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An alternative proof of the uniqueness of martingale-coboundary decomposition of strictly stationary processes

Takehiko Morita

Abstract. P. Samek and D. Volný, in the paper "Uniqueness of a martingale-coboundary decomposition of a stationary processes" (1992), showed the uniqueness of martingale-coboundary decomposition of strictly stationary processes. The original proof is given by reducing the problem to the ergodic case. In this note we give another proof without such reduction.

Keywords: strictly stationary process; martingale-coboundary decomposition

Classification: 28D05, 60G10, 60G42

1. Introduction

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let T be an invertible measure-preserving transformation on $(\Omega, \mathcal{A}, P)$. We are interested in a strictly stationary process $\{X_n\}_{n\in\mathbb{Z}}$ defined by $X_n = f \circ T^n$ for $n \in \mathbb{Z}$, where f is a real-valued measurable function. Given a sub σ-algebra $\mathcal{M}$ of $\mathcal{A}$ satisfying $\mathcal{M} \subset T^{-1}\mathcal{M}$, we consider the filtration $\mathbb{M} = \{\mathcal{M}_n\}_{n\in\mathbb{Z}}$ with $\mathcal{M}_n = T^{-n}\mathcal{M}$ for $n \in \mathbb{Z}$. We say that the process $\{X_n\}_{n\in\mathbb{Z}}$ has a martingale-coboundary decomposition with respect to the filtration $\mathbb{M}$ if there exist a real-valued measurable function g and an integrable function m such that the equation $f = m + g - g \circ T$ holds a.s. (almost surely) and $\{m \circ T^n\}_{n\in\mathbb{Z}}$ is a sequence of martingale differences with respect to the filtration $\mathbb{M}$, i.e. $m \circ T^n$ is $\mathcal{M}_{n+1}$-measurable and the conditional expectation $E[m \circ T^n \,|\, \mathcal{M}_n]$ of $m \circ T^n$ given $\mathcal{M}_n$ vanishes a.s. for each $n \in \mathbb{Z}$. Note that in our situation it is enough to assume that m is $\mathcal{M}_1$-measurable and $E[m \,|\, \mathcal{M}] = 0$ a.s. P. Samek and D. Volný proved the following theorem in [3].

Theorem 1.1. *Let f and $\mathbb{M}$ be as above. Suppose that there exist real-valued measurable functions g_1, g_2 and integrable functions m_1, m_2 such that*

$$f = m_1 + g_1 - g_1 \circ T \ \text{ a.s.} \qquad \text{and} \qquad f = m_2 + g_2 - g_2 \circ T \ \text{ a.s.}$$

hold and $\{m_1 \circ T^n\}_{n\in\mathbb{Z}}$ and $\{m_2 \circ T^n\}_{n\in\mathbb{Z}}$ are sequences of martingale differences with respect to the filtration $\mathbb{M}$. Then we have

$$m_1 = m_2 \ \text{ a.s.} \qquad \text{and} \qquad (g_1 - g_2) \circ T = g_1 - g_2 \ \text{ a.s.}$$

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In [3] the theorem is proved by reducing it to the case when T is ergodic. So their method needs to translate the problem to an appropriately chosen factor of T having the ergodic decomposition, i.e. to a factor defined on a space admitting the regular conditional probabilities. The purpose of this note is to give an alternative proof of Theorem 1.1 which is free from the method of ergodic decomposition.

As mentioned in [3] necessary and sufficient conditions for the existence of such a decomposition are discussed in [4] in connection with the central limit theorem for stationary processes. D. Volný in [4] studied the case when both f and g are integrable as well as the case when both f and g are square-integrable in the decomposition. We note that the point of the uniqueness result in [3] is that it is given without integrability of g. In fact, under the integrability of g, one can find that the proof of Theorem 1.1 is a sort of exercise of Birkhoff's ergodic theorem and Doob's martingale convergence theorem.

2. Lemmas

We start with introducing the notation and some well known facts which we use frequently. As references we give [1] and [5]. For A, $B \in \mathcal{A}$, if $P(A \setminus B) = 0$, we write as $A \subset B$ a.s. If both $A \subset B$ a.s. and $B \subset A$ a.s. hold, i.e. $P(A\Delta B) = 0$, we write as $A = B$ a.s. For an integrable function h, we put $h^* = \limsup_{n\to\infty}(1/n) \times \sum_{k=0}^{n-1} h \circ T^k$. By the Birkhoff individual ergodic theorem, we know that h^* is integrable, $h^* = \lim_{n\to\infty}(1/n)\sum_{k=0}^{n-1} h \circ T^k$ a.s., $h^* \circ T = h^*$ a.s., and $Eh = Eh^*$, where $E[X]$ denotes the expectation of a random variable X with respect to P (for simplicity we often write it as EX). For the later convenience we note that $h^* = E[h \,|\, \mathcal{I}]$ a.s., where $\mathcal{I}$ denotes the sub σ-algebra of $\mathcal{A}$ consisting of all the elements $A \in \mathcal{A}$ satisfying $T^{-1}A = A$. Indeed, h^* has a version which is $\mathcal{I}$-measurable. In addition, if $A \in \mathcal{I}$, we have $\int_A h^* \,\mathrm{d}P = \lim_{n\to\infty}(1/n)\sum_{k=0}^{n-1}\int_A h \circ T^k \,\mathrm{d}P = \lim_{n\to\infty}(1/n)\sum_{k=0}^{n-1}\int_{T^{-k}A} h \circ T^k \,\mathrm{d}P = \int_A h \,\mathrm{d}P$ since T preserves P and $A = T^{-1}A$.

Throughout the paper we use shorthand notation $(X \in B)$ to denote the set $\{\omega \in \Omega\colon X(\omega) \in B\}$ for a random variable X and a set $B \in \mathcal{A}$. For $A \in \mathcal{A}$, we put $A^* = (I(A)^* > 0)$, where $I(A)$ is the indicator of the set A. Clearly, A^* satisfies $A^* = T^{-1}A^*$ a.s. Finally, for random variables $Y_1, \dots, Y_n$ and $a, b \in \mathbb{R}$ with $a < b$, we denote by $H_n(a, b; Y_1, \dots, Y_n)$ the upcrossing number from a to b of $Y_1, Y_2, \dots, Y_n$.

The following is crucial in our argument.

Lemma 2.1. *Let g be a real-valued measurable function. If $P((g \neq g \circ T)) > 0$, we can find s, $r \in \mathbb{R}$ with $s < r$ such that $P((g < s)^* \cap (g \geq r)) > 0$.*

Proof: Assume that $P((g \neq g\circ T)) > 0$. First we show that there exists an $r \in \mathbb{R}$ such that $P((g < r)^* \cap (g \geq r)) > 0$. For if $P((g < r)^* \cap (g \geq r)) = 0$ holds for every $r \in \mathbb{R}$, we see that $g = g\circ T$ a.s. as follows. By definition $0 \leq I((g < r))^* \leq 1$

and $(g < r)^* = (I((g < r))^* > 0)$. Therefore the Birkhoff ergodic theorem yields

$$P((g < r)) = \int_\Omega I((g < r))^* \, \mathrm{d}P = \int_{(g<r)^*} I((g < r))^* \, \mathrm{d}P \le P((g < r)^*). \tag{2.1}$$

On the other hand $P((g < r)^* \cap (g \ge r)) = 0$ implies that $(g < r)^* \subset (g < r)$ a.s. Therefore we have $(g < r)^* = (g < r)$ a.s. by the inequality (2.1). Thus we obtain $(g < r) = T^{-1}(g < r)$ a.s. since $I((g < r))^* = I((g < r))^* \circ T$ a.s. holds. Hence we see that $P((g < r)^* \cap (g \ge r)) = 0$ for every $r \in \mathbb{R}$ implies that $(g < r) = T^{-1}(g < r)(= (g \circ T < r))$ a.s. for every $r \in \mathbb{R}$. By taking the complement we also have $(g \ge r) = (g \circ T \ge r)$ a.s. for every $r \in \mathbb{R}$. Therefore we see $g = g \circ T$ a.s. as follows. For $r \in \mathbb{R}$ we have $\emptyset = (g < r \le g) = (g \circ T < r \le g) = (g < r \le g \circ T)$ a.s. This yields that $P((g < g \circ T)) = P\big(\bigcup_{r \in \mathbb{Q}} (g < r \le g \circ T)\big) = 0$ and $P((g \circ T < g)) = P\big(\bigcup_{r \in \mathbb{Q}} (g \circ T < r \le g)\big) = 0$. Consequently, we arrive at $g = g \circ T$ a.s.

Now we can take $r \in \mathbb{R}$ satisfying $P((g < r)^* \cap (g \ge r)) > 0$. Note that $(g < r - 1/n) \uparrow (g < r)$ as $n \uparrow \infty$. Since $I((g < r - 1/n))^*$ is increasing a.s. as $n \uparrow \infty$ and $E[I((g < r - 1/n))^*] = P((g < r - 1/n)) \uparrow P((g < r)) = E[I((g < r))^*]$ as $n \uparrow \infty$, it is easy to see that $I((g < r - 1/n))^* \uparrow I((g < r))^* = \sup_{k \in \mathbb{N}} I((g < r - 1/k))^*$ a.s. as $n \uparrow \infty$. Thus we have $(g < r)^* = (I((g < r))^* > 0) = (\sup_{n \in \mathbb{N}} I((g < r - 1/n))^* > 0) = \bigcup_{n \in \mathbb{N}} (I(g < r - 1/n))^* > 0) = \bigcup_{n \in \mathbb{N}} (g < r - 1/n)^*$ a.s. Obviously this yields $(g < r)^* \cap (g \ge r) = \bigcup_{n \in \mathbb{N}} (g < r - 1/n)^* \cap (g \ge r)$ a.s. Thus we can find an $N \in \mathbb{N}$ satisfying $P((g < r - 1/N)^* \cap (g \ge r)) > 0$. We obtain the desired result with $s = r - 1/N$. □

The next lemma is a modification of Lemma in [3], which does work without assuming the ergodicity of T.

Lemma 2.2. *Let g be a real-valued measurable function. If $P((g \ne g \circ T)) > 0$, we can find a, $b \in \mathbb{R}$ with $a < b$ such that the upcrossing number $H_n(a, b) = H_n(a, b; g - g \circ T, \dots, g - g \circ T^n)$ from a to b of $g - g \circ T, \dots, g - g \circ T^n$ satisfies*

$$\limsup_{n \to \infty} \frac{1}{n} E H_n(a, b) > 0.$$

Proof: By Lemma 2.1 we can take s, $r \in \mathbb{R}$ with $s < r$ such that $P((g < s)^* \cap (g \ge r)) > 0$. Therefore we can find an interval $[u, v) \subset [r, \infty)$ such that $P((g < s)^* \cap (u \le g < v)) > 0$. We note that we can choose the interval small enough that the following can hold.

$$a = v - u < r - s, \qquad \text{consequently} \quad b = u - s \ge r - s > a. \tag{2.2}$$

Since $(g < s)^* = (I((g < s))^* > 0)$, there exist $N \in \mathbb{N}$ and $C \subset (g < s)^* \cap (u \le g \le v)$ such that the following hold.

$$P(C) > 0 \qquad \text{and} \qquad \sum_{k=1}^{N} I((g < s)) \circ T^k \ge 1. \tag{2.3}$$

From now we follow the argument in [3] for a while. By using (2.2) and (2.3) we see that

$$
\begin{aligned}
C \cap T^{-nN} C &\subset \left(u \le g < v,\ u \le g \circ T^{nN} < v,\ \text{and } \sum_{k=1}^{N} I((g < s)) \circ T^{nN+k} \ge 1\right) \\
&\subset (u \le g < v,\ u \le g \circ T^{nN}, \quad \text{and} \quad g \circ T^{nN+k} < s \quad \text{for some } k,\ 1 \le k \le N) \\
&\subset (g - g \circ T^{nN} < a, \quad \text{and} \quad g - g \circ T^{nN+k} > b \quad \text{for some } k,\ 1 \le k \le N) \\
&\subset (H_{(n+1)N}(a,b) \ge H_{nN}(a,b) + 1).
\end{aligned}
$$

holds for $n \ge 1$. Therefore we have

$$
\sum_{k=1}^{n} I(C \cap T^{-nN} C) \le \sum_{k=1}^{n} I((H_{(k+1)N}(a,b) \ge H_{kN}(a,b) + 1)) \le H_{(n+1)N}(a,b).
$$

By dividing each side by n after integration, we obtain

$$
\frac{1}{n} \sum_{k=1}^{n} P((C \cap T^{-nN} C)) \le \frac{1}{n} E H_{(n+1)N}(a,b). \tag{2.4}
$$

On the other hand by the ergodic theorem and the Schwartz inequality we have

$$
\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P((C \cap T^{-nN} C)) &= \int_\Omega I(C) \cdot E[I(C) \,|\, \mathcal{I}_N] \, \mathrm{d}P = \int_\Omega (E[I(C) \,|\, \mathcal{I}_N])^2 \, \mathrm{d}P \\
&\ge (E[E[I(C) \,|\, \mathcal{I}_N]])^2 = (E[I(C)])^2 = P(C)^2 > 0,
\end{aligned}
$$

where $\mathcal{I}_N$ is the sub σ-algebra of $\mathcal{A}$ consisting of all the elements $A \in \mathcal{A}$ satisfying $T^{-N} A = A$. Combining this with the inequality (2.4), we have

$$
\limsup_{n\to\infty} \frac{1}{n} E H_n(a,b) \ge \frac{P(C)^2}{N} > 0.
$$

Now the proof of Lemma 2.2 is complete. □

3. Proof of Theorem 1.1

Let g_1, g_2 and m_1, m_2 be functions satisfying the assumptions in Theorem 1.1. By considering the real-valued measurable function $g = g_1 - g_2$ and the integrable function $m = m_1 - m_2$, it suffices to show the following.

Proposition 3.1. *Let g be a real-valued measurable function and let m be an integrable function. Suppose that*

$$
m = g - g \circ T \quad \text{a.s.}
$$

holds and $\{m \circ T^n\}_{n\in\mathbb{Z}}$ *is a sequence of martingale differences with respect to* $\mathbb{M}$. *Then* $g = g \circ T$ *a.s. and consequently,* $m = 0$ *a.s.*

Proof: Consider the process $\{M_n\}_{n=1}^{\infty}$ defined by

$$M_n = \sum_{k=0}^{n-1} m \circ T^k = \sum_{k=1}^{n} (g - g \circ T^k) = g - g \circ T^n.$$

Clearly $\{M_n\}_{n=1}^{\infty}$ is a martingale and $\{m \circ T^n\}_{n=0}^{\infty}$ is a uniformly integrable sequence of martingale differences. Therefore by Theorem 2.22 on page 42 of [2], we have

$$\lim_{n\to\infty} \frac{1}{n} E|M_n| = 0.$$

Next, for any $a,\ b \in \mathbb{R}$ with $a < b$ we consider the upcrossing number $H_n(a,b) = H_n(a,b;M_1,\dots,M_n)$ from a to b of $M_1,\dots,M_n$. Then Doob's upcrossing inequality yields

$$EH_n(a,b) \le \frac{E[(M_n - a)^+]}{b-a} \le \frac{E|M_n| + |a|}{b-a}.$$

Thus we obtain

$$\frac{1}{n} EH_n(a,b) \le \frac{E|M_n| + |a|}{(b-a)n} \to 0, \qquad n \to \infty,$$

for any a, $b \in \mathbb{R}$ with $a < b$.

Therefore if we notice that $M_n = g - g \circ T^n$ for each $n \ge 1$, we see that it is impossible to find a, $b \in \mathbb{R}$ with $a < b$ such that $\limsup_{n\to\infty} (1/n) EH_n(a,b) = 0$ holds. Hence by Lemma 2.2 we conclude that $P((g \ne g \circ T)) = 0$. □

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T. Morita:

Department of Mathematics, Graduate School of Science, Osaka University, 1-1 Machikaneyama, Toyonaka, Osaka 560–0043, Japan

E-mail: take@math.sci.osaka-u.ac.jp

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