y)$.

The 1-convergent operators are precisely the unconditionally converging operators and the ∞-convergent operators are precisely the completely continuous operators. If $p < q$, then $C_q(X, Y) \subseteq C_p(X, Y)$.

A bounded subset A of X^* is called a V-subset of X^* provided that

$$\sup_{x^* \in A} |x^*(x_n)| \to 0$$

for each wuc series $\sum x_n$ in X.

A. Pelczyński introduced property (V) in his fundamental paper, see [21]. The Banach space X has property (V) if every V-subset of X^* is relatively weakly compact. The following results were also established in [21]: reflexive Banach spaces and $C(K)$ spaces have property (V); the Banach space X has property (V) if and only if every unconditionally converging operator T from X to any Banach space Y is weakly compact; every quotient space of a Banach space with property (V) has property (V); if X has property (V), then X^* is weakly sequentially complete.

The bounded subset A of X^* is called an *L-subset* of X^* if each weakly null sequence (x_n) in X tends to 0 uniformly on A.

The Banach space X has the *reciprocal Dunford–Pettis* (RDP) *property* if every completely continuous operator T from X to any Banach space Y is weakly compact. The space X has the RDP property if and only if every L-subset of X^* is relatively weakly compact, see [19]. Banach spaces with property (V) of A. Pełczyński, in particular reflexive spaces and $C(K)$ spaces, have the RDP property, see [21]. A Banach space X does not contain l_1 if and only if every L-subset of X^* is relatively compact, see [10].

Let $1 \le p < \infty$. A bounded subset A of X^* is called a *weakly-p-L-set*, see [18], if for all weakly p-summable sequences (x_n) in X,

$$\sup_{x^* \in A} |x^*(x_n)| \to 0.$$

The weakly-1-L-subsets of X^* are precisely the V-subsets. If $p < q$, then a weakly-q-L-set is a weakly-p-L-set, since $l_p^w(X) \subseteq l_q^w(X)$.

Let $1 \le p < \infty$. A Banach space X has the *reciprocal Dunford–Pettis property of order p* or RDP_p (or the *weak reciprocal Dunford–Pettis property of order p* or wRDP_p) if every weakly-p-L-subset of X^* is relatively weakly compact (or weakly precompact, respectively), see [18].

If $p < q$ and X has the RDP_p property, then X has the RDP_q property. If X has property (V), then X has property RDP_p, see [18]. If X has the RDP_p property, then X has the RDP property (since any L-subset of X^* is a weakly-p-L-set).

A Banach space X has the RDP_p (or wRDP_p) property if and only if every p-convergent operator $T\colon X \to Y$ has a weakly compact (or weakly precompact, respectively) adjoint, see [18].

Suppose that $1 \le p < \infty$. An operator $T\colon X \to Y$ is called *p-summing* (or *absolutely p-summing*) if there is a constant $c \ge 0$ such that for any $m \in \mathbb{N}$ and any $x_1, x_2, \cdots, x_m$ in X,

$$\left(\sum_{i=1}^{m} \|T(x_i)\|^p\right)^{1/p} \le c \sup\left\{\left(\sum_{i=1}^{m} |\langle x^*, x_i\rangle|^p\right)^{1/p} : x^* \in B_{X^*}\right\}.$$

The least c for which the previous inequality always holds is denoted by $\pi_p(T)$, see [8, page 31]. The set of all p-summing operators from X to Y is denoted by $\Pi_p(X, Y)$. The operator $T\colon X \to Y$ is *p-summing* if and only if $(Tx_n) \in l_p(Y)^{\mathrm{strong}}$ whenever $(x_n) \in l_p^w(X)$, see [8, page 34], [7, page 59].

A topological space S is called *dispersed* (or *scattered*) if every nonempty closed subset of S has an isolated point. A compact Hausdorff space K is dispersed if and only if $l_1 \not\hookrightarrow C(K)$, see [23].

The Banach space X has the Dunford–Pettis property (DPP) if every weakly compact operator $T\colon X \to Y$ is completely continuous. Equivalently, X has the DPP if and only if $x_n^*(x_n) \to 0$ whenever (x_n^*) is weakly null in X^* and (x_n) is weakly null in X [6, Theorem 1]. If X is a $C(K)$ space or an L_1-space, then X has the DPP. The reader can check [7], [6], and [9] for results related to the DPP.

The Banach-Mazur distance $d(E,F)$ between two isomorphic Banach spaces E and F is defined by $\inf(\|T\|\|T^{-1}\|)$, where the infinum is taken over all isomorphisms T from E onto F. A Banach space E is called an $\mathcal{L}_\infty$-space (or $\mathcal{L}_1$-space), see [4], if there is a $\lambda \geq 1$ so that every finite dimensional subspace of E is contained in another subspace N with $d(N, l_\infty^n) \leq \lambda$ (or $d(N, l_1^n) \leq \lambda$, respectively) for some integer n. Complemented subspaces of $C(K)$ spaces (or $L_1(\mu)$ spaces) are $\mathcal{L}_\infty$-spaces (or $\mathcal{L}_1$-spaces, respectively), see [4, Proposition 1.26]. The dual of an $\mathcal{L}_1$- space (or $\mathcal{L}_\infty$-space) is an $\mathcal{L}_\infty$-space (or $\mathcal{L}_1$-space, respectively), see [4, Proposition 1.27].

The $\mathcal{L}_\infty$-spaces, $\mathcal{L}_1$-spaces, and their duals have the DPP, see [4, Corollary 1.30].

3. Property RDP_p in spaces of compact operators

In this section we consider property RDP_p in the projective tensor product $X \otimes_\pi Y$. We begin by noting that there are examples of Banach spaces X and Y such that $X \otimes_\pi Y$ has property RDP_p. If $1 < q' < p < \infty$, then $L(l_p, l_{q'}) = K(l_p, l_{q'})$ (by a result of Pitt [24], [9, page 247]). If q is the conjugate of q', then $l_p \otimes_\pi l_q$ is reflexive (by [26, Theorem 4.19], [9, page 248]), and thus has the RDP_p property. Then the spaces $X = l_p$ and $Y = l_q$ are as desired.

In the proofs of Theorems 4 and 5 we will need the following results.

Theorem 1 ([16]). *Suppose that $L(X,Y) = K(X,Y)$ and H is a subset of $K(X,Y)$ such that:*

(i) *The set $H(x)$ is relatively weakly compact for all $x \in X$.*
(ii) *The set $H^*(y^*)$ is relatively weakly compact for all $y^* \in Y^*$.*

Then H is relatively weakly compact.

Theorem 2 ([16]). *Let H be a bounded subset of $K(X,Y)$ such that:*

(i) *The set $H(x)$ is weakly precompact for each $x \in X$.*
(ii) *The set $H^*(y^*)$ is relatively weakly compact for each $y^* \in Y^*$.*

Then H is weakly precompact.

Lemma 3 ([17]). *Let $1 \leq p < \infty$. Suppose that $L(X, Y^*) = \Pi_p(X, Y^*)$. If (x_n) is weakly p-summable in X and (y_n) is bounded in Y, then $(x_n \otimes y_n)$ is weakly p-summable in $X \otimes_\pi Y$.*

Theorem 4. *Let $1 \leq p < \infty$. Suppose that $L(X, Y^*) = K(X, Y^*) = \Pi_p(X, Y^*)$. If X and Y have property RDP_p, then $X \otimes_\pi Y$ has property RDP_p.*

Proof: Let H be a weakly-p-L-subset of $L(X, Y^*) = K(X, Y^*) = \Pi_p(X, Y^*)$ and let (T_n) be a sequence in H. We will verify the conditions (i) and (ii) of Theorem 1. Let $x \in X$. We show that $\{T_n(x) \colon n \in \mathbb{N}\}$ is a weakly-p-L-subset of Y^*. Suppose (y_n) is weakly p-summable in Y. Let $T \in L(X, Y^*) \simeq (X \otimes_\pi Y)^*$, see [9, page 230]. Since T is weakly compact, $T^{**}(X^{**}) \subseteq Y^*$. If $x^{**} \in X^{**}$, then $\sum_n |\langle x^{**}, T^*(y_n)\rangle|^p = \sum_n |\langle T^{**}(x^{**}), y_n\rangle|^p < \infty$. Thus $(T^*(y_n))$ is weakly p-summable in X^*. Hence

$$\sum_n |\langle T, x \otimes y_n\rangle|^p = \sum_n |\langle x, T^*(y_n)\rangle|^p < \infty.$$

Thus $(x \otimes y_n)$ is weakly p-summable in $X \otimes_\pi Y$. Since (T_n) is a weakly-p-L-set,

$$\langle T_n, x \otimes y_n\rangle = \langle T_n(x), y_n\rangle \to 0.$$

Therefore $\{T_n(x) : n \in \mathbb{N}\}$ is a weakly-p-L-subset of Y^*, hence relatively weakly compact.

Let $y^{**} \in Y^{**}$. We show that $\{T_n^*(y^{**}) : n \in \mathbb{N}\}$ is a weakly-p-L-subset of X^*. Suppose (x_n) is weakly p-summable in X. For $n \in \mathbb{N}$,

$$\langle T_n^*(y^{**}), x_n\rangle = \langle y^{**}, T_n(x_n)\rangle \le \|y^{**}\| \, \|T_n(x_n)\|.$$

We show that $\|T_n(x_n)\| \to 0$. Suppose that $\|T_n(x_n)\| \not\to 0$. Without loss of generality assume that $\langle T_n(x_n), y_n\rangle > \varepsilon$ for some sequence (y_n) in B_Y and some $\varepsilon > 0$. By Lemma 3, $(x_n \otimes y_n)$ is weakly p-summable in $X \otimes_\pi Y$. Since $\{T_n : n \in \mathbb{N}\}$ is a weakly-p-L-set,

$$\langle T_n, x_n \otimes y_n\rangle = \langle T_n(x_n), y_n\rangle \to 0.$$

This contradiction shows that $\|T_n(x_n)\| \to 0$. Hence $\{T_n^*(y^{**}) : n \in \mathbb{N}\}$ is a weakly-p-L-subset of X^*, thus relatively weakly compact. By Theorem 1, H is relatively weakly compact. □

Theorem 5. *Let $1 \le p < \infty$. Suppose that $L(X, Y^*) = K(X, Y^*) = \Pi_p(X, Y^*)$. If X has property* RDP_p *and Y has property* wRDP_p*, then $X \otimes_\pi Y$ has property* wRDP_p*.*

Proof: Let H be an weakly-p-L-subset of $(X \otimes_\pi Y)^* \simeq L(X, Y^*) = K(X, Y^*)$ and let (T_n) be a sequence in H. The proof of Theorem 4 shows that $\{T_n(x) : n \in \mathbb{N}\}$ is a weakly-p-L-subset of Y^*, and thus weakly precompact. Similarly, $\{T_n^*(y^{**}) : n \in \mathbb{N}\}$ is a weakly-p-L-subset of X^*, thus relatively weakly compact. By Theorem 2, H is weakly precompact. □

Observation 1. If $l_1 \hookrightarrow X$, then X^* does not have the Schur property (since $l_1 \hookrightarrow X$, $L_1 \hookrightarrow X^*$, see [7, page 212]).

Corollary 6. *Let $1 \le p < \infty$. Suppose $L(X, Y^*) = \Pi_p(X, Y^*)$, and X and Y have property* RDP_p*. If X^* (or Y^*) has the Schur property, then $X \otimes_\pi Y$ has property* RDP_p*.*

Proof: Let $T : X \to Y^*$ be an operator. Then T is p-summing, and thus weakly compact and completely continuous, see [8, Theorem 2.17]. If X^* has the Schur property, then $l_1 \not\hookrightarrow X$ (by Observation 1). Thus T is compact by a result of Odell, see [25, page 377]. If Y^* has the Schur property, then T is compact (since it is also weakly compact). Then $L(X, Y^*) = K(X, Y^*)$. Apply Theorem 4. □

Observation 2. (i) Let $1 \le p \le 2$. If X is an $\mathcal{L}_\infty$-space and Y an $\mathcal{L}_p$-space, then every operator $T\colon X \to Y$ is 2-summing, see [8, Theorem 3.7].

(ii) If X and Y are $\mathcal{L}_\infty$-spaces, then $L(X, Y^*) = \Pi_p(X, Y^*)$, $2 \le p < \infty$. Indeed, by (i), every operator $T\colon X \to Y^*$ is 2-summing, and thus p-summing, $2 \le p < \infty$.

Observation 3. If X and Y are infinite dimensional $\mathcal{L}_\infty$-spaces, then $L(X, Y^*) = \mathrm{CC}(X, Y^*)$ by [8, Theorems 3.7 and 2.17].

Corollary 7. *Let $2 \le p < \infty$. Suppose X and Y are $\mathcal{L}_\infty$-spaces and $l_1 \not\hookrightarrow X$ (or $l_1 \not\hookrightarrow Y$). If X and Y have property* RDP_p*, then $X \otimes_\pi Y$ has property* RDP_p.

Proof: Suppose $l_1 \not\hookrightarrow X$. By Observation 2, $L(X, Y^*) = \Pi_p(X, Y^*)$. By Observation 3, $L(X, Y^*) = \mathrm{CC}(X, Y^*)$. Since $l_1 \not\hookrightarrow X$, $\mathrm{CC}(X, Y^*) = K(X, Y^*)$, see [25, page 377]. Thus $L(X, Y^*) = K(X, Y^*) = \Pi_p(X, Y^*)$. By Theorem 4, $X \otimes_\pi Y$ has property RDP_p.

If $l_1 \not\hookrightarrow Y$, then the previous argument shows that $Y \otimes_\pi X$ has property RDP_p. Hence $X \otimes_\pi Y \simeq Y \otimes_\pi X$ has property RDP_p. □

Corollary 8. *Let $2 \le p < \infty$. Let $X = C(K_1)$, $Y = C(K_2)$, where K_1 and K_2 are infinite compact Hausdorff spaces and K_1 (or K_2) is dispersed. Then $X \otimes_\pi Y$ has property* RDP_p.

Proof: The $C(K)$ spaces are $\mathcal{L}_\infty$-spaces, see [4, Proposition 1.26], [8, Theorem 3.2]. Since $C(K)$ spaces have property (V), see [21], they have property RDP_p, see [18]. If K_1 (or K_2) is dispersed, then $l_1 \not\hookrightarrow C(K_1)$ (or $l_1 \not\hookrightarrow C(K_2)$), see [23]. Apply Corollary 7. □

Corollary 9. *Let $2 \le p < \infty$. Suppose X and Y are $\mathcal{L}_\infty$-spaces, $l_1 \not\hookrightarrow Y$, and Y has property* RDP_p*. Then $X^{**} \otimes_\pi Y$ has property* RDP_p.

Proof: Since X is an $\mathcal{L}_\infty$-space, X^{**} is complemented in some $C(K)$ space, see [4, Proposition 1.23]. Hence X^{**} has property (V) (since property (V) is inherited by quotients, see [21]). Then X^{**} has property RDP_p. Apply Corollary 7. □

Every $L_p(\mu)$ space is an $\mathcal{L}_p$-space, $1 \le p \le \infty$, see [8, Theorem 3.2].

Corollary 10. *Let $2 \le p < \infty$. Let X be a $C(K)$ space and $Y = l_r$, $r > 2$. Then $X \otimes_\pi Y$ has property* RDP_p.

Proof: Since X has property (V), it has property RDP_p. If q is the conjugate of r, then $1 < q < 2$. Every operator $T\colon C(K) \to l_q$, $1 < q < 2$, is compact ([27, page 100]). By Observation 2, $L(X, Y^*) = \Pi_p(X, Y^*)$. Apply Theorem 4. □

The fact that property RDP_p is inherited by quotients [18], immediately implies the following result.

Corollary 11. *Let $1 \le p < \infty$. Suppose that $L(X^*, Y^*) = K(X^*, Y^*) = \Pi_p(X^*, Y^*)$. If X^* and Y have property* RDP_p*, then the space $N_1(X, Y)$ of all nuclear operators from X to Y has property* RDP_p.

Proof: It is known that $N_1(X,Y)$ is a quotient of $X^* \otimes_\pi Y$, see [26, page 41]. By Theorem 4, $X^* \otimes_\pi Y$ has property RDP_p. Hence $N_1(X,Y)$ has property RDP_p. $\square$

Lemma 12. *Let $1 \le p < \infty$. If X has property wRDP_p, then $l_1 \overset{c}{\not\hookrightarrow} X$ and $c_0 \not\hookrightarrow X^*$.*

Proof: The identity map $i\colon l_1 \to l_1$ is completely continuous, thus p-convergent, and not weakly precompact. (Otherwise i is compact, a contradiction). Suppose l_1 has property wRDP_p. Then i^* is weakly precompact, see [18]. Thus i is weakly precompact, see [2, Corollary 2], a contradiction. Hence l_1 does not have property wRDP_p. Since property wRDP_p is inherited by quotients, it follows that if X has property wRDP_p, then $l_1 \overset{c}{\not\hookrightarrow} X$, and $c_0 \not\hookrightarrow X^*$, see [3]. $\square$

Theorem 13. *Let $1 \le p < \infty$. If $X \otimes_\pi Y$ has property RDP_p (or wRDP_p), then X and Y have property RDP_p (or wRDP_p, respectively) and at least one of them does not contain l_1.*

Proof: We only prove the result for property RDP_p. The other proof is similar. Suppose that $X \otimes_\pi Y$ has property RDP_p. Then X and Y have property RDP_p, since property RDP_p is inherited by quotients. We will show that $l_1 \not\hookrightarrow X$ or $l_1 \not\hookrightarrow Y$. Suppose that $l_1 \hookrightarrow X$ and $l_1 \hookrightarrow Y$. Hence $L_1 \hookrightarrow X^*$, see [22], [7, page 212]. Also, the Rademacher functions span l_2 inside of L_1, and thus $l_2 \hookrightarrow X^*$. Similarly $l_2 \hookrightarrow Y^*$. Then $c_0 \hookrightarrow K(X,Y^*)$, see [13], [20]. This contradiction concludes the proof. $\square$

Observation 4. If $l_1 \hookrightarrow X$ and $l_1 \hookrightarrow Y$, then $l_2 \hookrightarrow X^*$ and $l_2 \hookrightarrow Y^*$, and $c_0 \hookrightarrow K(X,Y^*)$, see [13], [20]. More generally, if $l_1 \hookrightarrow X$ and $l_p \hookrightarrow Y^*$, $p \ge 2$, then $c_0 \hookrightarrow K(X,Y^*)$, see [13], [20]. Thus $l_1 \overset{c}{\hookrightarrow} X \otimes_\pi Y$, see [3, Theorem 4], [7, Theorem 10, page 48]. Hence $X \otimes_\pi Y$ does not have property wRDP_p.

Next we present some results about the necessity of the condition $L(X,Y^*) = K(X,Y^*)$.

A separable Banach space X has an *unconditional compact expansion of the identity* (u.c.e.i) if there is a sequence (A_n) of compact operators from X to X such that $\sum A_n x$ converges unconditionally to x for all $x \in X$, see [15]. In this case, (A_n) is called an (u.c.e.i.) of X.

The space X has (Rademacher) *cotype* q for some $2 \le q \le \infty$ if there is a constant C such that for every n and every $x_1, x_2, \dots, x_n$ in X,

$$\Big(\sum_{i=1}^n \|x_i\|^q\Big)^{1/q} \le C\Big(\int_0^1 \|r_i(t)x_i\|^q \,\mathrm{d}t\Big)^{1/q},$$

where (r_n) are the Radamacher functions. A Hilbert space has cotype 2, see [7, page 118]. The dual of $C(K)$, $M(K)$, has cotype 2, see [1, page 142]. The $\mathcal{L}_p$-spaces have cotype 2, if $1 \le p \le 2$, see [7, page 118].

Observation 5. If $T\colon Y \to X^*$ be an operator such that $T^*|_X$ is compact (or weakly compact), then T is compact (or weakly compact, respectively). To see this, let $T\colon Y \to X^*$ be an operator such that $T^*|_X$ is compact (or weakly compact). Let $S = T^*|_X$. Suppose $x^{**} \in B_{X^{**}}$ and choose a net (x_α) in B_X which is w^*-convergent to x^{**}. Then $(T^*(x_\alpha)) \stackrel{w^*}{\to} T^*(x^{**})$. Now, $(T^*(x_\alpha)) \subseteq S(B_X)$, which is a relatively compact set (or relatively weakly compact). Then $(T^*(x_\alpha)) \to T^*(x^{**})$ (or $T^*(x_\alpha) \stackrel{w}{\to} T^*(x^{**})$, respectively). Hence $T^*(B_{X^{**}}) \subseteq \overline{S(B_X)}$, which is relatively compact (or relatively weakly compact, respectively). Therefore $T^*(B_{X^{**}})$ is relatively compact (or relatively weakly compact), and thus T is compact (or weakly compact, respectively). It follows that if $L(X, Y^*) = K(X, Y^*)$, then $L(Y, X^*) = K(Y, X^*)$.

Theorem 14. *Let $1 \leq p < \infty$. Assume that one of the following holds:*

(i) *If $T\colon X \to Y^*$ is an operator which is not compact, then there is a sequence (T_n) in $K(X, Y^*)$ such that for each $x \in X$, the series $\sum T_n(x)$ converges unconditionally to $T(x)$.*
(ii) *Either X^* or Y^* has an u.c.e.i.*
(iii) *The space X is an $\mathcal{L}_\infty$-space and Y^* is an $\mathcal{L}_1$-space.*
(iv) *The space $X = C(K)$, K a compact Hausdorff space, and Y^* is a space with cotype 2.*
(v) *The space X has the* DPP *and $l_1 \hookrightarrow Y$.*
(vi) *The spaces X and Y have the* DPP.

If $X \otimes_\pi Y$ has property wRDP$_p$, *then $L(X, Y^*) = K(X, Y^*)$.*

Proof: Suppose that $X \otimes_\pi Y$ has property wRDP$_p$. Then X and Y have property wRDP$_p$.

(i) Suppose $L(X, Y^*) \neq K(X, Y^*)$. Let $T\colon X \to Y^*$ be a noncompact operator. Let (T_n) be a sequence as in the hypothesis. By the uniform boundedness principle, $\left\{ \sum_{n \in A} T_n \colon A \subseteq \mathbb{N},\ A \text{ finite} \right\}$ is bounded in $K(X, Y^*)$. Then $\sum T_n$ is wuc and not unconditionally convergent (since T is noncompact). Hence $c_0 \hookrightarrow K(X, Y^*)$, see [3]. This contradiction shows that $L(X, Y^*) \neq K(X, Y^*)$.

(ii) Suppose that Y^* has an u.c.e.i. (A_n). Then $A_n\colon Y^* \to X^*$ is compact for each n and $\sum A_n y$ converges unconditionally to y for each $y \in Y^*$. Let $T\colon X \to Y^*$ be a noncompact operator. Hence $\sum A_n T(x)$ converges unconditionally to $T(x)$ for each $x \in X$ and $A_n T \in K(X, Y^*)$. Then $c_0 \hookrightarrow K(X, Y^*)$ (by (i)), a contradiction.

Similarly, if X^* has an u.c.e.i. and $L(X, Y^*) \neq K(X, Y^*)$, then $c_0 \hookrightarrow K(Y, X^*)$.

Suppose (iii) or (iv) holds. It is known that any operator $T\colon X \to Y^*$ is 2-absolutely summing, see [7, page 189], hence it factorizes through a Hilbert space. If $L(X, Y^*) \neq K(X, Y^*)$, then $c_0 \hookrightarrow K(X, Y^*)$, by [11, Remark 3], a contradiction.

(v) Suppose that X has the DPP and $l_1 \hookrightarrow Y$. By Theorem 13, $l_1 \not\hookrightarrow X$. Then X^* has the Schur property, see [6, Theorem 3]. Let $T\colon Y \to X^*$ be an operator. Then T is p-convergent (since X^* has the Schur property). Since Y has property

wRDP_p, T^* is weakly precompact, see [18]. Hence T is weakly precompact, see [2, Corollary 2]. Then T is compact, and thus $L(Y, X^*) = K(Y, X^*)$. Hence $L(X, Y^*) = K(X, Y^*)$, by Observation 5.

(vi) Suppose that X and Y have the DPP. Then $L(X, Y^*) = K(X, Y^*)$, either by (v) if $l_1 \hookrightarrow Y$, or since Y^* has the Schur property, see [6], if $l_1 \not\hookrightarrow Y$ (by an argument similar to the one in (v)). □

By Theorem 14, if one of the hypotheses (i)–(vi) holds and $L(X, Y^*) \neq K(X, Y^*)$, then $X \otimes_\pi Y$ does not have property wRDP_r, $1 \leq r < \infty$. Thus the space $l_p \otimes l_q$, where $1 < p \leq q' < \infty$ and q and q' are conjugate, does not have property wRDP_r, since the natural inclusion map $i\colon l_p \to l_{q'}$ is not compact.

The space $C(K) \otimes_\pi l_p$, with K not dispersed and $1 < p \leq 2$ does not have property wRDP_r, $1 \leq r < \infty$ (by Observation 4, since $l_1 \hookrightarrow C(K)$ and $l_2 \hookrightarrow l_p^*$).

For $1 < p_1,\ p_2 < \infty$, $L_{p_1}[0,1] \otimes_\pi L_{p_2}[0,1]$ does not have property wRDP_p, $1 \leq p < \infty$, by Lemma 12, since $l_1 \overset{c}{\hookrightarrow} L_{p_1}[0,1] \otimes_\pi L_{p_2}[0,1]$, see [26, Corollary 2.26].

References

[1] Albiac F., Kalton N. J., *Topics in Banach Space Theory*, Graduate Texts in Mathematics, 233, Springer, New York, 2006.

[2] Bator E. M., Lewis P. W., *Operators having weakly precompact adjoints*, Math. Nachr. **157** (1992), 99–103.

[3] Bessaga C., Pełczyński A., *On bases and unconditional convergence of series in Banach spaces*, Studia Math. 17 (1958), 151–164.

[4] Bourgain J., *New classes of $\mathcal{L}^p$-spaces*, Lecture Notes in Mathematics, 889, Springer, Berlin, 1981.

[5] Castillo J. M., Sanchez F., *Dunford–Pettis-like properties of continuous vector function spaces*, Rev. Mat. Univ. Complut. Madrid **6** (1993), no. 1, 43–59.

[6] Diestel J., *A survey of results related to the Dunford–Pettis property*, Proc. of the Conf. on Integration, Topology, and Geometry in Linear Spaces, Contemp. Math., 2, Amer. Math. Soc., Provicence, 1980, pages 15–60.

[7] Diestel J., *Sequences and Series in Banach Spaces*, Graduate Texts in Mathematics, 92, Springer, New York, 1984.

[8] Diestel J., Jarchow H., Tonge A., *Absolutely Summing Operators*, Cambridge Studies in Advanced Mathematics, 43, Cambridge University Press, Cambridge, 1995.

[9] Diestel J., Uhl J. J. Jr., *Vector Measures*, Mathematical Surveys, 15, American Mathematical Society, Providence, 1977.

[10] Emmanuele G., *A dual characterization of Banach spaces not containing l^1*, Bull. Polish Acad. Sci. Math. **34** (1986), no. 3–4, 155–160.

[11] Emmanuele G., *Dominated operators on $C[0,1]$ and the* (CRP), Collect. Math. **41** (1990), no. 1, 21–25.

[12] Emmanuele G., *On the reciprocal Dunford–Pettis property and projective tensor products*, Math. Proc. Cambridge Philos. Soc. **109** (1991), no. 1, 161–166.

[13] Emmanuele G., *A remark on the containment of c_0 in spaces of compact operators*, Math. Proc. Cambridge Philos. Soc. **111** (1992), no. 2, 331–335.

[14] Emmanuele G., Hensgen W., *Property* (V) *of Pelczyński in projective tensor products*, Proc. Roy. Irish Acad. Sect. A **95** (1995), no. 2, 227–231.

[15] Emmanuele G., John K., *Uncomplementability of spaces of compact operators in larger spaces of operators*, Czechoslovak Math. J. **47** (1997), no. 1, 19–31.

[16] Ghenciu I., *Property (wL) and the reciprocal Dunford–Pettis property in projective tensor products*, Comment. Math. Univ. Carolin. **56** (2015), no. 3, 319–329.

[17] Ghenciu I., *Dunford–Pettis like properties on tensor products*, Quaest. Math. **41** (2018), no. 6, 811–828.

[18] Ghenciu I., *The p-Gelfand–Phillips property in spaces of operators and Dunford–Pettis like sets*, Acta Math. Hungar. **155** (2018), 439–457.

[19] Ghenciu I., Lewis P., *The Dunford–Pettis property, the Gelfand-Phillips property, and L-sets*, Colloq. Math. **106** (2006), no. 2, 311–324.

[20] Ghenciu I., Lewis P., *The embeddability of c_0 in spaces of operators*, Bull. Pol. Acad. Sci. Math. **56** (2008), no. 3–4, 239–256.

[21] Pełczyński A., *Banach spaces on which every unconditionally converging operator is weakly compact*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **10** (1962), 641–648.

[22] Pełczyński A., *On Banach spaces containing $L_1(\mu)$*, Studia Math. **30** (1968), 231–246.

[23] Pełczyński A., Semadeni Z., *Spaces of continuous functions (III). Spaces $C(\Omega)$ for Ω without perfect subsets*, Studia Math. **18** (1959), 211–222.

[24] Pitt H. R., *A note on bilinear forms*, J. London Math. Soc. **11** (1936), no. 3, 174–180.

[25] Rosenthal H., *Point-wise compact subsets of the first Baire class*, Amer. J. Math. **99** (1977), no. 2, 362–378.

[26] Ryan R. A., *Introduction to Tensor Products of Banach Spaces*, Springer Monographs in Mathematics, Springer, London, 2002.

[27] Wojtaszczyk P., *Banach Spaces for Analysts*, Cambridge Studies in Advanced Mathematics, 25, Cambridge University Press, Cambridge, 1991.

I. Ghenciu:
Mathematics Department, University of Wisconsin-River Falls, 410 S 3rd St, River Falls, Wisconsin, 54022, U.S.A.

E-mail: ioana.ghenciu@uwrf.edu

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