A note on generalizations of semisimple modules

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Abstract. A left module M over an arbitrary ring is called an $\mathcal{RD}$-module (or an $\mathcal{RS}$-module) if every submodule N of M with $\mathrm{Rad}(M) \subseteq N$ is a direct summand of (a supplement in, respectively) M. In this paper, we investigate the various properties of $\mathcal{RD}$-modules and $\mathcal{RS}$-modules. We prove that M is an $\mathcal{RD}$-module if and only if $M = \mathrm{Rad}(M) \oplus X$, where X is semisimple. We show that a finitely generated $\mathcal{RS}$-module is semisimple. This gives us the characterization of semisimple rings in terms of $\mathcal{RS}$-modules. We completely determine the structure of these modules over Dedekind domains.

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1. Introduction

Throughout this study, all rings are associative with identity and all modules are unital left modules unless indicated otherwise. Let R be such a ring and M be an R-module. The notation $N \subseteq M$ means that N is a submodule of M. In [8, 17.1], a nonzero submodule $L \subseteq M$ is called *essential* in M, denoted as $L \trianglelefteq M$, if $L \cap N \neq 0$ for every nonzero submodule $N \subseteq M$. Dually, a proper submodule $S \subseteq M$ is called *small* in M, denoted by $S << M$, if $S + L \neq M$ for every proper submodule L of M [8, 19.1]. For a module M, $\mathrm{Rad}(M)$ (or $\mathrm{Soc}(M)$) indicates the radical (the socle, respectively) of M. The module M is said to be *radical* in case $\mathrm{Rad}(M) = M$. A nonzero module M is said to be *hollow* if every proper submodule is small in M, and it is said to be *local* if it is hollow and finitely generated. M is local if and only if it is finitely generated and $\mathrm{Rad}(M)$ is maximal. A ring R is called *local* if ${}_RR$ (or R_R) is a local module, that is, every proper submodule of ${}_RR$ is small in ${}_RR$ (see [8]).

By a *supplement* of N in M we mean a submodule K which is minimal in the collection of submodules L of M such that $M = N + L$. It is well known that K is a supplement of N in M if and only if $M = N + K$ and $N \cap K << K$. Clearly, every direct summand is a supplement (see [8, Section 41]). A module M is said to be *supplemented* if every submodule of M has a supplement in M, and it is said to be *strongly radical supplemented* if every submodule $N \subseteq M$ with $\mathrm{Rad}(M) \subseteq N$ has a supplement in M (see [3] and [8, page 349]). Every semisimple module is supplemented, and supplemented modules are strongly radical supplemented.

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It is shown in [5, Lemma 2.13] that a module M is semisimple if and only if every submodule of M is a supplement in the module M. Motivated by this characterization and the concept of strongly radical supplemented modules, we say that M is an $\mathcal{RS}$*-module* if every submodule N of M with $\mathrm{Rad}(M) \subseteq N$ is a supplement in M. Note that semisimple modules are $\mathcal{RS}$-modules. Also, an $\mathcal{RS}$-module which has zero radical is semisimple.

In [2], a module M is said to be a *md-module* (or *ms-module*) if every maximal submodule of M is a direct summand of (supplement in, respectively) M. We also say that a module M is an $\mathcal{RD}$*-module* if every submodule N of M with $\mathrm{Rad}(M) \subseteq N$ is a direct summand of M. It is clear that $\mathcal{RS}$-modules are *ms*-modules. Obviously, the same relationship between $\mathcal{RD}$-modules and *md*-modules also holds.

In this study, we investigate the various properties of $\mathcal{RD}$-modules and $\mathcal{RS}$-modules. A module M is an $\mathcal{RD}$-module if and only if $M = \mathrm{Rad}(M) \oplus X$, where X is a semisimple submodule of M. We prove that $\mathcal{RS}$-modules are semilocal. We show that a finitely generated $\mathcal{RS}$-module is semisimple. This gives us the characterization of semisimple rings in terms of $\mathcal{RS}$-modules. We determine the structure of $\mathcal{RS}$-modules over Dedekind domains, and using this we show that $\mathcal{RD}$-modules and $\mathcal{RS}$-modules coincide. We also prove that, over a Dedekind domain, every $\mathcal{RS}$-module is strongly $\oplus$-radical supplemented.

2. $\mathcal{RD}$-modules and $\mathcal{RS}$-modules

Let M be a module. By $\mathcal{P}(M)$ we will denote the sum of all radical submodules of M. Note that $\mathcal{P}(M)$ is the largest radical submodule of M and $\mathcal{P}(M) \subseteq \mathrm{Rad}(M)$.

Lemma 2.1. *A sum $\mathcal{P}(M)$ is an $\mathcal{RD}$-module for every module M.*

Proof: Since $\mathcal{P}(M)$ is a radical module, it suffices to prove that any radical module is an $\mathcal{RD}$-module. Let Y be radical, that is, $Y = \mathrm{Rad}(Y)$. Since Y has the trivial decomposition, it follows that Y is an $\mathcal{RD}$-module. □

In general it is not true that every $\mathcal{RD}$-module (consequently, every $\mathcal{RS}$-module) is semisimple. Consider the $\mathbb{Z}$-module $M =_{\mathbb{Z}} \mathbb{Q}$. Since M is injective, it is radical. By Lemma 2.1, M is an $\mathcal{RD}$-module. On the other hand, M is not semisimple.

The following Theorem gives the characterization of the radical of an $\mathcal{RS}$-module.

Theorem 2.2. *Let M be an $\mathcal{RS}$-module and* $\mathrm{Rad}(M) \subseteq N \subseteq M$. *Then,* $\mathrm{Rad}(N) = \mathcal{P}(M)$.

Proof: By the hypothesis, N is a supplement in M. In particular, $\mathrm{Rad}(M)$ is a supplement of some submodule K of M, that is, $M = K + \mathrm{Rad}(M)$ and $K \cap \mathrm{Rad}(M) << \mathrm{Rad}(M)$. Therefore, $\mathrm{Rad}(\mathrm{Rad}(M)) = \mathrm{Rad}(M) \cap \mathrm{Rad}(M) =$

$\mathrm{Rad}(M)$ according to [5, Theorem 2.3]. So $\mathrm{Rad}(M)$ is radical. Since $\mathcal{P}(M)$ is the largest radical submodule of M, we get $\mathrm{Rad}(M) = \mathcal{P}(M)$.

Now again applying [5, Theorem 2.3], we obtain that $\mathrm{Rad}(N) = N \cap \mathrm{Rad}(M) = \mathrm{Rad}(M) = \mathcal{P}(M)$. □

Recall that a module M is *reduced* provided $\mathcal{P}(M) = 0$, that is, every nonzero submodule of M has a maximal submodule. Note that for any module M since $\mathcal{P}(M/\mathcal{P}(M)) = 0$, the factor module $M/\mathcal{P}(M)$ is reduced.

Proposition 2.3. *Let M be a reduced module. Then, the following are equivalent:*

(1) *A module M is an $\mathcal{RD}$-module.*
(2) *A module M is an $\mathcal{RS}$-module.*
(3) *A module M is semisimple.*

PROOF: (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (1) are clear.

(2) $\Longrightarrow$ (3) Let M be an $\mathcal{RS}$-module. It follows from Theorem 2.2 that $\mathrm{Rad}(M) = \mathcal{P}(M)$. Then, since M is reduced, we get $\mathrm{Rad}(M) = 0$. So, by the hypothesis, every submodule of M is a supplement in M. Thus, M is semisimple by [5, Lemma 2.13]. □

Let R be a ring. A ring R is said to be a *left max ring* if every nonzero left R-module has maximal submodules. Left perfect rings (over which every module has a projective cover, see [8, 43.9]) and left V-rings (whose simple modules are injective, see [8, Section 23]) are left max rings.

Corollary 2.4. *Let R be a left max ring and M be a nonzero left R-module. If M is an $\mathcal{RS}$-module, then it is semisimple.*

PROOF: Let M be an $\mathcal{RS}$-module. Since R is a left max ring, every left R-module has a maximal submodule. So M is reduced. Applying Proposition 2.3, we have M is semisimple. □

Lemma 2.5. *Every factor module of an $\mathcal{RS}$-module is an $\mathcal{RS}$-module.*

PROOF: Let M be an $\mathcal{RS}$-module. For $\mathrm{Rad}(M/L) \subseteq N/L \subseteq M/L$ modules and the canonical projection $\Phi\colon M \longrightarrow M/L$, we have $\Phi(\mathrm{Rad}(M)) = (\mathrm{Rad}(M) + L)/L \subseteq \mathrm{Rad}(M/L) \subseteq N/L$ and so $\mathrm{Rad}(M) \subseteq N$. Since M is an $\mathcal{RS}$-module, we can write $M = U + N$ and $U \cap N << N$ for some submodule $U \subseteq M$. Therefore, $M/L = (U + L)/L + N/L$. Note that, by [8, 19.3 (4)],

$$\Phi(U \cap N) = \frac{U \cap N + L}{L} = \frac{(U + L) \cap N}{L} = \frac{U + L}{L} \cap \frac{N}{L} << \frac{N}{L}.$$

This means that N/L is a supplement in M/L. Hence, M/L is an $\mathcal{RS}$-module. □

Now we obtain the following result, which is crucial for our work.

Proposition 2.6. *For every $\mathcal{RS}$-module M, $M/\mathcal{P}(M)$ is semisimple.*

Proof: By Lemma 2.5, we deduce that $M/\mathcal{P}(M)$ is an $\mathcal{RS}$-module as a factor module of the $\mathcal{RS}$-module M. Since $M/\mathcal{P}(M)$ is reduced, it follows from Proposition 2.3 that $M/\mathcal{P}(M)$ is semisimple. □

Let M be a module and U, V be submodules of M. If $M = U + V$ and $U \cap V << M$, then V is said to be a *weak supplement* in M. The module M is said to be *weakly supplemented* (shortly, say, ws-*module*) if every submodule N of M is a weak supplement in M, it is said to be *weakly radical supplemented* (or briefly, wrs-*module*) provided every submodule N of M with $\mathrm{Rad}(M) \subseteq N$ is a weak supplement in M. For the properties and characterizations of wrs-modules, see the paper [7]. Clearly, every $\mathcal{RS}$-module is a wrs-module because supplements are weak supplements.

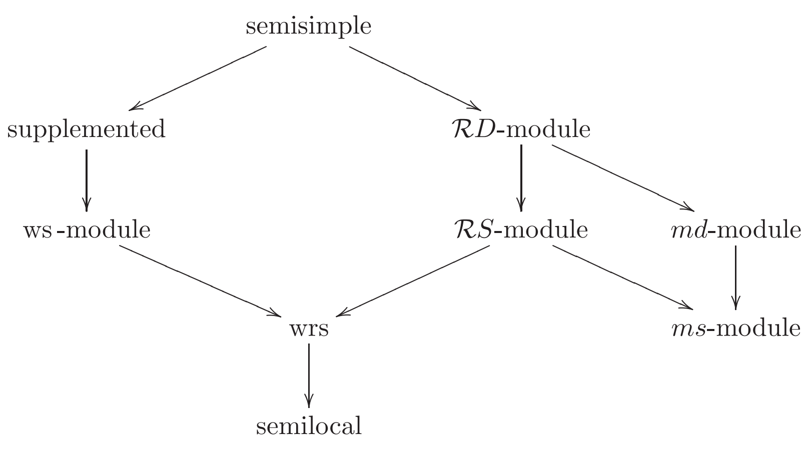
In [4], over an arbitrary ring a module M is said to be *semilocal* if $M/\mathrm{Rad}(M)$ is semisimple. It is shown in [4, Proposition 2.1] that a module M is semilocal if and only if every submodule K of M is a weak Rad-supplement in M, that is, $M = N + K$ and $N \cap K \subseteq \mathrm{Rad}(M)$ for some submodule N of M. Equivalently, N is a weak Rad-supplement in M whenever $\mathrm{Rad}(M) \subseteq N \subseteq M$. By [7, Corollary 2.10], semilocal modules are proper generalizations of wrs-modules, and so $\mathcal{RS}$-modules are clearly semilocal.

Now using Theorem 2.2 and Proposition 2.6 we prove the following fact, that is, $\mathcal{RS}$-modules are contained in the class of semilocal modules.

Corollary 2.7. *Every $\mathcal{RS}$-module is semilocal.*

Proof: Let M be an $\mathcal{RS}$-module. It follows from Theorem 2.2 that $\mathrm{Rad}(M) = \mathcal{P}(M)$. Then, by Proposition 2.6, we obtain that $M/\mathrm{Rad}(M) = M/\mathcal{P}(M)$ is semisimple. This means that M is semilocal. □

Now, we have the following implications between the classes of modules:



The class of $\mathcal{RD}$-modules over an arbitrary ring will be characterized in the following theorem which is frequently used in this study.

Theorem 2.8. *For a module M over an arbitrary ring, the following three statements are equivalent:*

(1) *A module M is an $\mathcal{RD}$-module.*
(2) *$M = \mathrm{Rad}(M) \oplus X$, where X is a semisimple submodule of M.*
(3) *$M = \mathcal{P}(M) \oplus X$, where X is a semisimple submodule of M.*

Proof: (1) $\Longrightarrow$ (2) (1) implies that $\mathrm{Rad}(M)$ is a direct summand of M. So we can write the decomposition $M = \mathrm{Rad}(M) \oplus X$ for some submodule X of M. Since all $\mathcal{RD}$-modules are a $\mathcal{RS}$-module, it follows from Theorem 2.2 and Proposition 2.6 that X is semisimple.

(2) $\Longrightarrow$ (3) Let $M = \mathrm{Rad}(M) \oplus X$, where X is semisimple. By [8, 21.6 (5)], we can write $\mathrm{Rad}(M) = \mathrm{Rad}(\mathrm{Rad}(M)) \oplus \mathrm{Rad}(X) = \mathrm{Rad}(\mathrm{Rad}(M)) \oplus 0 = \mathrm{Rad}(\mathrm{Rad}(M))$, and so $\mathrm{Rad}(M)$ is radical. Therefore, $\mathcal{P}(M) = \mathrm{Rad}(M)$ because $\mathcal{P}(M)$ is the largest radical submodule of M.

(3) $\Longrightarrow$ (1) Let $\mathrm{Rad}(M) \subseteq N \subseteq M$. Since X is semisimple, the intersection $X \cap N$ is a direct summand of X. Therefore, we can write $X = (X \cap N) \oplus Y$ for some submodule $Y \subseteq X$. Now $M = \mathcal{P}(M) \oplus X = N + X = N + (X \cap N + Y) = N + Y$. Note that $N \cap Y = N \cap (X \cap Y) = (X \cap N) \cap Y = 0$, thus the sum $N + Y$ is direct. Hence, M is an $\mathcal{RD}$-module. □

Proposition 2.9. *Every finitely generated $\mathcal{RS}$-module is semisimple.*

Proof: Let M be a finitely generated $\mathcal{RS}$-module. By Theorem 2.2, the radical module $\mathcal{P}(M)$ is a supplement in M. Since M is finitely generated, it follows from [8, 41.1 (2)] that $\mathcal{P}(M)$ is finitely generated. It means that $\mathcal{P}(M) = 0$. By Proposition 2.3, we get M is semisimple. □

Now we give the closure properties of $\mathcal{RD}$-modules in the following proposition.

Proposition 2.10. (1) *If a module M is an $\mathcal{RD}$-module, then so is every factor module.*

(2) *Let $M = \sum_{i \in I} M_i$, where each M_i is an $\mathcal{RD}$-module for any index set I. Then, M is an $\mathcal{RD}$-module.*

(3) *Every submodule U of an $\mathcal{RD}$-module M with $\mathrm{Soc}(M) \subseteq U$ is an $\mathcal{RD}$-module.*

(4) *A nonzero projective $\mathcal{RD}$-module M is semisimple.*

(5) *A finitely generated $\mathcal{RD}$-module is semisimple.*

Proof: (1) Let M be an $\mathcal{RD}$-module and $U \subseteq M$. If the factor module M/U of M is radical, then it follows from Lemma 2.1 that it is an $\mathcal{RD}$-module. Let $\mathrm{Rad}(M/U) \neq M/U$ and $\mathrm{Rad}(M/U) \subseteq N/U \subseteq M/U$. By the first part in proof of Lemma 2.5, we have $\mathrm{Rad}(M) \subseteq N$. Since M is an $\mathcal{RD}$-module, M has the decomposition $M = N \oplus K$ for some submodule $K \subseteq M$. Therefore, $M/U = N/U + U + K/U$. Now, $N/U \cap (U + K)/U = N \cap (U + K)/U =$

$(U + N \cap K)/U = 0$. So $M/U = N/U \oplus (U + K)/U$. It means that M/U is an $\mathcal{RD}$-module.

(2) Let $M = \sum_{i \in I} M_i$, where each M_i is an $\mathcal{RD}$-module for any index set I. Now, we consider the external direct sum $M' = \bigoplus_{i \in I} M_i$. So there exists an epimorphism $\Psi \colon M' \longrightarrow M$ via $\Psi((m_i)_{i \in I}) = \sum_{i \in I_0} m_i$, where I_0 is a finite subset of the index set I. By (1), it suffices to show that M' is an $\mathcal{RD}$-module. Applying Theorem 2.8, we obtain that $M_i = \mathrm{Rad}(M_i) \oplus X_i$ where each X_i is a semisimple submodule of M_i for every $i \in I$. Put $X = \bigoplus_{i \in I} X_i$. Therefore, X is semisimple as the direct sum of semisimple modules X_i. It follows from [8, 21.6 (5)] that $M' = \bigoplus_{i \in I} M_i = \bigoplus_{i \in I} (\mathrm{Rad}(M_i) \oplus X_i) = \mathrm{Rad}(M') + X$. It can be seen that the sum $\mathrm{Rad}(M') + X$ is direct. Hence, applying Theorem 2.8 twice, M' is an $\mathcal{RD}$-module.

(3) Let M be an $\mathcal{RD}$-module. Then, it follows from Theorem 2.8 that we have $M = \mathcal{P}(M) \oplus X$, where X is a semisimple submodule of M. Since U contains $\mathrm{Soc}(M)$, by the modular law, we can write $N = N \cap M = N \cap (\mathcal{P}(M) \oplus X) = N \cap \mathcal{P}(M) \oplus X = \mathcal{P}(N) \oplus X$. Hence, again applying Theorem 2.8, N is an $\mathcal{RD}$-module.

(4) By [8, 22.3 (2)] and Theorem 2.8, we get $\mathcal{P}(M) = 0$. Hence, M is semisimple.

(5) It follows from Proposition 2.9. □

Proposition 2.11. *Every finite sum of $\mathcal{RS}$-submodules of a module M is an $\mathcal{RS}$-module.*

Proof: The proof is standard. □

It is known that a ring R is semisimple if and only if every left R-module is semisimple. Now we generalize this fact in the next theorem, characterizing the rings over which modules are $\mathcal{RS}$-modules.

Theorem 2.12. *Let R be a ring. Then, R is semisimple if and only if every left R-module is an $\mathcal{RS}$-module.*

Proof: ($\Longrightarrow$) It is clear.

($\Longleftarrow$) By the assumption, we obtain that ${}_RR$ is an $\mathcal{RS}$-module. It follows from Proposition 2.10 that ${}_RR$ is semisimple. Hence, R is semisimple. □

We get the following Corollary.

Corollary 2.13. *The following statements are equivalent for a ring R.*

(1) *A ring R is semisimple.*
(2) *Every left R-module is an $\mathcal{RD}$-module.*
(3) *Every left R-module is an $\mathcal{RS}$-module.*

Proof: (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) are clear, and (3) $\Longrightarrow$ (1) follows from Theorem 2.12. □

3. $\mathcal{RS}$-modules over commutative domains

In this section, we shall consider commutative domains, and determine the structure of $\mathcal{RS}$-modules over these domains. In particular, we show that $\mathcal{RD}$-modules and $\mathcal{RS}$-modules coincide.

Let R be a commutative domain and M be an R-module. We denote by $\mathrm{Tor}(M)$ the set of all elements m of M for which there exists a nonzero element r of R such that $rm = 0$, i.e. $\mathrm{Ann}(m) \neq 0$. Then $\mathrm{Tor}(M)$, which is a submodule of M, is called the *torsion submodule* of M. If $M = \mathrm{Tor}(M)$, then M is called the *torsion module* and M is called *torsion-free* provided $\mathrm{Tor}(M) = 0$. Note that $\mathrm{Tor}(M/\mathrm{Tor}(M)) = 0$ for every module M over a commutative domain R.

Proposition 3.1. *Let R be a commutative domain which is not a field and M be a torsion-free R-module. Then, M is an $\mathcal{RD}$-module if and only if it is radical.*

Proof: ($\Longrightarrow$) Let M be an $\mathcal{RD}$-module. It follows from Theorem 2.8 that $M = \mathrm{Rad}(M) \oplus X$ for some semisimple submodule X of M. Note that $X \subseteq \mathrm{Tor}(M)$, and by the assumption, we obtain $X = 0$. Hence, M is radical.

($\Longleftarrow$) It follows from Lemma 2.1. □

Let R be a Dedekind domain and M be a left R-module. Since R is a Dedekind domain, by [1, Lemma 4.4], $\mathcal{P}(M)$ is (divisible) injective and so there exists a submodule A of M such that $M = \mathcal{P}(M) \oplus A$. By Lemma 2.1, we obtain that $\mathcal{P}(M)$ is an $\mathcal{RD}$-module. Using these facts, we have the following result, showing that over Dedekind domains $\mathcal{RD}$-modules and $\mathcal{RS}$-modules coincide.

Theorem 3.2. *Let R be a Dedekind domain and M be a left R-module. Then, the following statements are equivalent:*

(1) *A module M is an $\mathcal{RD}$-module.*
(2) *A module M is an $\mathcal{RS}$-module.*
(3) *A module M is a direct sum of a divisible R-module and a semisimple R-module.*

Proof: (1) $\Longrightarrow$ (2) is clear.

(2) $\Longrightarrow$ (3) Let $M = \mathcal{P}(M) \oplus A$ for some submodule A of M. Since M is an $\mathcal{RS}$-module, by Proposition 2.6, A is semisimple. This completes the proof of (2) $\Longrightarrow$ (3).

(3) $\Longrightarrow$ (1) Let $M = D \oplus A$, where D is a divisible submodule and A is a semisimple submodule of M. By [1, Lemma 4.4] and Lemma 2.1, the divisible module D is an $\mathcal{RD}$-module. Hence, M is an $\mathcal{RD}$-module as the direct sum of $\mathcal{RD}$-modules by (2) of Proposition 2.10. □

Note that, by Theorem 3.2 and [2, Proposition 6.4 and Theorem 6.11], we have the following strict containments of classes of modules:

$$\{\text{semisimple modules}\} \subset \{\mathcal{RD}\text{-modules}\} \subset \{md\text{-modules}\}$$

A module M is said to be *strongly $\oplus$-radical supplemented* if every submodule N of M containing the radical has a supplement that is a direct summand of M

(see [6]). It is clear that every $\mathcal{RD}$-module is strongly $\oplus$-radical supplemented. Using this fact and Theorem 3.2, we get this result:

Corollary 3.3. *Every $\mathcal{RS}$-module over a Dedekind domain is strongly $\oplus$-radical supplemented.*

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