True preimages of compact or separable sets for functional analysts

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To Krystyna – with love since 1962

Abstract. We discuss various results on the existence of 'true' preimages under continuous open maps between F-spaces, F-lattices and some other spaces. The aim of the paper is to provide accessible proofs of this sort of results for functional-analysts.

Keywords: preimage; open map; complete metric space; F-space; F-lattice; compact set; uniformly open map; surpositive operator; lower semicontinuous set-valued map

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1. Introduction and main results on compact preimages

The topological and functional-analytic terminology, and the relevant facts we use, are standard, as in [6], [7], and [1]; precise references or additional explanations will be given when necessary. Throughout, we agree that in any metric space, d denotes the metric, and $K(z,r)$ and $B(z,r)$ stand for the open and closed balls with center z and radius r, respectively. When dealing with an *F-space*, i.e., a complete metrizable topological vector space (TVS), or merely an F-normed space, we assume that its metric and topology are defined by an F-norm $\|\cdot\|$ (cf. [7, Sections 2.7, 2.8.]. After K. Nagami [13], given a map f from a space X to a space Y, whatever 'space' means, by a *preimage* under f, or an *f-preimage* of a set $B \subset Y$, we shall understand *any* set $A \subset X$ such that $f(A) = B$. Like ways of saying may be used in the case of sequences, series, or functions in place of sets. We are especially interested in *true preimages*—those which are of the same type as their originals. If f is onto, $f^{-1}(B)$ is clearly the largest preimage of B but, in general, is not a true preimage. We will be dealing, mostly, with continuous and open maps. The following two facts about maps f between Hausdorff spaces X, Y will be useful. Both are easy and maybe even of a 'folklore' status. A somewhat surprising application of the equivalence of (a) and (c) in the fact below will be seen in the proof of Theorem 4.2 (b).

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Fact 1.1. *Let $f\colon X \to Y$ be onto and consider the conditions:*

(a) *$f\colon X \to Y$ is open;*

(b) *for each $x \in X$ and each neighborhood U of x, $f(U)$ is a neighborhood of $f(x)$;*

(c) *for each $x \in X$ and each sequence (y_n) in Y converging to $f(x)$, there is a sequence (x_n) in X converging to x such that $y_n = f(x_n)$ for all $n \in \mathbb{N}$.*

Then (a) *and* (b) *mean the same;* (b) *implies* (c) *when X is first countable;* (c) *implies* (b) *when Y is first countable. When both X and Y are* TVS*'s and f is linear, one can use $x = 0$ in* (b) *and* (c), *and replace 'first countable' by 'metrizable'.*

Proof: We only show that (b) $\implies$ (c) and (c) $\implies$ (b) when X respectively Y is first countable. The first implication: Let $x \in X$, $y = f(x)$, and let (y_n) be any sequence in Y converging to y. Fix a decreasing base (U_k) of neighborhoods of x. Then one can find a strictly increasing sequence (n_k) of positive integers such that for each k, $y_n \in f(U_k)$ for all $n \geqslant n_k$. From this, and since f is onto, we easily deduce the existence of a sequence (x_n) in X such that $f(x_n) = y_n$ for all n, with $x_n \in U_k$ whenever $n_k < n \leqslant n_{k+1}$, $k \in \mathbb{N}$. It is evident that (x_n) is as required in (c). The second implication: Suppose it is false so that for a point $x \in X$ and some its neighborhood U the image $f(U)$ is not a neighborhood of $y = f(x)$. Hence, if (V_n) is a decreasing local base at y, then for every n there is a point $y_n \in V_n \setminus f(U)$. Clearly, this contradicts (c). □

Fact 1.2. *Let $f\colon X \to Y$ be continuous.*

(a) *If B is a closed subset of Y, and $B = f(A)$ for a set $A \subset X$, then also $B = f(\overline{A})$.*

(b) *If f is also open and onto, then for each set $B \subset Y$ and $A =: f^{-1}(B)$ one has $f(\overline{A}) = \overline{B}$.*

Proof: (a) We have $B = f(A) \subset f(\overline{A}) \subset \overline{f(A)} = B$.

(b) Since $B = f(A)$, the inclusion '$\subset$' holds by the continuity of f. For the converse inclusion, take any $y \in \overline{B}$, then choose $x \in X$ with $f(x) = y$, and a neighborhood U of x. Since f is open, $f(U)$ is a neighborhood of y. Select any $v \in B \cap f(U)$, and next $u \in U$ so that $v = f(u)$. Clearly, $u \in A$. Thus $A \cap U \neq \emptyset$. Consequently, $x \in \overline{A}$, and we are done. □

In this paper, we concentrate on results similar in spirit to the following.

Theorem 1.3. *If f is a continuous linear operator from an F-space X onto an F-space Y, then every compact set C in Y has a compact preimage K in X. If X is a locally convex F-space (i.e., a Fréchet space) and C is, moreover, convex or absolutely convex, then also K may be chosen of the same type.*

In Section 3, analogous results on separable preimages are given, and in Section 4 we move our setting to the case of F-lattices. At this point let us pause to recall that, by the open mapping theorem, a surjection f as in the theorem above is always open. This will be crucial for all what follows. As a simple application of Theorem 1.3, we note

Corollary 1.4. *Let X be an F-space, and $E_1, \dots, E_n$ be its closed subspaces with $X = E_1 + \dots + E_n$. Then for every compact set C in X there are compact sets $K_i \subset E_i$, $i = 1, \dots, n$, such that $C \subset K_1 + \dots + K_n$.*

Proof: The addition map $A\colon E_1 \times \dots \times E_n \to X$ is linear, continuous and onto. Applying Theorem 1.3, we find a compact set $K \subset E_1 \times \dots \times E_n$ so that $C = A(K)$. To finish, define K_i to be the projection of K into E_i, $i = 1, \dots, n$. □

We now make a few comments on Theorem 1.3. It may rightly be considered well-known, at least when one restricts to Fréchet spaces. Indeed, its occurrences in the standard functional-analytic texts are limited to such spaces. Let us also note that Fact 1.1 (c) and Theorem 1.3 are often stated only for the case of quotient maps $X \to X/N$, see e.g., [7, 9.4.5] or [8, Section 22.2 (7)]. But the general case follows from this easily due to the factorization $f = \hat{f} \circ q$, where $q\colon X \to X/\ker f$ is the quotient map, and $\hat{f}\colon X/\ker f \to Y$ is the isomorphism associated with f. Of course, the assertion of Theorem 1.3 is trivial when f is one-to-one, because then its inverse f^{-1} is continuous and it is enough to set $K = f^{-1}(C)$. Another 'quick' proof when both X and Y are Fréchet spaces or, more generally, when just $\ker f$ is a locally convex subspace of the F-space X, can be given by using a deep selection theorem of Bartle and Graves or its extension due to Michael, see [2, Chapter II, Corollary 7.1 and Proposition 7.1]. It asserts that in that case the map f has a continuous right inverse, that is, there is a continuous (one-to-one) map $g\colon Y \to X$ such that $f \circ g = \mathrm{id}_Y$. Once this is known, it is enough to set $K = g(C)$. For a very simple instance of a situation of this sort, see Fact 5.1 and its proof.

The usual proofs of Theorem 1.3 in the functional-analytic literature as, for instance, in [7, 9.4.5] depend strongly on a result (attributed to Grothendieck in [4, page 6] in the normed case, and to Dieudonné and Robertson-Robertson in [7, page 194] in the locally convex case), which in its most general form given in [7, 9.4.2] reads as follows.

Proposition 1.5. *For every precompact subset E of a metrizable* TVS *Z, there is a null sequence (z_n) in Z such that E is contained in its closed convex hull $\overline{\mathrm{co}}(z_n)$.*

See also [4, Chapter I, Theorem 5], noting that the proof given there for normed spaces can easily be adapted to the general situation. One may also consult our proof of Theorem 1.3 given in the next section. In order to make it easier for the reader to see what's new in our proof, let us recall how Proposition 1.5 was used in the usual proofs of Theorem 1.3 when X, and hence Y as well, were Fréchet spaces. First, applying that Proposition to the compact set C, one finds a null sequence (y_n) in Y so that $C \subset F := \overline{\mathrm{co}}(y_n)$. Next, using Fact 1.1 (c), one further finds a null sequence (x_n) in X such that $f(x_n) = y_n$ for each n. Then $E := \overline{\mathrm{co}}(x_n)$ is a compact set in X (see [7, 6.7.2] or [8, Section 20.6 (3)]), and it follows easily that $f(E) = F$. Finally, $K := E \cap f^{-1}(C)$ is as desired. This argument fails for general F-spaces for in that case one cannot be sure that the set E is compact and, consequently, one would only have $f(E) \subset F$. However, there is a way out of this difficulty, and thus our first contribution will be a modification of the argument just presented resulting in a 'functional-analytic' proof of Theorem 1.3 in its full generality. Finally, the status of Theorem 1.3 becomes completely clear when one consults the general topological literature, and with a bit of luck comes upon a remarkable result of N. Bourbaki [3, Chapter IX, Section 1, Proposition 18]. We state it, in the form most suitable for us, as the first theorem below. The other two theorems are close relatives of the first. We postpone the proofs to the next section. They are fairly elementary and quite similar, though not identical, to the original proof of Bourbaki. Additionally note that each of these theorems readily implies Theorem 1.3, modulo the open mapping theorem.

Theorem 1.6. *Let f be a continuous open map from a complete metric space X onto a Hausdorff space Y. Then every compact set in Y has a compact preimage in X.*

Theorem 1.7. *Let X be a complete metric space and Y a Hausdorff space. If $F\colon Y \to X$ is a lower semicontinuous set-valued map with nonempty closed values, then for every compact subset C of Y there is a compact subset K of X such that $C \subset F^{-1}(K)$.*

Recall that F *is lower semicontinuous* if for every open set $U \subset X$, the set $F^{-1}(U) := \{y \in Y\colon F(y) \cap U \neq \emptyset\}$ is open in Y. Before proceeding, let us agree to say that a map f from a metric space X to a metric space Y is *uniformly open* if for each $\varepsilon > 0$ there is $\delta > 0$ such that $B(f(x), \delta) \subset f\big(B(x, \varepsilon)\big)$ for all $x \in X$. Clearly, f is then an open map; in particular, its range $f(X)$ is an open subset of Y. In fact, $f(X)$ is also closed because, for δ as above, $d(f(x), Y \setminus f(X)) > \delta$ for all $x \in X$. Evidently, every open linear map between F-normed spaces is uniformly open. In view of Theorem 1.6 only the completeness of Y in the next result is of importance, but a direct proof of it would not be much shorter. We refer the reader to [6, Problem 5.5.8 (d)] for historical information on related earlier results, in particular, a 1934 theorem of F. Hausdorff.

Theorem 1.8. *Let f be a continuous and uniformly open map from a complete metric space X onto a metric space Y. Then for every precompact set C in Y one can find a compact set K in X such that $C \subset f(K)$. In consequence, the space Y is complete.*

Remarks 1.9.

(a) Theorem 1.6 follows right from Theorem 1.7 by taking $F(y) := f^{-1}(\{y\})$ for $y \in Y$, then noting that for this F the condition $C \subset F^{-1}(K)$ translates into $C \subset f(K)$, and finally replacing the provided compact set $K \subset X$ with $K \cap f^{-1}(C)$. As will be seen, to get a compact set $K \subset X$ so that $C \subset f(K)$ we only need to know that f is open and that $f^{-1}(\{y\})$ is closed for all $y \in C$.

(b) It is standard that a Hausdorff quotient X/N of an F-space X is an F-space, see e.g. [7, Section 4.4, Proposition 1]. Note that this also is a direct consequence of Theorem 1.8 applied to the quotient map $q\colon X \to X/N$.

(c) Theorem 1.6 was strengthened by E. Michael in [9, Corollary 1.2] who removed the continuity of f and replaced the completeness of X with a weaker requirement (M): *fibers $f^{-1}(\{y\})$ for $y \in Y$ are complete subsets of X.* He also introduced the concept of *compact-covering maps*, see [11], which was later intensively studied; for a sample of references see [10], [12], [13]. However, we will not use Michael's result in this paper, the reason being that its proof requires quite a deep knowledge of general topology while, on the other hand, it is hard to imagine situations involving the assumption (M) in the functional-analytic practice.

2. Proofs of the main results on compact preimages

PROOF OF THEOREM 1.3: Fix a sequence (ε_k) of positive reals with $\sum_k \varepsilon_k < \infty$. Since f is open, there is a sequence (δ_k) of positive reals with $\sum_k \delta_k < \infty$ and such that $B(0, 2\delta_{k-1}) \subset f(B(0,\varepsilon_k))$ for $k \geqslant 2$. Now, given a compact set C in Y, we repeat the proof of [4, Chapter I, Theorem 5] making only a few minor changes. For the sake of clarity, we include details of the two inductive procedures, (1) and (2), used in that proof. Part (3) is our own, and essential, ingredient of the reasoning.

(1) The set $2C$ is compact, hence admits a finite δ_1-net $(y_j : j \in J_1)$ so that $2C \subset \bigcup_{j\in J_1} B(y_j, \delta_1)$. Clearly, also the set

$$C_1 := \bigcup_{j\in J_1} (2C \cap B(y_j, \delta_1) - y_j)$$

is compact and, obviously, $C_1 \subset B(0,\delta_1)$. Assume that for some $k \geqslant 1$ the compact set $C_k \subset B(0,\delta_k)$ has already been defined. Then to proceed, choose any finite δ_{k+1}-net $(y_j : j \in J_{k+1})$ in the compact set $2C_k$, and put

$$C_{k+1} := \bigcup_{j \in J_{k+1}} (2C_k \cap B(y_j, \delta_{k+1}) - y_j).$$

This completes the first inductive procedure. Of course, we may assume that the finite index sets J_k occurring above form a sequence of consecutive intervals in $\mathbb{N}$ covering all of $\mathbb{N}$. We thus obtain a sequence (y_n) in Y such that $y_n \in B(0,\delta_{k-1})$ for all $n \in J_k$ and $k \geqslant 2$. Hence (y_n) is a null sequence.

(2) Take any $y \in C$. Then there exists $j_1 \in J_1$ such that $2y - y_{j_1} \in C_1$. Likewise, there is $j_2 \in J_2$ such that $2(2y - y_{j_1}) - y_{j_2} \in C_2$. Continue. This yields, for every $y \in C$ indices $j_k \in J_k$, $k \geqslant 1$, such that $y - \sum_{k=1}^{m} 2^{-k} y_{j_k} \in 2^{-m} C_m \subset B(0,\delta_m)$ for all $m \geqslant 1$. Hence $y = \sum_{k=1}^{\infty} 2^{-k} y_{j_k}$. In consequence, $C \subset \overline{\mathrm{co}}(y_n)$, but this is of no importance in the proof.

(3) Finally, choose (x_n) in X so that $x_n \in B(0,\varepsilon_k)$ whenever $n \in J_k$ and $k \geqslant 2$, and $f(x_n) = y_n$ for all $n \in \mathbb{N}$. Note that for any choice of indices $j_k \in J_k$, $k \geqslant 1$, the series $\sum_{k=1}^{\infty} 2^{-k} x_{j_k}$ converges in X. It is so because $\|2^{-k} x_{j_k}\| \leqslant \|x_{j_k}\| \leqslant \varepsilon_k$, and $\sum_k \varepsilon_k < \infty$. This gives rise to a map

$$h\colon J := \prod_{k=1}^{\infty} J_k \to X, \qquad \text{with} \;\; h((j_k)) := \sum_{k=1}^{\infty} 2^{-k} x_{j_k},$$

where the product space is compact (its factors are considered with the discrete topology), and h is easily verified to be continuous. Hence $K' := h(J)$ is a compact subset of X and $f(K') \supset C$. We finish by setting $K := K' \cap f^{-1}(C)$. The 'moreover' part follows by replacing K with its closed [absolutely] convex hull. □

We now proceed to the other results on compact preimages from Section 1. The basic technical ingredient is the following.

Lemma 2.1. *Let X be a complete metric space, (ε_k) a sequence of positive reals with $\sum_k \varepsilon_k < \infty$, and denote $\varepsilon'_k = \sum_{r=k}^{\infty} \varepsilon_r$. Moreover, let (F_k) be a sequence of closed subsets of X of the form*

$$F_k = \bigcup_{j \in J_k} B(x_{k\,j}, \varepsilon_k),$$

where each J_k is a nonempty finite set of indices. Assume that for each k the 'components' $B(x_{k+1\,j}, \varepsilon_{k+1})$ of F_{k+1} are linked to those of F_k so that

$$(*) \qquad B(x_{k+1\,j}, \varepsilon_{k+1}) \cap F_k \neq \emptyset \quad \textit{for all } j \in J_{k+1}.$$

Write

$$K = \bigcap_{k=1}^{\infty} F_k.$$

Then every sequence (x_k) *such that* $x_k \in F_k$ *for each* k *(in particular, every sequence* (x_k) *in* K*) has a subsequence convergent in* X *to a point, say* x*. Further, this* x *belongs to the set*

$$K' := \bigcap_{k=1}^{\infty} F'_k, \qquad \text{where } F'_k = \bigcup_{j \in J_k} B(x_{k\,j}, 2\varepsilon'_k),$$

and is also the limit of a sequence $(x_{k\,r_k})$ *for some* $r_k \in J_k$, $k \in \mathbb{N}$*. Clearly, when* $(x_k) \subset K$, x *also belongs to* K*. Consequently, the set* K *is compact.*

The reader has certainly noticed that compactness of both K and K' could be inferred directly from Hausdorff's compactness criterion. However, the assertions involving sequences will also be of importance for us in the forthcoming proofs.

Proof: It follows from $(*)$ that if $k < l$ and $x \in F_l$, then x is *linked* to one of the points $x_{k\,i}$, $i \in J_k$, in the sense that there are indices $i = j_k, j_{k+1}, \dots, j_l = j$ such that $x \in B(x_{l\,j_l}, \varepsilon_l)$ and $B(x_{r\,j_r}, \varepsilon_r) \cap B(x_{r-1\,j_{r-1}}, \varepsilon_{r-1}) \neq \emptyset$ for $k < r \leqslant l$. It is then easily seen that

$$(**) \qquad d(x_{k\,i}, x) \leqslant 2\varepsilon'_k.$$

In consequence, for every k each of the points x_l with $l > k$ is linked to one of the points $x_{k\,j}$ for some $j \in J_k$. Hence, since the set J_k is finite, for every infinite set $M \subset \mathbb{N}$ with $\min M > k$ there is an index $j \in J_k$ and an infinite set $N \subset M$ such that x_l is linked to $x_{k\,j}$ for every $l \in N$. Proceeding by induction, we find a sequence of indices (r_k) with $r_k \in J_k$, $k \in \mathbb{N}$, and a sequence of infinite sets $N_1 \supset N_2 \supset \dots$ with $k < \min N_k < \min N_{k+1}$, $k \in \mathbb{N}$, such that x_l is linked to $x_{k\,r_k}$ for all $l \in N_k$, $k \in \mathbb{N}$. Let $l_k := \min N_k$; note that then $l_k < l_{k+1} < \dots$ are all in N_k. By the estimate $(**)$ above for each k and all $n > k$

$$(+) \qquad d(x_{k\,r_k}, x_{l_n}) \leqslant 2\varepsilon'_k,$$

hence $d(x_{l_n}, x_{l_m}) \leqslant 4\varepsilon'_k$ for all $m, n > k$ so that the sequence (x_{l_n}) is Cauchy. Let x be its limit in the space X. Then from $(+)$ it follows that $x \in B(x_{k\,r_k}, 2\varepsilon'_k)$ for each k, hence $x \in K'$ and $x_{k\,r_k} \to x$. □

Remark 2.2. If the condition $(*)$ holds in a stronger form: $x_{k+1\,j} \in F_k$ for all $j \in J_{k+1}$, then $2\varepsilon'_k$ appearing in the definition of the set K', in $(**)$ and in $(+)$ can be replaced with ε'_k.

Proof of Theorem 1.6: For each $k \in \mathbb{N}$, let $\varepsilon_k = 2^{-k}$; note that then $\varepsilon'_k = 2\varepsilon_k$. Next, let $F_0 = f^{-1}(C)$ and note that $f(F_0) = C$. Proceed to construct, by induction, a sequence $(F_k)_{k \geqslant 1}$ of closed subsets of X such that for each $k \in \mathbb{N}$

$$F_k := \bigcup_{j \in J_k} B(x_{kj}, \varepsilon_k) \qquad \text{and} \qquad C \subset f(F_k),$$

where J_k is a finite set of indices, and $x_{kj} \in F_{k-1}$ for all $j \in J_k$. Then our sequence (F_k) will satisfy the assumption $(*)$ of Lemma 2.1. The inductive step from k to $k+1$ is performed as follows. Assume that F_k has already been defined for some $k \geqslant 0$. Since $C \subset f(F_k)$ and f is an open map, the sets $f(K(x, \varepsilon_{k+1}))$ for $x \in F_k$ are open and cover the compact set C. Hence, there is a finite number of points $x_{k+1\,j} \in F_k$, $j \in J_{k+1}$, such that $C \subset f(F_{k+1})$, where

$$F_{k+1} := \bigcup_{j \in J_{k+1}} B(x_{k+1\,j}, \varepsilon_{k+1}).$$

By the Hausdorff compactness criterion, the set

$$K := \bigcap_{k=1}^{\infty} \bigcup_{j \in J_k} B(x_{kj}, \varepsilon'_k)$$

is compact. (This also follows from Lemma 2.1, because $\sum_k \varepsilon'_k < \infty$.) We are going to show that $C \subset f(K)$. Take any point $y \in C$. Then, for each k, since $C \subset f(F_k)$, we may choose a point $x_k \in F_k$ such that $y = f(x_k)$. By Lemma 2.1 and Remark 2.2, the sequence (x_k) has a convergent subsequence (x_{l_n}), and its limit, say x, belongs to K. On the other hand, since $(x_k) \subset f^{-1}(\{y\})$ and the latter set is closed in X, $f(x) = y$. Thus $C \subset f(K)$, and to finish just replace K with $(f|K)^{-1}(C) = K \cap f^{-1}(C)$. □

Proof of Theorem 1.7: As in the proof of Theorem 1.6, we let $\varepsilon_k = 2^{-k}$ for each $k \in \mathbb{N}$ and using the fact that $\{F^{-1}(K(x, \varepsilon_k) \colon x \in X\}$ is an open cover of X, hence of C as well, arrive at a closed set

$$E_k = \bigcup_{j \in J_k} B(x_{kj}, \varepsilon_k)$$

in X so that $x_{kj} \in E_{k-1}$ for all $j \in J_k$ (with $E_0 = X$) and $C \subset F^{-1}(E_k)$. Now, take any point $y \in C$. Next, for every k select a point x_k in $F(y) \cap E_k$. By Lemma 2.1, there is a subsequence (x_{l_n}) convergent to a point x that belongs to the compact set

$$K := \bigcap_{k=1}^{\infty} \bigcup_{j \in J_k} B(x_{kj}, 2\varepsilon_k).$$

Since $(x_k) \subset F(y)$ and $F(y)$ is closed, $x \in F(y)$. Hence, $K \cap F(y) \neq \emptyset$. □

Proof of Theorem 1.8: The proof is very much like those above. Start by letting $\varepsilon_k = 2^{-k}$ for each k and by choosing $\delta_k > 0$ such that $B(f(x), \delta_k) \subset f(B(x, \varepsilon_k))$ for all $x \in X$. Then, using precompactness, select a finite number of points $y_{kj} \in C$, $j \in J_k$, so that

$$C \subset \bigcup_{j \in J_k} B(y_{kj}, \delta_k).$$

For $k \geqslant 2$ and $j \in J_k$ pick $p(k,j) \in J_{k-1}$ so that $y_{kj} \in B(y_{k-1\,p(k,j)}, \delta_{k-1})$. Construct by induction a sequence (F_k) of subsets of X such that for each $k \in \mathbb{N}$

$$F_k = \bigcup_{j \in J_k} B(x_{kj}, \varepsilon_k),$$

where $x_{kj} \in B(x_{k-1\,p(k,j)}, \varepsilon_{k-1})$ and $f(x_{kj}) = y_{kj}$ for all $j \in J_k$. Note that then the assumptions of Lemma 2.1 are satisfied.

Step 1. Select points $x_{1j} \in X$ so that $f(x_{1j}) = y_{1j}$ for $j \in J_1$.

Step $k \to k+1$. Since, for each $j \in J_{k+1}$

$$y_{k+1\,j} \in B(y_{k\,p(k+1,j)}, \delta_k) \subset f(B(x_{k\,p(k+1,j)}, \varepsilon_k)),$$

it is obvious that one may find points $x_{k+1\,j}$, as required. By the Hausdorff compactness criterion, the set

$$K := \bigcap_{k=1}^{\infty} \bigcup_{j \in J_k} B(x_{kj}, 2\varepsilon_k)$$

is compact. We show that $C \subset f(K)$. Take any point $y \in C$. Then, for every k, since

$$C \subset \bigcup_{j \in J_k} B(y_{kj}, \delta_k) \subset f(F_k),$$

there is a point $x_k \in F_k$ such that $y = f(x_k)$. By Lemma 2.1 and Remark 2.2, the sequence (x_k) has a convergent subsequence (x_{l_n}), and its limit, say x, belongs to K. Moreover, as in the preceding proof, $y = f(x)$. Thus $C \subset f(K)$, and hence C is relatively compact, i.e., its closure is compact. In consequence, any Cauchy sequence in Y is relatively compact, hence convergent. □

3. Separable preimages

Theorem 3.1. *Let f be a continuous linear operator from an F-space X onto an F-space Y. Then for every closed separable subset F of Y there exists a closed separable subset E of X such that $f(E) = F$. If, in addition, F is a linear subspace or a convex set, then E may be required to be of the same type.*

Proof: We first show that for every closed separable linear subspace M in Y there is a closed separable linear subspace L in X such that $f(L) = M$. Fix any sequence $\varepsilon_k > 0$ with $\sum_k \varepsilon_k < \infty$. Since f is an open map, there is a decreasing sequence $0 < \delta_k \to 0$ such that for each k

$$f(K(0,\varepsilon_k)) \supset K(0,\delta_k).$$

Choose a countable dense subset S of M, and then select a countable subset D_0 of X with $f(D_0) = S$. Applying the condition displayed above, we construct a sequence (D_k) of countable subsets of X so that for $k \geqslant 1$

$$D_k \subset K(0,\varepsilon_k) \quad \text{and} \quad f(D_k) = S \cap K(0,\delta_k).$$

Now, take any $y \in M$. Then there is $y_0 \in S$ with $\|y - y_0\| < \delta_1$, and next there is $y_1 \in S \cap K(0,\delta_1)$ with $\|y - y_0 - y_1\| < \delta_2$. Continuing in this manner, we find a sequence (y_k) in S such that $y_k \in S \cap K(0,\delta_k)$ and $\|y - (y_0 + y_1 + \cdots + y_k)\| < \delta_{k+1}$ for each $k \geqslant 1$. By the construction of the sets D_k, there is a sequence (x_k) in X such that $x_k \in D_k$ and $f(x_k) = y_k$ for each $k \geqslant 0$. Since $\|x_k\| < \varepsilon_k$ for $k \geqslant 1$ the series $\sum_{k=0}^{\infty} x_k$ converges in X, and denoting by x its sum we obviously have $f(x) = y$. Let L denote the (separable) closed linear span of the union D of the sets D_k for $k \geqslant 0$. We have just shown above that $f(L) \supset M$. Since $f(D) \subset S \subset M$, we conclude that $f(L) = M$. In the general case, let M denote the closed linear span of the set F. By what was proved above, there is a closed separable linear subspace L in X with $f(L) = M$. Then $E := L \cap f^{-1}(F)$ is as required. □

Theorem 3.2. *Let f be a continuous and uniformly open map from a complete metric space X onto a metric space Y. Then for every separable subset F of Y there is a separable subset E of X such that $f(E) = F$. If, in addition, F is closed, also E may be chosen to be closed.*

Proof: Fix any sequence $\varepsilon_k > 0$ with $\sum_k \varepsilon_k < \infty$, and next choose a decreasing sequence $0 < \delta_k \to 0$ so that

$$K(f(x), 2\delta_k) \subset f(K(x,\varepsilon_k)) \quad \text{for all } x \in X.$$

Choose a countable dense subset S in F, and then a countable subset D_0 of X such that $f(D_0) = S$. Applying the condition displayed above, we construct a sequence (D_k) of countable subsets of X of the form

$$D_k = \bigcup_{x \in D_{k-1}} D_k(x),$$

where $D_k(x) \subset K(x,\varepsilon_k)$ and $f(D_k(x)) = S \cap K(f(x), 2\delta_k)$ for each $x \in D_{k-1}$, $k \geqslant 1$. Now, take any $y \in F$. Then there is a sequence (y_k) in S such that

$d(y, y_k) < \delta_k$, and hence $d(y_k, y_{k+1}) < \delta_k + \delta_{k+1} \leqslant 2\delta_k$ for each $k \geqslant 1$. Select any $x_1 \in D_0$ so that $f(x_1) = y_1$. Then, since $y_2 \in S \cap K(f(x_1), 2\delta_1)$, select $x_2 \in D_1(x_1)$ so that $f(x_2) = y_2$. Continuing in this manner, we find a sequence (x_k) such that for each $k \geqslant 2$

$$x_k \in D_{k-1}(x_{k-1}) \subset K(x_{k-1}, \varepsilon_{k-1}) \quad \text{and} \quad f(x_k) = y_k.$$

Thus $d(x_{k-1}, x_k) < \varepsilon_k$, hence the sequence (x_k) is Cauchy in X, and has a limit, say x. Since f is continuous, $y_k = f(x_k) \to f(x) = y$. We have thus shown that if D denotes the union of the sets D_k for $k \geqslant 0$, then $f(\overline{D}) \supset F$. It follows that the set $E := \overline{D} \cap f^{-1}(F)$ is as required. □

Remark 3.3. An inspection of the proofs reveals that in the two results above one may replace 'separable' by 'of density character $\mathfrak{m} \geqslant \aleph_0$'.

4. True preimages in F-lattices

We now turn our attention to F-lattices, i.e., complete metrizable topological vector lattices, or Riesz spaces. In this case the most specific type of linear operators f between F-lattices X and Y that comes to mind are the *positive* ones, i.e., those for which $f(X_+) \subset Y_+$, and as our main goal we see proving the existence of true preimages in X_+ of compact subsets of Y_+. In view of this, a stronger requirement on f's pops up immediately: $f(X_+) = Y_+$; as was pointed out to me by W. Wnuk, it appears for instance in [14, Corollary on page 60]. We shall call such operators *surpositive* (the term suggested by I. Labuda). So far that goal has not been reached, cf. Problem 4.4. Our only results in this direction are nothing but simple observations, see the proposition right below and Section 5.

Proposition 4.1. *Let f be a surpositive linear operator from an F-lattice X to an F-lattice Y. Then f is continuous and onto, hence also open. Therefore, for each compact set $C \subset Y$ there is a compact set $K \subset X$ such that $C = f(K)$. If, additionally, f is one-to one on X_+ or, equivalently, on X, then each compact set in Y_+ has a compact preimage in X_+.*

Proof: Both statements are easily justified with the help of equalities $x = x^+ - x^-$, $y = y^+ - y^-$, and $f(x) = f(x^+) - f(x^-)$, $x \in X$, $y \in Y$, and an appeal to Theorem 1.3. □

In our next result, we assume more on f, and derive stronger conclusions. Part (b) deserves a special attention; it seems to be a new property of vector lattice homomorphisms.

Theorem 4.2. *Let f be a vector lattice (or Riesz) homomorphism from an F-lattice X onto another F-lattice Y. Then:*

(a) *Vector lattice homomorphism f is surpositive, continuous and open.*
(b) *Moreover, also the map $f|X_+ \colon X_+ \to Y_+$ is open.*
(c) *In consequence, for every compact set $C \subset Y_+$ there exists a compact set $K \subset X_+$ such that $C = f(K)$.*

PROOF: Recalling that $f(|x|) = |f(x)|$ for all $x \in X$, (a) is clear. To see (c), apply the preceding result or Theorem 1.3 to get a compact set L in X with $C = f(L)$. Since the map $x \to |x|$ is continuous in X, the set $K := \{|x| \colon x \in L\}$ is compact, and is as required. To see (b), use Fact 1.1 twice: Take any $x \in X_+$, and any sequence (y_n) in Y_+ converging to $y = f(x)$. As $f \colon X \to Y$ is open, there is a sequence (x_n) in X converging to x and such that $y_n = f(x_n)$ for all n. Then $(|x_n|) \subset X_+$, $|x_n| \to x$, and $f(|x_n|) = y_n$ for all n. One may also arrive at (c) from (b) by using Theorem 1.6. □

Remarks 4.3.

(a) A surjective positive operator between F-lattices need not be surpositive. The following simple example to this effect has kindly been provided by W. Wnuk: Define $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ by $f(u,v) = (u+v, v)$. Then f is a positive linear bijection. But it is not surpositive for whenever $0 \leqslant u < v$, then $(u,v) = f(u-v, v) \in \mathbb{R}^2_+ \setminus f(\mathbb{R}^2_+)$.
(b) Important: The theorem above is in particular applicable to the quotient map $q : X \to X/I$, where I is a closed ideal in the F-lattice X.

Problem 4.4. In the setting of Proposition 4.1, is $f \colon X_+ \to Y_+$ open or, does every compact subset of Y_+ have a compact preimage in X_+?

We conclude with a result on separable preimages in the context of F-lattices.

Theorem 4.5. *Let f be a vector lattice homomorphism from an F-lattice X onto another F-lattice Y. Then for every closed separable vector sublattice W of Y there is a closed separable vector sublattice V of X such that $W = f(V)$.*

PROOF: By Theorem 3.2, there is a closed separable linear subspace L in X such that $f(L) = W$. Denote by U the closed vector sublattice of X generated by L. Applying [5, Proposition 2.6], one easily verifies that U is separable. Obviously, $W \subset f(U)$. Finally, $V := (f|U)^{-1}(W) = U \cap f^{-1}(W)$ is as required. □

5. Compact preimages in finite dimensions

The elementary fact below could well be placed at the very beginning of the Introduction; we have chosen to state it here for the editorial reasons. Surely, the reader will have no difficulty in extending it to complex euclidean spaces, as well as to general finite-dimensional Hausdorff topological vector spaces or lattices, in the latter case making use of Yudin's theorem, see e.g., [5, Theorem 2.7]. Note that this gives a partial positive answer to Problem 4.4.

Fact 5.1. *Let f be a linear map from $X = \mathbb{R}^m$ into $Y = \mathbb{R}^n$, $m, n \in \mathbb{N}$. If f is onto, then every compact set C in Y has a compact preimage K in X. Likewise, if $Y_+ \subset f(X_+)$, then every compact set C in Y_+ has a compact preimage K in X_+.*

Proof: Denote by (v_k) the standard basis in Y, and by (v_k^*) the corresponding coordinate functionals. By the assumption on f, for each k there exists u_k in X or in X_+, respectively, such that $f(u_k) = v_k$. Define $g\colon \mathbb{R}^n \to \mathbb{R}^m$ by $g(y) = \sum_k v_k^*(y) u_k$. Then g is linear, continuous, and $f \circ g = \mathrm{id}_Y$, obviously, $K = f(C)$ is as desired. □

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