Comment.Math.Univ.Carolin. 60,4 (2019) 581–604

Vector product and composition algebras in braided monoidal additive categories

Ross Street

In respectful memory of Věra Trnková

Abstract. This is an account of some work of Markus Rost and his students Dominik Boos and Susanne Maurer. It concerns the possible dimensions for composition (also called Hurwitz) algebras. We adapt the work to the braided monoidal setting.

Keywords: string diagram; vector product; bilinear form; braiding; monoidal dual

Classification: 18D10, 15A03, 17A75, 11E20

Introduction

This is a fairly detailed account of some work of M. Rost [9] and his students D. Boos [2] and S. Maurer [8]. I am grateful to J. Baez for alerting us to this material in a seminar at Macquarie University on 22 January 2003. The stimulus for revisiting this material was an invitation to participate in the Workshop on Diagrammatic Reasoning in Higher Education during November 2018 in Newcastle, New South Wales[1]. It seemed a great opportunity to demonstrate the joy and power of string diagrams for proving substantial algebraic facts.

The contribution of the present paper is to adapt and generalize the ideas of M. Rost [9] to the braided monoidal additive setting and to keep the diagrams closer to those of A. Joyal and the author in [5], [6]. In this context, the goals are to prove that there are not many dimensions in which vector product algebras can exist and that the category of vector product algebras is equivalent to the category of composition algebras. These structures have their origins in A. Hurwitz' papers [3], [4]. Recent relevant work includes J. C. Baez [1] and B. W. Westbury [10]. Background category theory can be found in S. Mac Lane [7].

DOI 10.14712/1213-7243.2019.024

The author gratefully acknowledges the support of Australian Research Council Discovery Grant DP160101519.

[1]See `https://carma.newcastle.edu.au/meetings/drhe/`.

1. Axiomatics

Definition 1. A *vector product algebra (vpa)* over a commutative ring R is an R-module V equipped with a symmetric non-degenerate bilinear form $\bullet\colon V \otimes V \to R$ and a linear map $\wedge\colon V \otimes V \to V$ such that

$$x \wedge y = -y \wedge x, \tag{1.1}$$

$$(x \wedge y) \bullet z = (z \wedge x) \bullet y, \tag{1.2}$$

$$(x \wedge y) \wedge z + x \wedge (y \wedge z) = 2(x \bullet z)y - (x \bullet y)z - (y \bullet z)x. \tag{1.3}$$

We call "$\bullet$" the inner or dot product and "$\wedge$" the exterior or vector product. Condition (1.1) expresses the antisymmetry of "$\wedge$" and (1.2) the cyclic symmetry; together they mean that $(x \wedge y) \bullet z$ is an alternating function of the variables. Condition (1.3) may seem strange or unfamiliar; however, we have the following observation.

Proposition 1. *Assume* $2 = 1 + 1$ *is cancellable in* R. *Then*

(a) (1.1) *is equivalent to* (1.4),

$$x \wedge x = 0; \tag{1.4}$$

(b) (1.2) *is equivalent to* (1.5),

$$(x \wedge y) \bullet z = x \bullet (y \wedge z); \tag{1.5}$$

(c) *in the presence of* (1.1), *equation* (1.3) *is equivalent to* (1.6),

$$(x \wedge y) \wedge x = (x \bullet x)y - (x \bullet y)x. \tag{1.6}$$

Proof: (a) Put $x = y$ in (1.1) to obtain $2x \wedge x = 0$; then cancel the 2. Conversely, apply (1.4) to $x + y$ in place of x, use linearity, and apply (1.4) twice more to obtain $x \wedge y + y \wedge x = 0$.
(b) Using (1.2) twice, we have $(x \wedge y) \bullet z = (z \wedge x) \bullet y = (y \wedge z) \bullet x = x \bullet (y \wedge z)$. Conversely, using (1.5) at the second step, $(z \wedge x) \bullet y = y \bullet (z \wedge x) = (y \wedge z) \bullet x = x \bullet (y \wedge z)$.
(c) Substitute $y = x$ in (1.3) to obtain $0 + x \wedge (x \wedge z) = 2(x \bullet z)x - (x \bullet x)z - (x \bullet z)x = (x \bullet z)x - (x \bullet x)z$; but $x \wedge (x \wedge z) = -(x \wedge z) \wedge x$ by (1.1), yielding (1.6). Conversely, replace x by $x + z$ in (1.6) then x by $x - z$ in (1.6) and subtract the two results. We obtain twice (1.3); cancel the 2. □

Remark. In other words, for such a ring, (1.4), (1.5), (1.6) can be taken as alternative axioms for a vpa. Notice that Properties (1.4) and (1.6) involve expressions in which terms have variables repeated. This causes a problem for defining the concept in a monoidal category since the diagonal function $V \to V \otimes V$, $x \mapsto x \otimes x$ is not linear. The reason (1.6) should seem more familiar than (1.3) is that as undergraduates we learn the property

$$(x \wedge y) \wedge z = (x \bullet z)y - (x \bullet y)z \tag{1.7}$$

for vectors in $\mathbb{R}^3$. This condition (1.7) does not have repeated variables. Note that (1.6) is a special case of (1.7). Moreover, (1.1) and (1.7) imply (1.3) over any commutative ring.

Definition 2.
- (com) A vpa V is called *commutative* when the wedge operation is zero: for all $x, y \in V$, $x \wedge y = 0$.
- (ass) A vpa V is called *associative* when (1.7) holds.

Remark. The words "commutative" and "associative" do not mean the operations of V have these properties in the usual sense. They apply to the following multiplication on $R \oplus V$:

$$(\alpha, x)(\beta, y) = (\alpha\beta - x \bullet y, \alpha y + \beta x + x \wedge y). \tag{1.8}$$

Indeed, (1.8) is the formula for multiplication of quaternions when $V = \mathbb{R}^3$ with usual inner and vector products.

2. Monoidal concepts

For a commutative ring R, the category Mod_R of R-modules (called R-vector spaces when R is a field) and linear functions becomes a monoidal category with tensor product given by tensoring over R; so, up to isomorphism, R is the unit object for tensoring.

Let $\mathscr{V}$ denote any monoidal category in which the tensor product functor is denoted by $\otimes \colon \mathscr{V} \times \mathscr{V} \to \mathscr{V}$ and the unit object by I. Here is the notion of non-degeneracy for a morphism $A \otimes B \to I$ in $\mathscr{V}$. A little later we will consider the notion of symmetry for such a morphism when $A = B$.

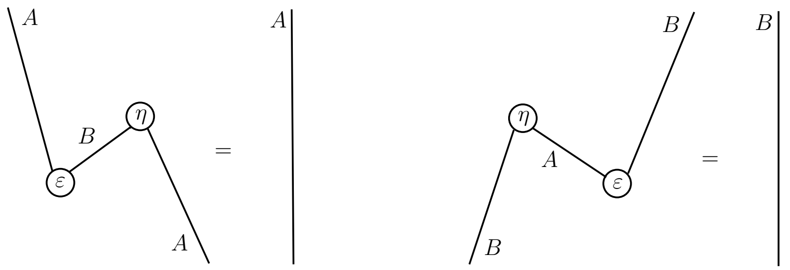
Definition 3. A morphism $\varepsilon \colon A \otimes B \to I$ is called a *counit* for an adjunction (or duality) $A \dashv B$ when, for all objects X and Y, the function

$$\mathscr{V}(X, Y \otimes A) \;\to\; \mathscr{V}(X \otimes B, Y), \tag{2.1}$$

sending $X \xrightarrow{f} Y \otimes A$ to the composite $X \otimes B \xrightarrow{f \otimes 1_B} Y \otimes A \otimes B \xrightarrow{1_Y \otimes \varepsilon} Y$, is a bijection. Taking $X = I$ and $Y = B$, we obtain a unique morphism $\eta \colon I \to B \otimes A$, called the *unit* of the adjunction, which maps to the identity of B under the function (2.1).

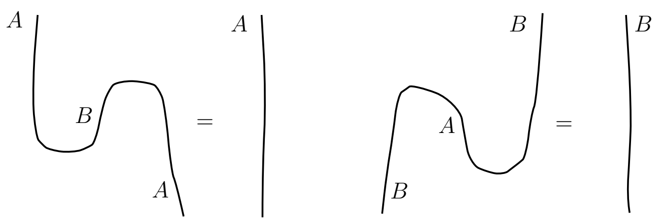
Proposition 2. *A morphism $\varepsilon \colon A \otimes B \to I$ is a counit for an adjunction $A \dashv B$ if and only if there exists a morphism $\eta \colon I \to B \otimes A$ satisfying the two equations* *depicted in* (2.2).

(2.2)



When there is no ambiguity, we denote counits by cups ∪ and units by caps ∩. So (2.1) becomes the more geometrically "obvious" operation of pulling the ends of the strings as in (2.3). These are sometimes called the *snake equations*.

(2.3)

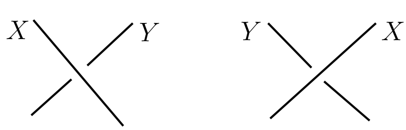


Now suppose the monoidal category is *braided*. Then we have isomorphisms

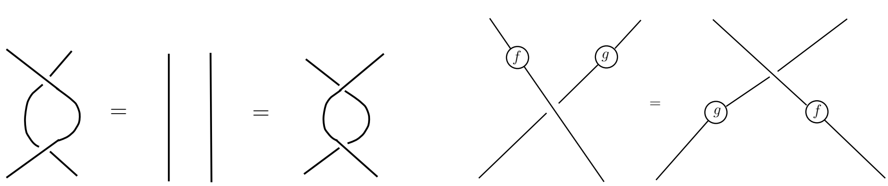
(2.4) $$c_{X,Y} \colon X \otimes Y \longrightarrow Y \otimes X$$

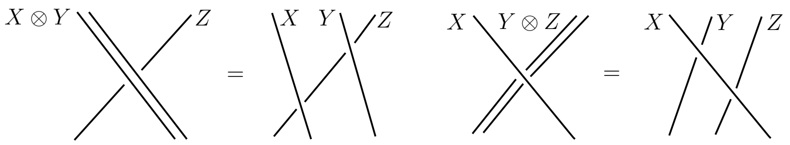
which we depict by a left-over-right crossing of strings in three dimensions; the inverse is a right-over-left crossing.

(2.5)

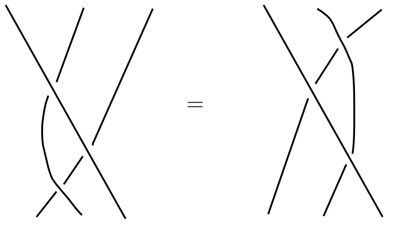


The braiding axioms reinforce the view that it behaves like a crossing.





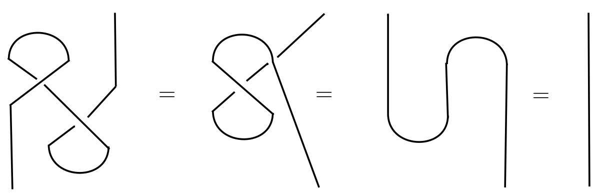
The following Reidemeister move or Yang–Baxter equation is a consequence.



We will refer to these properties as *the geometry of braiding* as justified by [5].

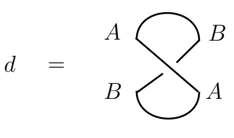
Proposition 3. *If $\mathscr{V}$ is braided and $A \dashv B$ with counit and unit depicted by $\cup$ and $\cap$ then $B \dashv A$ with counit and unit depicted by*

Proof: One of the snake equations is proved by the calculation



while the other is dual. □

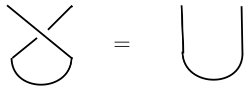
Definition 4. Objects with duals have *dimension*: if $A \dashv B$ then the dimension $d = d_A$ of A is the following element of the commutative ring $\mathscr{V}(I, I)$.



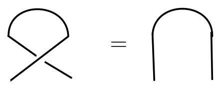
Definition 5. A morphism $g\colon A \otimes A \to X$ is called *symmetric* when

$$A \otimes A \xrightarrow{c_{A,A}} A \otimes A \xrightarrow{g} X = A \otimes A \xrightarrow{g} X.$$

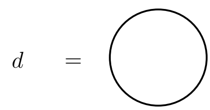
(Clearly the condition is equivalent to using the inverse braiding in place of the braiding.) In particular, a self-duality $A \dashv A$ with counit $\cup$ is called *symmetric* when



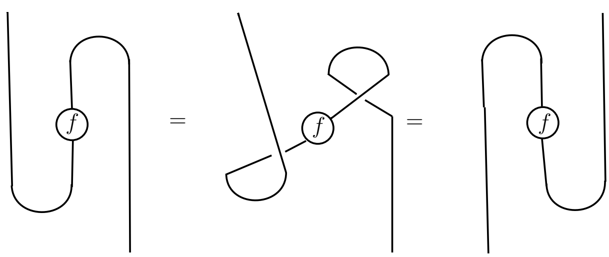
By Proposition 3 and uniqueness of units, it follows that



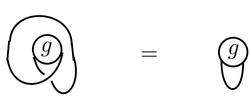
For a symmetric self-dual object A, the dimension is simply



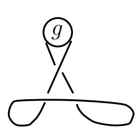
Proposition 4. *If $A \dashv A$ and $B \dashv B$ are symmetric self-dualities and $f \colon A \to B$ is a morphism then*



Proposition 5. *If $A \dashv A$ is a symmetric self-duality and $g \colon I \to A \otimes A$ is a morphism then*

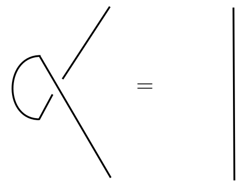


Proof: Both sides are equal to:

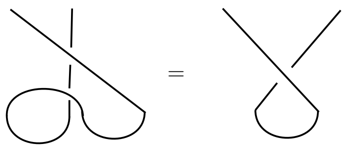


□

Proposition 6. *If $A \dashv A$ is a symmetric self-duality then the following Reidemeister move holds*



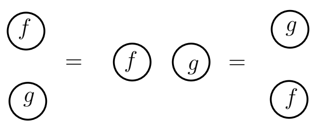
Proof: By dragging the bottom strings to the right and up over the top string we see that the proposition is the same as



However, applying the snake equation and the geometry of braiding to the left-hand side, we are left with the expression for symmetry of the counit. □

Proposition 7. *If $\mathscr{V}$ is a braided monoidal category then the set $\mathscr{V}(I, I)$ of endomorphisms of the tensor unit I is a commutative monoid under composition. In fact, composition agrees with tensor for such endomorphisms.*

Proof:



□

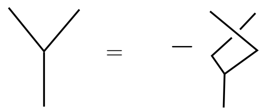
If $\mathscr{V}$ is a braided monoidal *additive* category then there is an addition in each hom set $\mathscr{V}(X, Y)$ such that

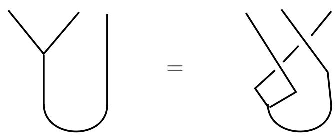
$$\begin{aligned} h \circ 0 \circ k &= 0, & h \circ (f+g) \circ k &= h \circ f \circ k + h \circ g \circ k \ , \\ U \otimes 0 \otimes V &= 0, & U \otimes (f+g) \otimes V &= U \otimes f \otimes V + U \otimes g \otimes V \ . \end{aligned} \tag{2.6}$$

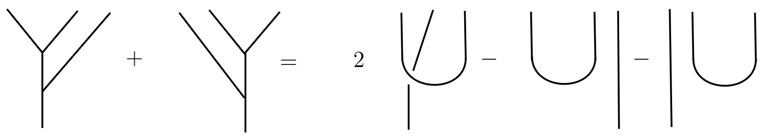
When ambiguity is possible, we sometimes write $0_{X,Y}$ for the zero 0 in the abelian group $\mathscr{V}(X, Y)$. In particular, $\mathscr{V}(I, I)$ is a commutative ring; the multiplicative identity is depicted by an empty string diagram $\emptyset$ as distinct from the zero 0_{II}.

3. Scarcity of vector product algebras

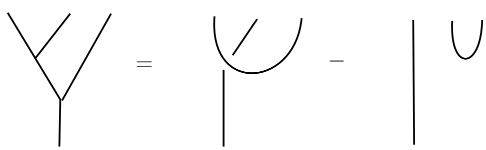
Definition 6. A *vector product algebra (vpa)* in a braided monoidal additive category $\mathscr{V}$ is an object V equipped with a symmetric self-duality $V \dashv V$ (depicted by a cup ∪) and a morphism $\wedge\colon V\otimes V\to V$ (depicted by a Y) such that the following three conditions hold.

(3.1) 

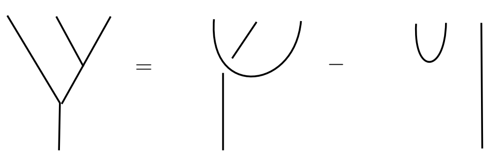
(3.2) 

(3.3) 

A vpa is *associative* when it satisfies

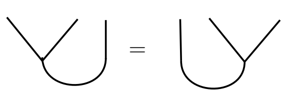
(3.4) 

Using (3.1) and (3.2), we see that (3.4) is equivalent to (3.5).

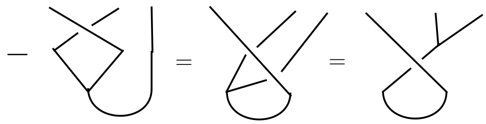
(3.5) 

By adding (3.4) and (3.5) we obtain (3.3). So (3.3) is redundant in the definition of associative vpa.

Proposition 8. *The following is a consequence of* (3.1) *and* (3.2).

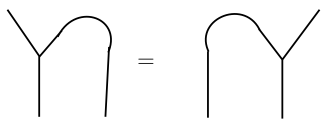


Proof: Using (3.1) and (3.2) for the first equality below then the geometry of braiding for the second, we have

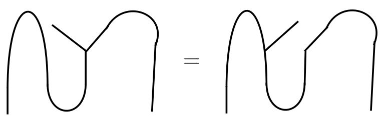


However, the left-hand side is equal to the left-hand side of the equation in the proposition by (3.1) while the right-hand sides are equal by symmetry of inner product $\cup$. □

Corollary 9.

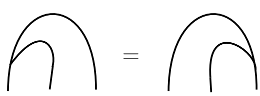


Proof: Proposition 8 yields

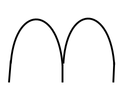


and the snake equations (2.3) yield the result. □

Corollary 10.

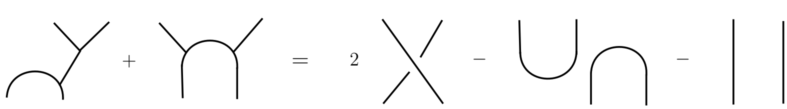


Proof: Using Corollary 9, we see that both sides are equal to



and we have the result. □

Corollary 11.



Proof: Attach a $\cap$ on the right input strings in all five terms of (3.4) and apply Corollary 9 to the two terms on the left-hand side. □

Corollary 12. *Condition* (3.3) *is equivalent to the following equation.*

Proof: Attach a $\cup$ on the right side of the output strings of all five terms of (3.3). The right-hand side is what we want. On the left-hand side apply symmetry of "•" to the first term, and the alternating form axioms (3.1) and (1.2) to the second term. Each step can be reversed. □

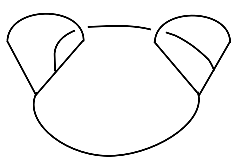
Theorem 13. *For any associative vector product algebra V in any braided monoidal additive category $\mathscr{V}$, the dimension $d = d_V$ satisfies the equation*

$$d(d-1)(d-3) = 0$$

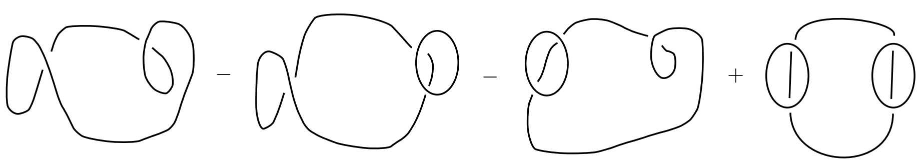
in the endomorphism ring $\mathscr{V}(I, I)$ of the tensor unit I.

Proof: We perform two string calculations each beginning with the following element of $\mathscr{V}(I, I)$.

(3.6)



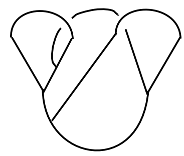
Using (3.5) twice and (2.6), we obtain



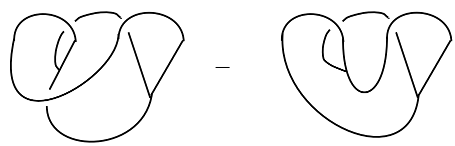
in which, using Proposition 6 and the geometry of braiding, each term reduced to a union of disjoint circles:

$$d - dd - dd + ddd = d(d-1)^2.$$

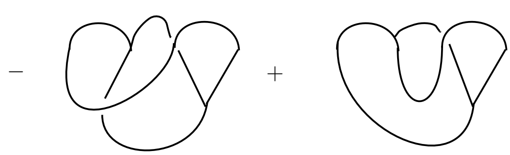
Return now to (3.6) and apply Proposition 8 to obtain



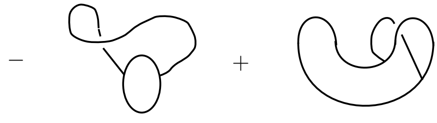
in which we see we can apply (3.4) to obtain the following difference.



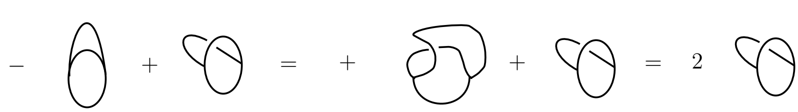
In both terms we can apply (3.1).



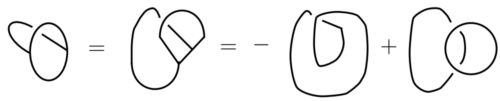
Now use geometry in the first term and Proposition 8 in the second.



Apply Proposition 6 to the first term and the snake identity to the second to obtain

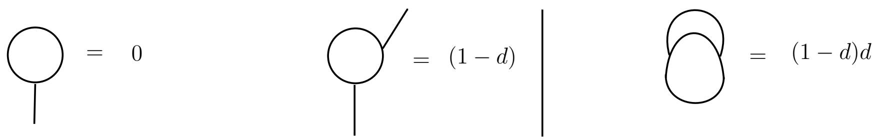


where the first equality uses (3.1) and the second uses Proposition 8 on the first terms. However, using Proposition 8 and then (3.5), we obtain

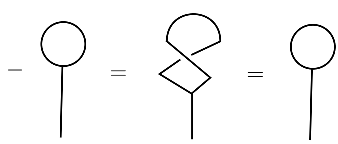


so, using Proposition 6 and geometry, we again obtain terms which consist of one and two disjoint circles. This shows that (3.6) is equal to $2(-d+d^2)$. The two calculations therefore imply $d(d-1)^2 = 2d(d-1)$. So $0 = d(d-1)(d-1-2) = d(d-1)(d-3)$ as required. □

Proposition 14. *Let V be any vector product algebra in an additive braided monoidal category $\mathscr{V}$ such that 2 is cancellable in the abelian group $\mathscr{V}(I, V)$. Then the following three equations hold.*

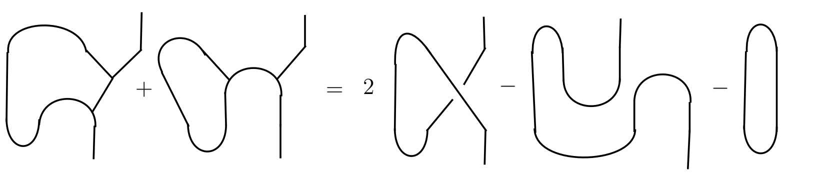


PROOF: Using the asymmetry (3.1) of "∧" and the symmetry of "•", we obtain



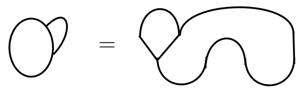
which proves the first equation after transposing and cancelling a 2.

For the second equation, contract Corollary 11 on the left to obtain:



in which the second term on the left is 0 using Corollary 9, the first term on the right is 2 1_V by Proposition 6, the second term on the right is 1_V using the snake identities, and the final term is $d1_V$. This proves the second equation of the proposition.

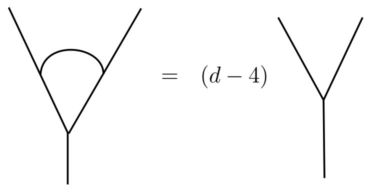
Using Corollary 9, the left-hand side of the third equation is equal to the left-hand side of the equation:



which is true by Proposition 8. The second equation of the proposition now gives the third equation. □

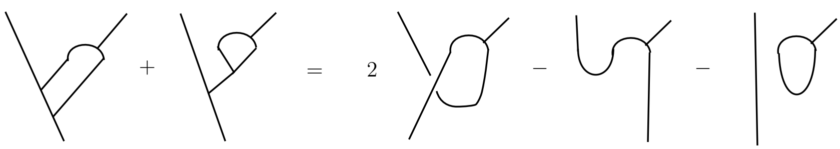
Proposition 15 (T. A. Springer). *For any vector product algebra V with 2 cancellable in $\mathscr{V}(I,V)$,*

(3.7)

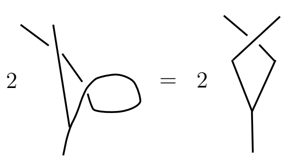


Proof: From (3.3), we obtain

(3.8)



The first term on the left is the left-hand side l (say) of the equation in the proposition. Proposition 14 applies to the second term on the left yielding $(1-d)\ \wedge$. Applying Corollary 9 to the first term on the right-hand side and using some geometry, we obtain the left-hand side of



which is equal to the right-hand side by Proposition 6, and this is equal to $-2\ \wedge$ by (3.1). The third term on the right-hand side of (3.8) is equal to $-\ \wedge$ by a snake equation. The fourth term of the right-hand side of (3.8) is 0 as an easy consequence of the first identity of Proposition 14.

This leaves us with the equation

$$l + (1-d)\ \wedge = -2\wedge\ -\ \wedge,$$

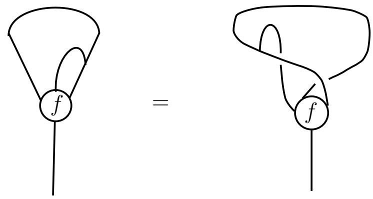
proving that $l = (d-4)\ \wedge$, as required. □

Proposition 16. *For any vector product algebra* V *and any morphism*

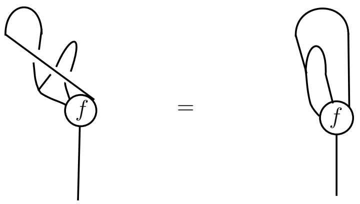
$$f\colon V\otimes V\otimes V\to X,$$

the following identity holds.

(3.9)



PROOF: Applying Corollary 9 and geometry of braiding to the right-hand side yields the left-hand side of:



where the equality holds by Proposition 6 and the geometry of braiding. Now the result follows by Corollary 10. □

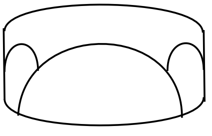
Theorem 17. *For any vector product algebra* V *in any braided monoidal additive category* $\mathscr{V}$ *such that* 2 *can be cancelled in* $\mathscr{V}(I,V)$ *and* $\mathscr{V}(I,I)$*, the dimension* $d=d_V$ *satisfies the equation*

$$d(d-1)(d-3)(d-7)=0$$

in the endomorphism ring $\mathscr{V}(I,I)$ *of the tensor unit* I*.*

PROOF: We perform two string calculations each beginning with the following element of $\mathscr{V}(I,I)$.

(3.10)

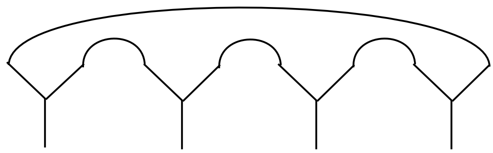


The first calculation involves noticing that the diagram of Proposition 15 occurs twice in (3.10). Using that fact and the third equation of Proposition 14, we obtain the value

(3.11) $$(d-4)^2(1-d)d$$

for (3.10).

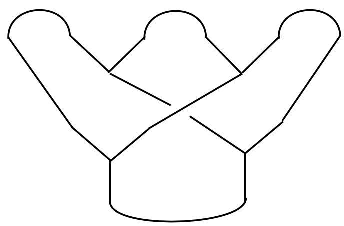
The second calculation begins by considering the equation Ξ obtained by mounting the following morphism on the top of each term of the equation of Corollary 12.



The reader should draw the diagram for equation Ξ. The first term on the left-hand side of Ξ is then none other than the morphism (3.10) in the form of the left-hand side of Proposition 16 for an appropriate $f\colon V \otimes V \otimes V \to I$. Notice also that the second term of the left-hand side of Ξ is equal to the right hand side of Proposition 16 for the same f. Consequently, the left-hand side of Ξ is twice the value of (3.10).

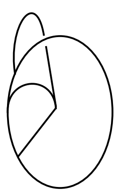
Applying Proposition 5 to the first term of the right-hand side of Ξ and using symmetry, we obtain twice the morphism (3.12).

(3.12)



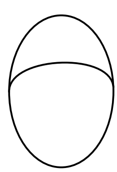
Applying vpa axioms leads to minus twice the morphism (3.13)

(3.13)



in which we can recognize the left-hand side of Proposition 15. Applying that Springer proposition and using vpa axioms, we find that the first term on the right-hand side of Ξ is twice the morphism

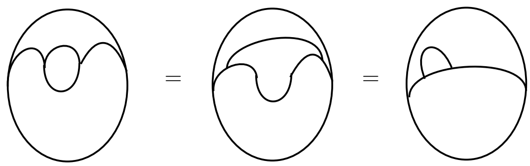
(3.14) $(d-4)$



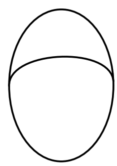
By the third equation of Proposition 14 we now have that the first term of the right-hand side of Ξ is

$$2(d-4)(1-d)d.$$

Beginning with minus the second term of the right-hand side of Ξ we have the calculation:

(3.15) 

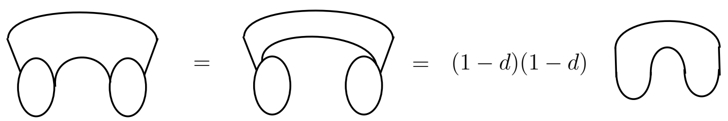
and the second equation of Proposition 14 applies to yield

(3.16) $(1-d)$ 

resulting in the value of the second term of the right-hand side of Ξ being

$$-(1-d)(1-d)d = -d(1-d)^2. \tag{3.17}$$

The third term on the right-hand side of Ξ is minus the morphism

(3.18) 

showing that we again obtain the value (3.17).

Putting this all together in Ξ and cancelling a 2, we obtain

$$(d-4)^2(1-d)d = (d-4)(1-d)d - d(1-d)^2$$

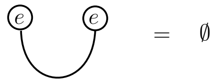
A little algebra turns this into:

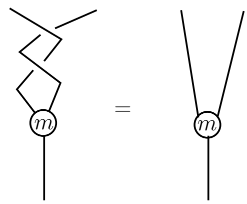
$$\begin{aligned} 0 &= d(d-1)((d-4)^2 - (d-4) + (1-d)) \\ &= d(d-1)(d^2 - 10d + 21) \\ &= d(d-1)(d-3)(d-7) \end{aligned}$$

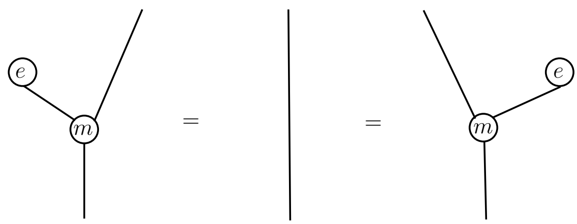
as claimed. □

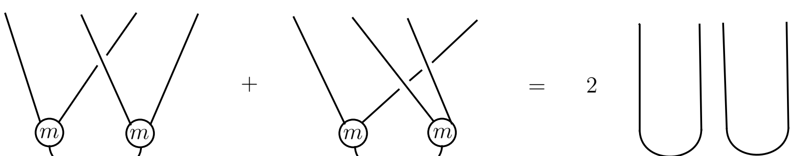
4. Composition algebras

Definition 7. A *composition algebra (ca)* in a braided monoidal additive category $\mathscr{V}$ is an object A equipped with a symmetric self-duality $A \dashv A$, a *multiplication* $m\colon A \otimes A \to A$ and an *identity* $e\colon I \to A$ such that the following conditions hold.

(4.1) 

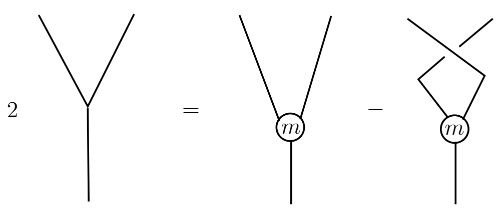
(4.2) 

(4.3) 

(4.4) 

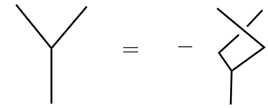
Assume that idempotents split in $\mathscr{V}$. Assume also that multiplication by 2 is invertible in the ring $\mathscr{V}(I,I)$ (and hence in each hom abelian group $\mathscr{V}(X,Y)$).

Define $\wedge\colon A \otimes A \to A$ by the following equation where it is depicted by a Y-shaped string diagram.

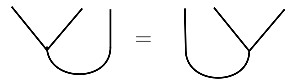
(4.5) 

Proposition 18. *The following two equations hold.*

(4.6)



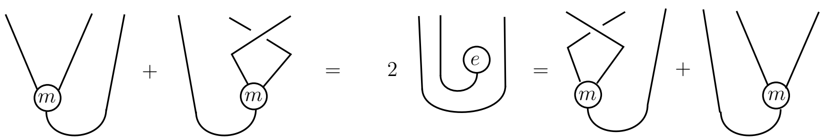
(4.7)



PROOF: From (4.2), we easily deduce (4.6).

By putting e on the right-hand input strings in (4.4), using the inverse braiding on the middle two strings, and applying (4.3), we obtain the left equality in the following diagram.

(4.8)

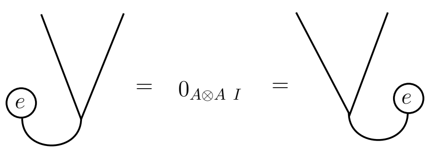


The right-hand inequality is obtained similarly from (4.4) this time by putting e on the third strings, putting the braiding on the first and last strings, then using (4.3) and the symmetry of the duality. Transposing terms in the equality of the left-hand side of (4.8) with the right-hand side, we obtain twice equation (4.7). □

Applying (4.7) of the proposition and using the unit condition (4.3), we obtain:

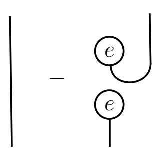
Corollary 19.

(4.9)



Notice that (4.1) says that $I \xrightarrow{e} A$ is a split monomorphism with left inverse $p = (A \xrightarrow{1_A \otimes e} A \otimes A \dot{\to} I)$. This gives the idempotent (4.10) on A.

(4.10)

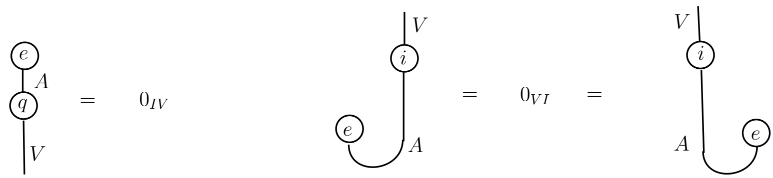


Splitting the idempotent, we obtain a direct sum diagram

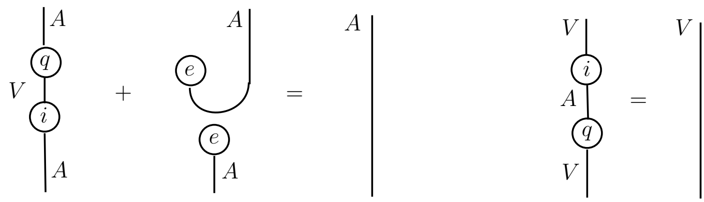
(4.11)
$$I \underset{p}{\overset{e}{\rightleftarrows}} A \underset{i}{\overset{q}{\rightleftarrows}} V$$

exhibiting $A \cong I \oplus V$. This is expressed by the following equations.

(4.12)

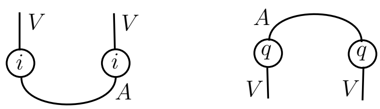


(4.13)

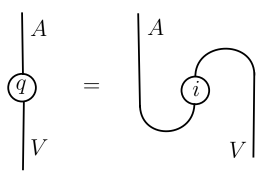


An easy exercise using these equations shows:

Proposition 20. *There is a self-duality $V \dashv V$ with counit and unit supplied by*



With this choice of duality for V,



Notice that Corollary 19 now tells us that $\wedge\colon A \otimes A \to A$ factors through $i\colon V \to A$; indeed,

$$\wedge = e\ p \wedge + i\ q\wedge = 0 + i\ q\wedge = i\ q \wedge .$$

Risking ambiguity, we define a wedge for V by

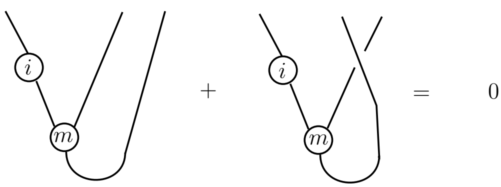
(4.14) $$\wedge := (V \otimes V \xrightarrow{i \otimes i} A \otimes A \xrightarrow{\wedge} A \xrightarrow{q} V)$$

so that the following square commutes.

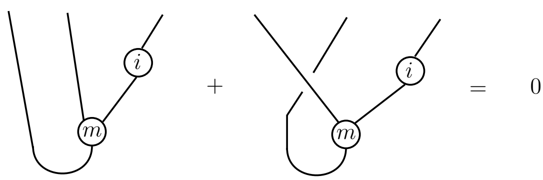
(4.15)
$$\begin{array}{ccc} V \otimes V & \xrightarrow{i \otimes i} & A \otimes A \\ \wedge \downarrow & & \downarrow \wedge \\ V & \xrightarrow[i]{} & A \end{array}$$

Proposition 21. *We have*

(i)



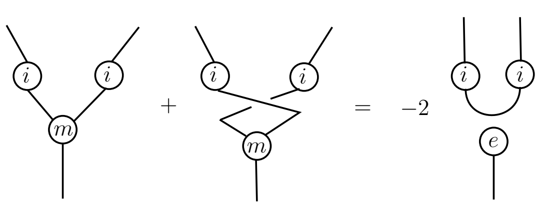
(ii)



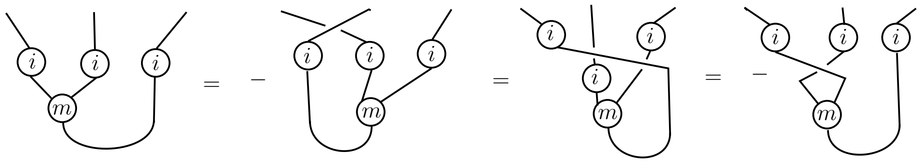
PROOF: For (i) put i on the first and e on the second input strings of axiom (4.4) for a composition algebra. Then use (4.3) and (4.12) to obtain the result.

For (ii) put i on the last and e on the third input strings of axiom (4.4). Then use (4.3) and (4.12) to obtain the result. □

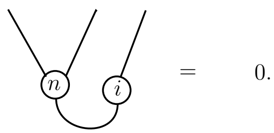
Proposition 22. *We have*



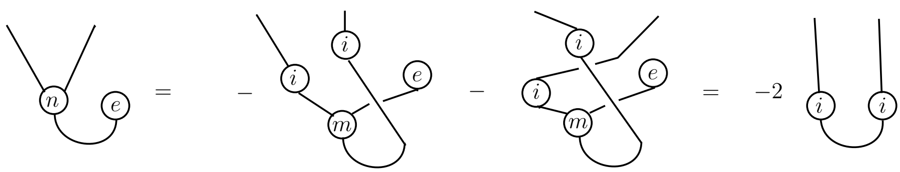
PROOF: Let us temporarily write $n\colon V \otimes V \to A$ for the left-hand side of the equation. Using Proposition 21 and the counit symmetry, we have the following calculation. The first step uses Proposition 21 (i) and symmetry, the second step uses Proposition 21 (ii) and symmetry, while the last step uses Proposition 21 (i) again.



Consequently,



Moreover, using Proposition 21 (i) twice, and then (4.3) and symmetry, we obtain

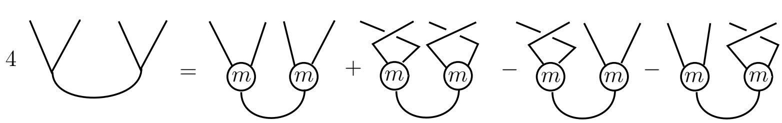


Now using the first equation of the direct sum property (4.13) and the formula for q in Proposition 20, we see that n is equal to the right-hand side of the equation in our proposition. □

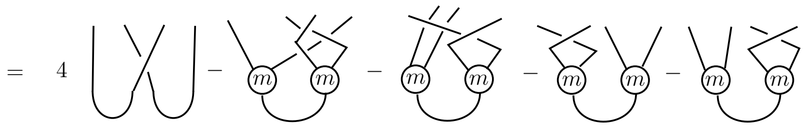
Proposition 23. *For any composition algebra A in $\mathscr{V}$, the object V as defined by* (4.11)*, equipped with the self-duality of Proposition* 20 *and the wedge* (4.14)*, is a vector product algebra in $\mathscr{V}$.*

PROOF: It remains to prove the three axioms for a vector product algebra. The first two axioms are taken care of by Proposition 18: (3.1) follows from (4.6) while (3.2) follows from (4.7) and symmetry of the self duality.

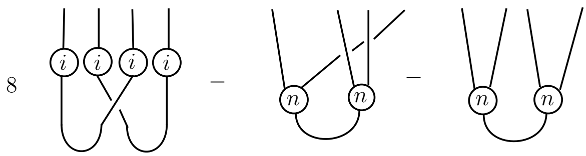
To prove the remaining axiom (3.3), we prove the equivalent form in Corollary 12. Begin with 4 times the first term on the left-hand side of Corollary 12 where, for now, the strings are all labelled by A. Substitute the formula (4.5) for $\wedge$ in the two places to obtain a sum of four terms, two of which are negative.



Apply composition algebra (4.4) to each of the positive terms on the right-hand side to obtain



The second term on the left-hand side of Corollary 12 is obtained from the first by composing with the braiding $c_{A,A\otimes A\otimes A}$. So we also obtain an expression for 4 times this second term as a combination of five terms where the first coefficient is +4 and the other four are −1. Now add the two five term expressions. The positive terms in each expression are the same so we obtain an 8 as coefficient. The negative terms factorize. (In sorting out the string diagrams for this, the reader must note that "•" and m do not see the difference between preceding with the braiding or with its inverse.) If we precede this expression by $i \otimes i \otimes i \otimes i$ and again denote the left-hand side of Proposition 22 by n, we find that the left-hand side of Corollary 12 is equal to



from which the result follows by using the formula for n in Proposition 22. □

Corollary 24. $(d_A - 1)(d_A - 2)(d_A - 4)(d_A - 8) = 0$.

PROOF: As the dimension of $X \oplus Y$ is $d_X + d_Y$, we see from (4.11) that $d_A = 1 + d_V$. The result now follows from Proposition 23 and Theorem 17. □

Let **CA** denote the category of composition algebras; morphisms $f\colon A \to B$ are those which preserve the operations:

$$\big(A \otimes A \xrightarrow{f\otimes f} B \otimes B \xrightarrow{\bullet} I\big) = \big(A \otimes A \xrightarrow{\bullet} I\big),$$
$$\big(A \otimes A \xrightarrow{f\otimes f} B \otimes B \xrightarrow{m} B\big) = \big(A \otimes A \xrightarrow{\bullet} A \xrightarrow{f} B\big),$$
$$\big(I \xrightarrow{e} A \xrightarrow{f} B\big) = \big(I \xrightarrow{e} B\big).$$

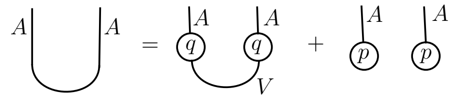
Let **VPA** denote the category of composition algebras; morphisms $h\colon V \to W$ are those which preserve the operation "•" as above and "∧" in the usual sense.

Theorem 25. *Let $\mathscr{V}$ be a braided monoidal additive category with finite direct sums and splitting of idempotents. Assume also that multiplication by 2 is invertible in the ring $\mathscr{V}(I, I)$. Then the functor Φ:* **CA** $\to$ **VPA**, *taking each* *composition algebra A to the vpa V of Proposition* 23 *and each morphism to its restriction, is an equivalence of categories.*

Proof: That Φ is fully faithful follows from (4.11), (4.5) and Proposition 22 which show how $\wedge$ and m can be defined in terms of each other.

It remains to show that Φ is essentially surjective on objects. Take a vpa V. Using the direct sums in $\mathscr{V}$, there exists an A as defined by the direct sum diagram (4.11). A symmetric self-duality for A is defined by

(4.16)

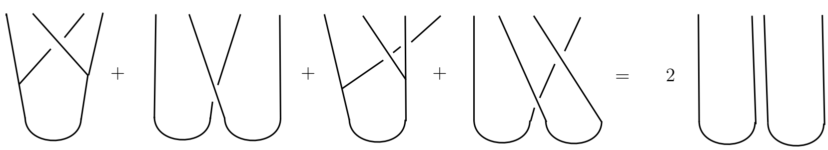


while we have the identity e as part of diagram (4.11). We define the multiplication $m\colon A \otimes A \to A$ by insisting that e is an identity for that multiplication together with the following two equations.

(4.17)

Axioms (4.1), (4.2), (4.3) are then obvious. In axiom (4.4), we replace the four counits for $A \dashv A$ occurring in the equation by the right-hand side of (4.16). Then it suffices to check the result for the sixteen cases obtained by attaching either of the direct sum injections e or i to the four input strings. The only case that needs attention is when all four input strings have i attached; this produces the following condition.

(4.18)



However, if you take the vpa axiom (3.3), move the negative terms to the left-hand side, drag the bottom string in each term up to the right, apply the braiding to the top middle two strings of each term, and manipulate a little using the earlier vpa axioms and Proposition 8, we obtain (4.18). Finally, observe that $\Phi(A) \cong V$. $\square$

References

[1] Baez J. C., *The octonions*, Bull. Amer. Math. Soc. (N.S.) **39** (2002), no. 2, 45–205.

[2] Boos D., *Ein tensorkategorieller Zugang zum Satz von Hurwitz*, Diplomarbeit ETH Zürich, Zürich, 1998 (Deutsch).

[3] Hurwitz A., *Über die Komposition der quadratischen Formen von beliebig vielen Variablen*, Nachrichten Ges. der Wiss. Göttingen (1898), 309–316; in: Mathematische Werke. Bd. II, Zahlentheorie, Algebra und Geometrie, Birkhäuser, Basel, 1963, 565–571 (Deutsch).

[4] Hurwitz A., *Über die Komposition der quadratischen Formen*, Math. Ann. **88** (1922), no. 1–2, 1–25 (Deutsch).

[5] Joyal A., Street R., *The geometry of tensor calculus I*, Adv. Math. **88** (1991), no. 1, 55–112.

[6] Joyal A., Street R., *Braided tensor categories*, Adv. Math. **102** (1993), no. 1, 20–78.

[7] Mac Lane S., *Categories for the Working Mathematician*, Graduate Texts in Mathematics, 5, Springer, New York, 1971.

[8] Maurer S., *Vektorproduktalgebren*, Diplomarbeit Universität Regensburg, Regensburg, 1998 (Deutsch).

[9] Rost M., *On the dimension of a composition algebra*, Doc. Math. **1** (1996), no. 10, 209–214.

[10] Westbury B. W., *Hurwitz' theorem on composition algebras*, available at arXiv:1011.6197 [math.RA] (2010), 33 pages.

R. Street:
Macquarie University, E3B, Eastern Rd, Macquarie Park, Sydney, New South Wales 2109, Australia

E-mail: ross.street@mq.edu.au

(*Received* December 23, 2018*, revised* March 15, 2019)