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Strong functors on many-sorted sets

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Dedicated to the memory of Věra Trnková

Abstract. We show that, on a category of many-sorted sets, the only functors that admit a cartesian strength are those that are given componentwise.

Keywords: strong functor; strong monad; many-sorted set

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1. Introduction

This note is inspired by Věra Trnková's pioneering work on set functors [4], and her later work with J. Adámek on functors on many-sorted sets [1].

Strong functors and strong monads arise in many settings. For example, in computer science they are used in denotational semantics [3], as well as programming languages such as Haskell.

On **Set**, the situation is extremely straightforward: every endofunctor admits a unique strength, so the notions of endofunctor and strong endofunctor coincide.

Our task is to examine the situation for the category $\mathbf{Set}^I$, where I is a set. This category is known as "many-sorted sets". It turns out that the situation here is very different. The only endofunctors on this category that are strong are those that (up to isomorphism) are of the form $\prod_{i\in I} H_i$ for a family $(H_i)_{i\in I}$ of endofunctors on **Set**. For example, the monad on $\mathbf{Set}^2$ generated by a unary operation from sort 0 to sort 1 sends (X,Y) to $(X, Y+X)$, and this (even as a mere endofunctor) is not strong.

We first review the basic facts about strong functors in Section 2. Then in Section 3 we see how, on a product category $\prod_{i\in I}\mathcal{C}_i$, under suitable assumptions, the notion of strong functor reduces to that of strong functor on the categories $(\mathcal{C}_i)_{i\in I}$. This gives the desired result for $\mathbf{Set}^I$.

2. Strong functors

We begin with the basic definition of strength.

Definition 1 ([2])**.** Let $\mathcal{C}$ be a monoidal category.

(1) A *strong endofunctor* H on $\mathcal{C}$ (more precisely: left strong) is an endofunctor equipped with a *strength*, i.e. a family of maps $t_{a,b}\colon a\otimes Hb \longrightarrow H(a\otimes b)$

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natural in $a, b \in \mathcal{C}$ and satisfying

$$\begin{array}{ccc}
Ha & \xleftarrow{\quad H\lambda_a \quad} & H(1 \otimes a) \\
{\scriptstyle \lambda_a}\uparrow & \nearrow{\scriptstyle t_{1,a}} & \\
1 \otimes Ha & &
\end{array}$$

$$\begin{array}{ccc}
(a \otimes b) \otimes Hc & \xrightarrow{\qquad t_{a\otimes b,c} \qquad} & H((a \otimes b) \otimes c) \\
{\scriptstyle \alpha_{a,b,Hc}}\downarrow & & \downarrow{\scriptstyle H\alpha_{a,b,c}} \\
a \otimes (b \otimes Hc) \xrightarrow[a\otimes t_{b,c}]{} a \otimes H(b \otimes c) & \xrightarrow[t_{a,b\otimes c}]{} & H(a \otimes (b \otimes c))
\end{array}$$

(2) The identity strong endofunctor on $\mathcal{C}$ is the identity endofunctor with strength given at $a, b \in \mathcal{C}$ by $\mathsf{id}_{a\otimes b}$.

(3) The *composite* (in diagrammatic order) of strong endofunctors (H, s) and (K, t) on $\mathcal{C}$ is the endofunctor KH with strength given at $a, b \in \mathcal{C}$ by

$$a \otimes KHb \xrightarrow{t_{a,Hb}} K(a \otimes Hb) \xrightarrow{Ks_{a,b}} KH(a \otimes b).$$

(4) For strong endofunctors (H, s) and (K, t) on $\mathcal{C}$, a natural transformation $\gamma\colon H \longrightarrow K$ is *strong* when

$$\begin{array}{ccc}
a \otimes Hb & \xrightarrow{s_{a,b}} & H(a \otimes b) \\
{\scriptstyle a\otimes\gamma_b}\downarrow & & \downarrow{\scriptstyle \gamma_{a\otimes b}} \\
a \otimes Kb & \xrightarrow{t_{a,b}} & K(a \otimes b)
\end{array}$$

Definition 2.

(1) Let $\mathcal{C}$ be a category. The category[1] of endofunctors on $\mathcal{C}$ and natural transformations, strict monoidal via composition (diagrammatic order, let us say), is written $\mathsf{Endo}(\mathcal{C})$. A *monad* on $\mathcal{C}$ is a monoid in $\mathsf{Endo}(\mathcal{C})$, and a *monad morphism* is a monoid morphism.

(2) Let $\mathcal{C}$ be a monoidal category. The category of strong endofunctors and strong natural transformations, strict monoidal via composition (diagrammatic order), is written $\mathsf{StrEndo}(\mathcal{C})$. A *strong monad* on $\mathcal{C}$ is a monoid in $\mathsf{StrEndo}(\mathcal{C})$, and a strong monad morphism is a monoid morphism.

Definition 3. A monoidal category $\mathcal{C}$ is said to be *strength-compliant* when every natural transformation between strong endofunctors is strong, i.e. the forgetful

[1] In a suitably large sense.

functor

$$\mathcal{U}_{\mathcal{C}} \colon \mathsf{StrEndo}(\mathcal{C}) \longrightarrow \mathsf{Endo}(\mathcal{C})$$

is fully faithful.

Proposition 4. *Let $\mathcal{C}$ be a monoidal category. If $\mathcal{C}$ is strength-compliant, then any endofunctor on $\mathcal{C}$ admits at most one strength.*

Proof: Let H be an endofunctor with strengths s and t. Then id_H is a strong natural transformation $(H, s) \longrightarrow (H, t)$, so $s = t$. □

A category $\mathcal{C}$ with a terminal object is *well-pointed* when the functor M: $\mathcal{C} \longrightarrow \mathbf{Set}$ sending X to $\mathcal{C}(1, X)$ is faithful. The following is adapted from [3, Proposition 3.4], with the same proof.

Proposition 5. *Let $\mathcal{C}$ be a cartesian category that is well-pointed.*

(1) *The category $\mathcal{C}$ is strength-compliant.*
(2) *An endofunctor H on $\mathcal{C}$ is strong if and only if for all $a, b \in \mathcal{C}$ the function*

$$\begin{aligned} \tilde{t}_{a,b} \colon\ M(a \times Fb) &\longrightarrow MF(a \times b) \\ \langle x, y \rangle &\mapsto \left(1 \xrightarrow{\ y\ } Fb \xrightarrow{F\langle\langle\rangle; x, \mathsf{id}_b\rangle} F(a \times b) \right) \end{aligned}$$

is in the range of M, and then $t_{a,b}$ is the preimage of $\tilde{t}_{a,b}$.

Proposition 6. **Set***, with cartesian structure, is strength-compliant, and any endofunctor on it admits a unique strength.*

3. Product categories

We turn now to endofunctors on product categories.

Lemma 7. *Let $(\mathcal{C}_i)_{i\in I}$ be a family of categories with an initial object. The functor*

$$\begin{aligned} \prod_{i\in I} \mathsf{Endo}(\mathcal{C}_i) &\longrightarrow \mathsf{Endo}\Big(\prod_{i\in I} \mathcal{C}_i\Big), \\ (H_i)_{i\in I} &\mapsto \prod_{i\in I} H_i \end{aligned}$$

is a coreflective embedding, i.e. fully faithful with a right adjoint.

Proof: Recall that for any adjunction

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \quad (F \dashv G)$$

the left adjoint F is fully faithful if and only if the unit is an isomorphism.

Let $j \in I$. We first define the coreflection

$$\mathcal{C}_j \underset{\mathsf{Pad}_j}{\overset{\pi_j}{\rightleftarrows}} \prod_{i \in I} \mathcal{C}_i \quad (\top)$$

as follows.

- π_j is the projection $b \mapsto b_j$.
- Pad_j sends $a \in \mathcal{C}_j$ to the tuple whose ith component is a if $i = j$ and 0 otherwise.
- The unit sends $a \in \mathcal{C}_j$ to id_a.
- The counit ε_j sends $b \in \prod_{i \in I} \mathcal{C}_i$ to the map $\mathsf{Pad}_j \pi_j b \longrightarrow b$ whose ith component is id_{b_j} if $i = j$ and $[\,]\colon 0 \longrightarrow b_i$ otherwise.

This gives rise to the coreflection

$$[\mathcal{C}_j, \mathcal{C}_j] \underset{-.\mathsf{Pad}_j}{\overset{-.\pi_j}{\rightleftarrows}} \Big[\prod_{i \in I} \mathcal{C}_i, \mathcal{C}_j\Big]. \quad (\bot)$$

Taking $\prod_{j \in I}$ of the above gives the desired coreflection

$$\begin{array}{ccc}
\prod_{j \in I} [\mathcal{C}_j, \mathcal{C}_j] & \underset{\prod_{j \in I}(-.\mathsf{Pad}_j)}{\overset{\prod_{j \in I}(-.\pi_j)}{\rightleftarrows}} \ (\bot) & \prod_{j \in I} \Big[\prod_{i \in I} \mathcal{C}_i, \mathcal{C}_j\Big] \\
\Big\| = & & \Big| \cong \\
\prod_{i \in I} \mathsf{Endo}(\mathcal{C}_i) & \underset{(\pi_i.-.\mathsf{Pad}_j)_{i \in I}}{\overset{\prod}{\rightleftarrows}} \ (\bot) & \mathsf{Endo}\big(\prod_{i \in I} \mathcal{C}_i\big)
\end{array}$$

where the unit is identity, and the counit sends $G \in \mathsf{Endo}\big(\prod_{i \in I} \mathcal{C}_i\big)$ to the natural transformation whose bth component for $b \in \prod_{i \in I} \mathcal{C}_i$ is

$$\big(G_i \varepsilon_{i,b} \colon G_i \mathsf{Pad}_i b_i \longrightarrow G_i b\big)_{i \in I}$$

writing $G_i \stackrel{\text{def}}{=} \pi_i G$. □

In a monoidal category $\mathcal{C}$, an object 0 is *right-distributive initial* when $0 \otimes a$ is initial for all $a \in \mathcal{C}$. This implies that 0 is initial.

Proposition 8. *Let $(\mathcal{C}_i)_{i\in I}$ be a family of monoidal categories with a right-distributive initial object. The functor*

$$\begin{aligned}\prod_{i\in I}\mathsf{StrEndo}(\mathcal{C}_i) &\longrightarrow \mathsf{StrEndo}\Big(\prod_{i\in I}\mathcal{C}_i\Big),\\ (H_i)_{i\in I} &\mapsto \prod_{i\in I} H_i\end{aligned}$$

is an equivalence.

Proof: Firstly, for $j\in I$ we define a lift

$$\begin{array}{ccc}\mathsf{StrEndo}\big(\prod_{i\in I}\mathcal{C}_i\big) & \xrightarrow{\ \mathcal{Q}_j\ } & \mathsf{StrEndo}(\mathcal{C}_j)\\ {\scriptstyle \mathcal{U}_{\prod_{i\in I}\mathcal{C}_i}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{U}_{\mathcal{C}_j}}\\ \mathsf{Endo}\big(\prod_{i\in I}\mathcal{C}_i\big) & \xrightarrow{\ \pi_j.-.\mathsf{Pad}_j\ } & \mathsf{Endo}(\mathcal{C}_j)\end{array}$$

It sends (G,t) to the functor $a\mapsto G_j(\mathsf{Pad}_j a)$ with strength given at a,b by

$$\begin{aligned}&a\otimes G_j\mathsf{Pad}_j b =\\ &(\mathsf{Pad}_j a\otimes G\mathsf{Pad}_j b)_j \xrightarrow{(t_{\mathsf{Pad}_j a,\mathsf{Pad}_j b})_j} G_j(\mathsf{Pad}_j a\otimes \mathsf{Pad}_j b)\xrightarrow{G_j m^j_{a,b}} G_j\mathsf{Pad}_j(a\otimes b)\end{aligned}$$

where we write

$$m^j_{a,b}\colon\ \mathsf{Pad}_j a\otimes \mathsf{Pad}_j b\cong \mathsf{Pad}_j(a\otimes b)$$

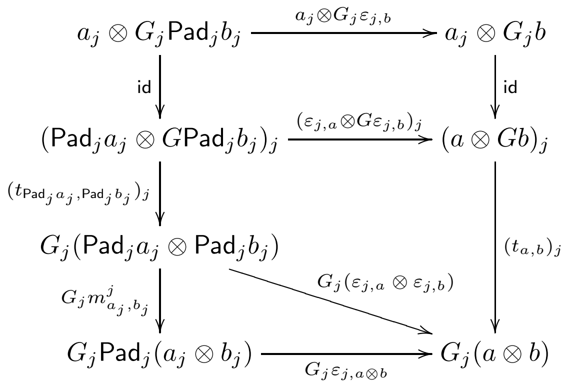
for the isomorphism whose ith component is $\mathsf{id}_{a\otimes b}$ if $i=j$ and the unique map $0\otimes 0\longrightarrow 0$ otherwise.

We then obtain a lift of the coreflection in Lemma 7:

$$\begin{array}{ccc}\prod_{i\in I}\mathsf{StrEndo}(\mathcal{C}_i) & \underset{(\mathcal{Q}_i)_{i\in I}}{\overset{\prod}{\rightleftarrows}}\ \bot & \mathsf{StrEndo}\big(\prod_{i\in I}\mathcal{C}_i\big)\\ {\scriptstyle \prod_{i\in I}\mathcal{U}_{\mathcal{C}_i}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{U}_{\prod_{i\in I}\mathcal{C}_i}}\\ \prod_{i\in I}\mathsf{Endo}(\mathcal{C}_i) & \underset{(\pi_i.-.\mathsf{Pad}_i)_{i\in I}}{\overset{\prod}{\rightleftarrows}}\ \bot & \mathsf{Endo}\big(\prod_{i\in I}\mathcal{C}_i\big)\end{array}$$

by verifying that the components of the unit and counit of the upper adjunction are strong natural transformations, as follows. The unit is just the identity. The counit at $G\in\mathsf{StrEndo}\big(\prod_{i\in I}\mathcal{C}_i\big)$ is strong because of the following diagram for

$a, b \in \prod_{i \in I} \mathcal{C}_i$ and $j \in I$.



It remains to prove that the counit is an isomorphism. That means we must show for $(G, t) \in \mathsf{StrEndo}\left(\prod_{i \in I} \mathcal{C}_i\right)$ and $b \in \prod_{i \in I} \mathcal{C}_i$ and $j \in I$ that the map

$$G_j \varepsilon_{j,b} \colon G_j \mathsf{Pad}_j b_j \longrightarrow G_j b$$

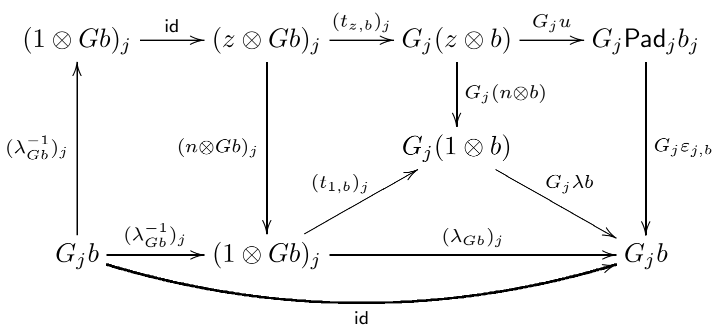
is an isomorphism. We write

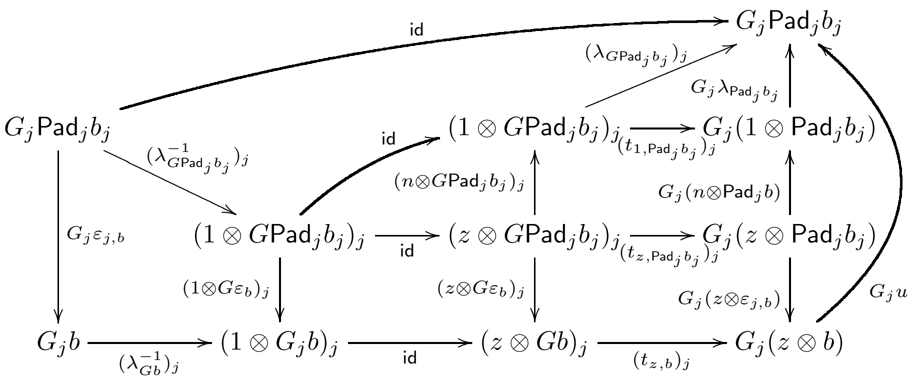
- z for the object in $\prod_{i \in I} \mathcal{C}_i$ whose ith component is 1 for $i = j$ and 0 otherwise
- $n \colon z \longrightarrow 1$ for the map whose ith component is identity for $i = j$ and the unique map $0 \longrightarrow 1$ otherwise
- $u \colon z \otimes b \cong \mathsf{Pad}_j b_j$ for the map whose ith component is λ_{b_j} for $i = j$ and the unique map $0 \otimes b_i \longrightarrow 0$ otherwise.

Then the inverse of $G_j \varepsilon_{j,b}$ is

$$G_j b \xrightarrow{(\lambda^{-1}_{Gb})_j} (1 \otimes Gb)_j \xrightarrow{\mathsf{id}} (z \otimes Gb)_j \xrightarrow{(t_{z,b})_j} G_j(z \otimes b) \xrightarrow{G_j u} G_j \mathsf{Pad}_j b_j$$

as shown by the following commutative diagrams:





□

Corollary 9. *Let $(\mathcal{C}_i)_{i\in I}$ be a family of monoidal categories that are strength-compliant and have a right-distributive initial object.*

(1) *$\prod_{i\in I}\mathcal{C}_i$ is strength-compliant and has a right-distributive initial object.*
(2) *An endofunctor on $\prod_{i\in I}\mathcal{C}_i$ is strong if and only if it is isomorphic to $\prod_{i\in I}H_i$ for some $H\in\prod_{i\in I}\mathsf{StrEndo}(\mathcal{C}_i)$.*

Proof:

(1) Consider the commutative square of strict monoidal functors:

$$\begin{array}{ccc}
\prod_{i\in I}\mathsf{StrEndo}(\mathcal{C}_i) & \xrightarrow{\quad\prod\quad} & \mathsf{StrEndo}\big(\prod_{i\in I}\mathcal{C}_i\big) \\
{\scriptstyle \prod_{i\in I}\mathcal{U}_{\mathcal{C}_i}}\downarrow & & \downarrow{\scriptstyle \mathcal{U}_{\prod_{i\in I}\mathcal{C}_i}} \\
\prod_{i\in I}\mathsf{Endo}(\mathcal{C}_i) & \xrightarrow[\quad\prod\quad]{} & \mathsf{Endo}\big(\prod_{i\in I}\mathcal{C}_i\big)
\end{array}$$

The left and lower functors are fully faithful, and the upper functor is an equivalence, so the right functor is fully faithful.

(2) From Proposition 8.

□

Theorem 10.

(1) *$\mathbf{Set}^I$ with cartesian structure is strength-compliant.*
(2) *An endofunctor on $\mathbf{Set}^I$ is strong if and only if it is isomorphic to $\prod_{i\in I}H_i$ for some family $(H_i)_{i\in I}$ of endofunctors on $\mathbf{Set}$.*

Proof:

(1) From Proposition 6 and Corollary 9 (1).
(2) From Proposition 6 and Corollary 9 (2).

□

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