Some classes of perfect strongly annihilating-ideal graphs associated with commutative rings

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Abstract. Let R be a commutative ring with identity and $A(R)$ be the set of ideals with nonzero annihilator. The strongly annihilating-ideal graph of R is defined as the graph $\mathrm{SAG}(R)$ with the vertex set $A(R)^* = A(R) \setminus \{0\}$ and two distinct vertices I and J are adjacent if and only if $I \cap \mathrm{Ann}(J) \neq (0)$ and $J \cap \mathrm{Ann}(I) \neq (0)$. In this paper, the perfectness of $\mathrm{SAG}(R)$ for some classes of rings R is investigated.

Keywords: strongly annihilating-ideal graph; perfect graph; chromatic number; clique number

Classification: 13A15, 13B99, 05C99, 05C25

1. Introduction

One of the most important and active areas in algebraic combinatorics is study of graphs associated with rings. This field has attracted many researches during the past 20 years. There are many papers on assigning a graph to a ring, for instance see [2], [3], [4], [10] and [11].

Throughout this paper, R is a commutative ring with unity. The sets of all zero-divisors, all ideals of R, nilpotent elements, minimal prime ideals of R and jacobson radical of R are denoted by $Z(R)$, $\mathbb{I}(R)$, $\mathrm{Nil}(R)$, $\mathrm{Min}(R)$ and $\mathrm{Jac}(R)$, respectively. For a subset T of a ring R we let $T^* = T \setminus \{0\}$. An ideal with nonzero annihilator is called an *annihilating ideal*. The set of annihilating ideals of R is denoted by $A(R)$. A nonzero ideal I of R is called *essential* if I has a nonzero intersection with every other nonzero ideal of R. An element e of the ring R is called an *idempotent* if $e^2 = e$. Two idempotents $e, f \in R$ are called *orthogonal* if $ef = 0$. For a ring R, $\mathrm{Soc}(R)$ is the sum of all minimal ideals of R and R is called *perfect* if it contains no infinite set of orthogonal idempotents and $\mathrm{Soc}(R)$ is an essential ideal of R. The ring R is said to be *reduced* if it has no nonzero nilpotent element. For any undefined notation or terminology in ring theory, we refer the reader to [9].

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Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $\overline{G}$, we mean the complement graph of G. A complete bipartite graph with part sizes m and n is denoted by $K_{m,n}$. Also, a complete graph of order n is denoted by K_n. If $U \subseteq V(G)$, then by $N(U)$ we mean the set of all neighbors of U in G. The graph $H = (V_0, E_0)$ is a subgraph of G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph* by V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E\colon u, v \in V_0\}$. A graph G is *empty* if it has no edges. Let G_1 and G_2 be two disjoint graphs. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with the vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv\colon\ u \in V(G_1),\ v \in V(G_2)\}$. A *clique* of G is a maximal complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G. For a graph G, let $\chi(G)$ denote the *vertex chromatic number* of G, i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. Clearly, for every graph G, $\omega(G) \leq \chi(G)$. A graph G is said to be *weakly perfect* if $\omega(G) = \chi(G)$. A *perfect graph* G is a graph in which every induced subgraph is weakly perfect. For any undefined notation or terminology in graph theory, we refer the reader to [12].

Let R be a commutative ring with $1 \neq 0$. The *annihilating-ideal graph* of R, denoted by $\mathbb{AG}(R)$, is a graph with the vertex set $A(R)^*$ and two distinct vertices I and J are adjacent if and only if $IJ = 0$, see [5] for more details. Coloring of annihilating-ideal graph was investigated in [1]. The *strongly annihilating-ideal graph*, denoted by $\mathrm{SAG}(R)$, is a graph with the vertex set $A(R)^*$ and two distinct vertices I and J are adjacent if and only if $I \cap \mathrm{Ann}(J) \neq (0)$ and $J \cap \mathrm{Ann}(I) \neq (0)$. The strongly annihilating-ideal graph, as a generalization of annihilating-ideal graph, was first introduced and studied in [11]. In [8], it was proved that strongly annihilating-ideal graph of a reduced ring is weakly perfect. In this paper, we prove a stronger result; indeed it is shown that strongly annihilating-ideal graphs of both reduced rings and perfect rings are perfect.

2. Strongly annihilating-ideal graph of a reduced ring is perfect

In this section, we show that $\mathrm{SAG}(R)$ is perfect for every reduced ring R with $\omega(\mathrm{SAG}(R)) < \infty$. In 2004 M. Chudnovsky et al. [6] settled a long standing conjecture regarding perfect graphs and provided a characterization of perfect graphs.

Theorem 2.1 (The strong perfect graph theorem [6]). *A graph G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length at least* 5.

Example 2.1.

(1) Every complete graph and complete bipartite graph is perfect.

(2) Let G_1 be a complete graph and $G_2 = C_n$ be a cycle of length at least 5 and let $G = G_1 \vee G_2$. If n is odd, then $\omega(\mathrm{SAG}(G)) = |G_1| + 2$ and $\chi(\mathrm{SAG}(G)) = |G_1| + 3$, i.e., G is not perfect. If n is even, then every induced subgraph of G is weakly perfect and thus G is perfect.

Let $R \cong D_1 \times D_2 \times \cdots \times D_n$, where every D_i is an integral domain and $S = F_1 \times F_2 \times \cdots \times F_n$, where every F_i is a field. Next, we show that $\mathrm{SAG}(R)$ is perfect if and only if $\mathrm{SAG}(S)$ is perfect. First, we need the following results.

Let G be a graph and $x \in V(G)$ a vertex, and let G' be obtained from G by adding a vertex x' and joining it to x and all the neighbours of x. We say that G' is obtained from G by expanding the vertex x to an edge $x - x'$.

Lemma 2.1 ([7, Lemma 5.5.5]). *Any graph obtained from a perfect graph by expanding a vertex is again perfect.*

Remark 2.1. Let G be a graph and $x \in V(G), A \subseteq V(G)$. By Lemma 2.1, if for every $y \in A$, $N(x) = N(y)$ or $N[x] = N[y]$, then G is perfect if and only if $G \setminus \{A \setminus \{x\}\}$ is perfect.

Lemma 2.2. *Let R be a reduced ring and $I, J \in V(\mathrm{SAG}(R))$. If $\mathrm{Ann}(I) = \mathrm{Ann}(J)$, then $N(I) = N(J)$.*

Proof: Suppose that $K - I$ is an edge of $\mathrm{SAG}(R)$. Then by [11, Lemma 2.1], $\mathrm{Ann}(I) \nsubseteq \mathrm{Ann}(K)$ and $\mathrm{Ann}(K) \nsubseteq \mathrm{Ann}(I)$. Since $\mathrm{Ann}(I) = \mathrm{Ann}(J)$, we deduce that $\mathrm{Ann}(K) \nsubseteq \mathrm{Ann}(J)$ and $\mathrm{Ann}(J) \nsubseteq \mathrm{Ann}(K)$. This means that $K - J$ is an edge of $\mathrm{SAG}(R)$ and thus $N(I) \subseteq N(J)$. Similarly, $N(J) \subseteq N(I)$, as desired. □

Let $F_1, \ldots, F_n$ be fields and $D_1, \ldots, D_n$ be integral domains. It is worth mentioning that perfectness of strongly annihilating-ideal graphs induced by $\prod F_i$ and $\prod D_i$ does not depend on concrete fields and domains.

Corollary 2.1. *Let $R \cong D_1 \times \cdots \times D_n$, where every D_i is an integral domain $S = F_1 \times \cdots \times F_n$, where every F_i is a field. Then $\mathrm{SAG}(R)$ is perfect if and only if $\mathrm{SAG}(S)$ is perfect.*

Proof: Assume that $I = I_1 \times \cdots \times I_n$ and $J = J_1 \times \cdots \times J_n$ are vertices of $\mathrm{SAG}(R)$. Define the relation "$\sim$" on $V(\mathrm{SAG}(R))$ as follows: $I \sim J$, whenever "$I_i = 0$ if and only if $J_i = 0$" for every $1 \leq i \leq n$. It is easily seen that "$\sim$" is an equivalence relation on $V(\mathrm{SAG}(R))$ and thus $V(\mathrm{SAG}(R)) = \bigcup_{i=1}^{2^n-2} [I]_i$, where $[I]_i$ is the equivalence class of I. (We note that the number of equivalence classes is $2^n - 2$.) Let $[I]$ be the equivalence class of I and $J, K \in [I]$. Then $\mathrm{Ann}(J) = \mathrm{Ann}(K)$ and thus by Lemma 2.2, $N(J) = N(K)$. This, together with I is not adjacent to J, implies that $\mathrm{SAG}(R)$ is perfect if and only if $\mathrm{SAG}(R) \setminus \{[I] \setminus \{I\}\}$ is perfect, by Remark 2.1. If we continue this procedure for every equivalence class $[I]$ ($2^n - 2$ times), then we conclude that $\mathrm{SAG}(R)$ is perfect if and only if $\mathrm{SAG}(R)[A]$ is perfect, where

$$A = \{I = I_1 \times \cdots \times I_n \in V(\mathrm{SAG}(R))\colon\ I_i \in \{0, D_i\} \text{ for every } 1 \le i \le n\}.$$

It is straightforward to check that $\mathrm{SAG}(R)[A] \cong \mathrm{SAG}(S)$ and thus $\mathrm{SAG}(R)$ is perfect if and only if $\mathrm{SAG}(S)$ is perfect. □

To prove our main result in this section, we need two following lemmas.

Lemma 2.3. *Let $R = F_1 \times \cdots \times F_n$, where every F_i is a field and $I, J \in V(\mathrm{SAG}(R))$. Then $I - J$ is an edge of $\mathrm{SAG}(R)$ if and only if $I \nsubseteq J$ and $J \nsubseteq I$.*

Proof: First, assume that $I - J$ is an edge of $\mathrm{SAG}(R)$. If $I \subseteq J$ or $J \subseteq I$, then $\mathrm{Ann}(J) \subseteq \mathrm{Ann}(I)$ or $\mathrm{Ann}(I) \subseteq \mathrm{Ann}(J)$, a contradiction, by [11, Lemma 2.1].

The converse is clear. □

Lemma 2.4. *Let $R = F_1 \times \cdots \times F_n$, where every F_i is a field. Then $\mathrm{SAG}(R)$ is perfect.*

Proof: By Theorem 2.1, it is enough to show that $\mathrm{SAG}(R)$ and $\overline{\mathrm{SAG}(R)}$ contain no induced odd cycle of length at least 5. Consider the following claims:

Claim 1. $\mathrm{SAG}(R)$ contains no induced odd cycle of length at least 5. Assume to the contrary,

$$I_0 - I_1 - \cdots - I_{n-1} - I_0$$

is an induced odd cycle of length at least 5 in $\mathrm{SAG}(R)$. Since I_0 is not adjacent to I_2, by Lemma 2.3 we can assume that $I_0 \subseteq I_2$. Now, if $I_3 \subseteq I_0$, then $I_3 \subseteq I_2$, a contradiction. So $I_0 \subseteq I_3$. Next, we show that $I_1 \subseteq I_{n-1}$. For this, since I_1 is not adjacent to I_3, by Lemma 2.3, we conclude that $I_1 \subseteq I_3$ or $I_3 \subseteq I_1$. If $I_3 \subseteq I_1$, then $I_0 \subseteq I_1$, as $I_0 \subseteq I_3$. This is a contradiction, by Lemma 2.3. So $I_0 \subseteq I_3$. If we continue this procedure for $I_4, \ldots, I_{n-1}$, then we get $I_1 \subseteq I_{n-1}$. If we start the above argument from I_2, on $I_4, \ldots I_{n-1}, I_0$, then we get $I_2 \subseteq I_0$. This is a contradiction as $I_0 \subseteq I_2$, and so $\mathrm{SAG}(R)$ contains no induced odd cycle of length at least 5.

Claim 2. $\overline{\mathrm{SAG}(R)}$ contains no induced odd cycle of length at least 5. Assume to the contrary,

$$I_1 - I_2 - \cdots - I_n - I_1$$

is an induced odd cycle of length at least 5 in $\overline{\mathrm{SAG}(R)}$. By Lemma 2.3, we may assume that $I_1 \subseteq I_2$. If $I_2 \subseteq I_3$, then $I_1 \subseteq I_3$, a contradiction. Thus $I_1 \subseteq I_2$ and $I_3 \subseteq I_2$. If $I_4 \subseteq I_3$, then $I_4 \subseteq I_2$, a contradiction. So $I_3 \subseteq I_4$. If $I_4 \subseteq I_5$, then

$I_3 \subseteq I_4$ implies that $I_3 \subseteq I_5$, a contradiction. Thus $I_3 \subseteq I_4$ and $I_5 \subseteq I_4$. Since n is odd, if we continue this procedure, then $I_{n-2} \subseteq I_{n-1}$, $I_n \subseteq I_{n-1}$. If $I_1 \subseteq I_n$, then since $I_n \subseteq I_{n-1}$, $I_1 \subseteq I_{n-1}$, a contradiction. So $I_n \subseteq I_1$ and since $I_1 \subseteq I_2$, we deduce that $I_n \subseteq I_2$, a contradiction. Therefore, $\overline{\mathrm{SAG}(R)}$ contains no induced odd cycle of length at least 5. By Claims 1 and 2, $\mathrm{SAG}(R)$ is perfect. □

Using Lemma 2.4 and Corollary 2.1, we have the following immediate corollary.

Corollary 2.2. *Let R be a ring such that $R \cong D_1 \times \cdots \times D_n$, where D_i is an integral domain for every $1 \le i \le n < \infty$. Then $\mathrm{SAG}(R)$ is a perfect graph.*

We are now in a position to state the main result of this section.

Theorem 2.2. *Let R be a reduced ring and $\omega(\mathrm{SAG}(R)) < \infty$. Then $\mathrm{SAG}(R)$ is a perfect graph.*

Proof: Since $\omega(\mathrm{SAG}(R)) < \infty$, by [8, Lemma 2.5], $|\mathrm{Min}(R)| < \infty$. Let $\mathrm{Min}(R) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ and $T = R/\mathfrak{p}_1 \times \cdots \times R/\mathfrak{p}_n$. Define a ring homomorphism $\varphi\colon R \longrightarrow R/\mathfrak{p}_1 \times \cdots \times R/\mathfrak{p}_n$ with $\varphi(r) = (r + \mathfrak{p}_1, \dots, r + \mathfrak{p}_n)$. Since $\mathfrak{p}_i$ are distinct minimal prime ideals, $\mathfrak{p}_i \nsubseteq \mathfrak{p}_j$ for each $i \neq j$. Hence there exists nonzero ideals S_i of $R/\mathfrak{p}_i$ such that $S_1 \times \cdots \times S_n \subseteq \varphi(R) \cong R$ and $S_1 \times \cdots \times S_n$ is ideal of both rings $\varphi(R)$ and T. We put

$$A = \{I_1 \times \cdots \times I_n \in V(\mathrm{SAG}(\varphi(R)))\colon\ I_i \in \{0, S_i\} \text{ for every } 1 \le i \le n\},$$
$$B = \{J_1 \times \cdots \times J_n \in V(\mathrm{SAG}(T))\colon\ J_i \in \{\mathfrak{p}_i, R/\mathfrak{p}_i\} \text{ for every } 1 \le i \le n\}.$$

Now, we can easily get $\mathrm{SAG}(\varphi(R))[A] \cong \mathrm{SAG}(T)[B]$ and thus $\mathrm{SAG}(\varphi(R))[A]$ is perfect if and only if $\mathrm{SAG}(T)[B]$ is perfect. On the other hand, by proof of Corollary 2.1, we can easily get $\mathrm{SAG}(\varphi(R))$ and $\mathrm{SAG}(T)$ are perfect if and only if $\mathrm{SAG}(\varphi(R))[A]$ and $\mathrm{SAG}(T)[B]$ are perfect, respectively. This, together with Corollary 2.2, implies that $\mathrm{SAG}(\varphi(R))$ is a perfect graph and hence $\mathrm{SAG}(R)$ is a perfect graph. □

3. Strongly annihilating-ideal graph of a perfect ring is perfect

The main aim of this section is to show that $\mathrm{SAG}(R)$ is perfect in case R is a perfect ring. It is known that if $R/\mathrm{Jac}(R)$ is a semisimple ring, then $\mathrm{Soc}(R) = \mathrm{Ann}(\mathrm{Jac}(R))$ (see [13, part 21.15]). Also, to prove our main result it is very important that $\mathrm{Soc}(R)$ to be an essential ideal of R. Since the perfect rings have both of these properties, we may show that the strongly annihilating-ideal graph of a perfect ring is perfect.

Lemma 3.1. *Let R be a perfect ring and $I \subseteq \mathrm{Jac}(R)$. Then $\mathrm{Ann}(I)$ is an essential ideal of R.*

Proof: Since R is perfect, $\mathrm{Ann}(\mathrm{Jac}(R))(=\mathrm{Soc}(R))$ is an essential ideal of R. Since $I \subsetneq \mathrm{Jac}(R)$, $\mathrm{Ann}(\mathrm{Jac}(R)) \subseteq \mathrm{Ann}(I)$ and so $\mathrm{Ann}(I)$ is an essential ideal of R. □

Lemma 3.2. *Let R be a perfect ring and $I, J \in A(R)^*$. Then the following statements hold.*

(1) If $I \nsubseteq J$, then $I \cap \mathrm{Ann}(J) \neq (0)$. In particular, if $I \nsubseteq J$ and $J \nsubseteq I$, then $I - J$ is an edge of $\mathrm{SAG}(R)$.

(2) If $I \subseteq J$ and $I \cap \mathrm{Ann}(J) \neq (0)$, then $I - J$ is an edge of $\mathrm{SAG}(R)$.

Proof: (1) Let $I = I_1 \times \cdots \times I_n$ and $J = J_1 \times \cdots \times J_n$. Since $I \nsubseteq J$, with no loss of generality, assume that $I_1 \nsubseteq J_1$. This implies that $J_1 \neq R_1$ and thus $\mathrm{Ann}(J_1)$ is an essential ideal of R_1, by Lemma 3.1. Hence $I_1 \cap \mathrm{Ann}(J_1) \neq (0)$. Let $0 \neq a_1 \in I_1 \cap \mathrm{Ann}(J_1)$. Then $(a_1, 0, \ldots, 0) \in I \cap \mathrm{Ann}(J)$ and so $I \cap \mathrm{Ann}(J) \neq (0)$. The "in particular" statement is now clear.

(2) Since $I \cap \mathrm{Ann}(J) \neq (0)$, we need only to show that $J \cap \mathrm{Ann}(I) \neq (0)$. Let $0 \neq a \in I \cap \mathrm{Ann}(J)$. Since $I \subseteq J$, $a \in J$. Also, $aJ = (0)$ and $I \subseteq J$ imply that $aI = (0)$. Thus $a \in J \cap \mathrm{Ann}(I)$ and so $I - J$ is an edge of $\mathrm{SAG}(R)$. □

Theorem 3.1. *Let R be a perfect ring. Then $\mathrm{SAG}(R)$ is perfect.*

Proof: Since R is perfect, then there exists a positive integer n such that $R \cong R_1 \times \cdots \times R_n$, where every R_i is a local ring. The argument here is a refinement of the proof of Corollary 2.1 and Lemma 2.4. By Theorem 2.1, it is enough to show that $\mathrm{SAG}(R)$ and $\overline{\mathrm{SAG}(R)}$ contain no induced odd cycle of length at least 5. Consider the following claims:

Claim 1. $\mathrm{SAG}(R)$ contains no induced odd cycle of length at least 5. Assume to the contrary,

$$I_0 - I_1 - \cdots - I_{n-1} - I_0$$

is an induced odd cycle of length at least 5 in $\mathrm{SAG}(R)$. Since I_0 is not adjacent to I_2, by Part (1) of Lemma 3.2, $I_0 \subseteq I_2$ or $I_2 \subseteq I_0$. Without loss of generality, we may assume that $I_0 \subseteq I_2$. Since I_0 is not adjacent to I_3, by Lemma 3.2, $I_0 \subseteq I_3$ or $I_3 \subseteq I_0$. If $I_3 \subseteq I_0$, then since $I_0 \subseteq I_2$, $I_3 \subseteq I_2$. Since $I_3 \cap \mathrm{Ann}(I_2) \neq (0)$ and $I_3 \subseteq I_0$, $I_0 \cap \mathrm{Ann}(I_2) \neq (0)$. This, together with Part (2) of Lemma 3.2, implies that I_0 is adjacent to I_2, a contradiction. Thus $I_0 \subseteq I_3$. Now, by a refinement of the proof of Lemma 2.4, we get $I_1 \subseteq I_{n-1}$ and $I_2 \subseteq I_0$. This is a contradiction as $I_0 \subseteq I_2$, and so $\mathrm{SAG}(R)$ contains no induced odd cycle of length at least 5.

Claim 2. $\overline{\mathrm{SAG}(R)}$ contains no induced odd cycle of length at least 5. Assume to the contrary,

$$I_1 - I_2 - \cdots - I_n - I_1$$

is an induced odd cycle of length at least 5 in $\overline{\mathrm{SAG}(R)}$. By Lemma 3.2, we may assume that $I_1 \subseteq I_2$. If $I_2 \subseteq I_3$, then since $I_1 \cap \mathrm{Ann}(I_3) \neq (0)$, we conclude that $I_2 \cap \mathrm{Ann}(I_3) \neq (0)$. By Lemma 3.2, I_2 is adjacent to I_3 in $\mathrm{SAG}(R)$, a contradiction. Thus $I_1 \subseteq I_2$ and $I_3 \subseteq I_2$. If $I_4 \subseteq I_3$, then $I_3 \cap \mathrm{Ann}(I_2) \neq (0)$, as $I_4 \cap \mathrm{Ann}(I_2) \neq (0)$ and thus by Lemma 3.2, I_2 is adjacent to I_3 in $\mathrm{SAG}(R)$, a contradiction. So $I_3 \subseteq I_4$. If $I_4 \subseteq I_5$, then $I_3 \subseteq I_4$ and $I_3 \cap \mathrm{Ann}(I_5) \neq (0)$ imply that $I_4 \cap \mathrm{Ann}(I_5) \neq (0)$ and thus by Lemma 3.2, I_4 is adjacent to I_5 in $\mathrm{SAG}(R)$, a contradiction. Thus $I_3 \subseteq I_4$ and $I_5 \subseteq I_4$. Since n is odd, by continuing this procedure $I_{n-2} \subseteq I_{n-1}$ and $I_n \subseteq I_{n-1}$. If $I_1 \subseteq I_n$, then $I_1 \cap \mathrm{Ann}(I_{n-1}) \neq (0)$ implies that $I_n \cap \mathrm{Ann}(I_{n-1}) \neq (0)$ and thus by Lemma 3.2, I_n is adjacent to I_{n-1} in $\mathrm{SAG}(R)$, a contradiction. Hence $I_n \subseteq I_1$ and so $I_n \cap \mathrm{Ann}(I_2) \neq (0)$. Thus $I_1 \cap \mathrm{Ann}(I_2) \neq (0)$. By Lemma 3.2, I_1 is adjacent to I_2 in $\mathrm{SAG}(R)$, a contradiction. Therefore, $\overline{\mathrm{SAG}(R)}$ contains no induced odd cycle of length at least 5.

By Claims 1 and 2, the proof is complete. □

We have not found any examples of a non-domain ring R such that $\mathrm{SAG}(R)$ is not perfect. The lack of such counterexamples, together with the fact that $\mathrm{SAG}(R)$ is perfect if R is reduced (with $\omega(\mathrm{SAG}(R)) < \infty$) or perfect motivates the following conjecture.

Conjecture 3.1. *Let R be a ring. Then* $\mathrm{SAG}(R)$ *is perfect.*

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