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Noncompact perturbation of nonconvex noncompact sweeping process with delay

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Abstract. We prove an existence theorem of solutions for a nonconvex sweeping process with nonconvex noncompact perturbation in Hilbert space. We do not assume that the values of the orient field are compact.

Keywords: nonconvex sweeping process; functional differential inclusion; uniformly ϱ-prox-regular set

Classification: 34A60, 34B15, 47H10

1. Introduction

The aim of the paper is to prove the existence of a solution of the following nonconvex noncompact sweeping process with delay

$$(1)\qquad \begin{cases} u'(t) \in -N^P_{C(t,u(t))}(u(t)) + F(t,\tau(t)u) & \text{a.e. for } t \in [0,T];\\ u(t) = \varphi(t) & \text{for } t \in [-r,0];\\ u(t) \in C(t,u(t)) & \text{for } t \in [0,T], \end{cases}$$

where C is a set-valued mapping with ϱ-prox-regular closed (not necessarily compact) values in a Hilbert space H, $N^P_{C(t,u(t))}(u(t))$ is the proximal normal cone of $C(t,u(t))$ at the point $u(t)$, F is a set-valued mapping with nonconvex and noncompact values in H and φ is a given continuous function.

In the setting when the values of C are assumed to be convex or the complement of the interior of a convex set, the problem has been considered by several authors, see [6], [10], [14], [13], [15], [16], and the references therein.

Recently, using important properties of uniformly ϱ-prox-regular sets developed in [5], [2], [18], the existence of solutions of the sweeping process with convex or nonconvex perturbation is established, see for example [1], [9], [12]. There are also papers by J. F. Edmond and L. Thibault in Mathematical programming and by M. Sene and L. Thibault in Set-valued and variational analysis with single-valued Lipschitz perturbation. Remark that, in all the cited papers, the compactness assumption on the perturbation is widely used.

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In recent papers, M. Atialioubrahim in [2] established the existence of solution for (1) when the values of C are nonconvex and compact in Hilbert space H, while in Theorem 3.1 in [4] M. Bounkhel and C. Castaing established the existence of solution for (1) when $F = 0$ and the values of C are convex (not necessarilly compact) and satisfies a compactness condition involving a measure of non compactness.

In this paper, our main purpose is to obtain the existence of solution of (1) in the case when the values of C are closed uniformly ϱ-prox-regular (noncompact), but a condition involving the Kuratowski/Hausdorf measure of noncompactness is used in Theorem 3.1, and also the perturbation F are nonconvex, noncompact, integrably bounded, measurable with respect to the first argument and Lipschitz continuous with respect to the second argument.

2. Preliminaries

Let H be a real separable Hilbert space with the norm $\|\cdot\|$ and the scalar inner product $\langle\cdot,\cdot\rangle$. For I a segment in $\mathbb{R}$, we denote by $C(I,H)$ the Banach space of continuous functions from I to H equipped with the norm $\|y\|_\infty := \sup\{\|y(t)\|\colon t \in I\}$. For r a positive number, we put $C_r = C([-r,0],H)$ and for any $t \in [0,T]$, $T > 0$, we define the operator $\tau(t)\colon C([-r,T],H) \longrightarrow C_r$ with $(\tau(t)g)(s) = g(t+s)$ for all $s \in [-r,0]$.

For $x \in H$ and $a > 0$, let $\mathbb{B}(0,a) := \{y \in H\colon \|y\| < a\}$ be the open ball centered at 0 with radius a and $\overline{\mathbb{B}}(0,a)$ be the closure of $\mathbb{B}(0,a)$, $\mathbb{B}(0,1)$ be the open unit ball centered at 0. For $x \in H$ and for nonempty subsets A, B of H, we denote $d(x,B) = \inf\{\|y-x\|\colon y \in B\}$, $e(A,B) = \sup\{d(x,B)\colon x \in A\}$ the excess of the set A over the set B and $h(A,B) = \max\{e(A,B), e(B,A)\}$ the Hausdorff distance between the set A and B. For measurability purpose, H (or $\Omega \subset H$) is endowed with the σ-algebra $\mathcal{B}(H)$ (or $\mathcal{B}(\Omega)$, respectively) of Borel subsets for the strong topology and the segment I is endowed with Lebesgue measure and the σ-algebra of Lebesgue measurable subsets. A set-valued mapping is said to be measurable if its graph is measurable. For more details on measurability theory, we refer the reader to book [7] of C. Castaing and M. Valadier.

We need first to recall some notations and definitions that will be used in the paper.

Let $V\colon H \longrightarrow \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function and x be any point where V is finite. The proximal subdifferential $\partial^P V(x)$ of V at x is the set of all $y \in H$ for which there exist δ, $\sigma > 0$ such that for all $x' \in x + \delta\overline{\mathbb{B}}$

$$\langle y, x'-x\rangle \le V(x') - V(x) + \sigma\|x'-x\|^2.$$

Let S be a nonempty closed subset of H and x be a point in S. We recall, see [8], that the proximal normal cone of S at x is defined by $N_S^P(x) := \partial^P \Psi_S(x)$, where Ψ_S denotes the indicator function of S, i.e.

$$\Psi_S(x) = \begin{cases} 0 & \text{if } x \in S, \\ \infty & \text{otherwise.} \end{cases}$$

Recall now, that for a given $\varrho \in (0, \infty]$, a subset S is uniformly ϱ-prox-regular, see [17], (or equivalently ϱ-proximally smooth, see [8]) if and only if every nonzero proximal normal to S can be realized by a ϱ-ball. This means that for all $\overline{x} \in S$ and all $\zeta \in N_S^P(\overline{x}) \backslash \{0\}$, one has

$$\left\langle \frac{\zeta}{\|\zeta\|}, x - \overline{x} \right\rangle \leq \frac{1}{2\varrho} \|x - \overline{x}\|^2$$

for all $x \in S$. We make the convention $1/\varrho = 0$ for $\varrho = \infty$. Recall that for $\varrho = \infty$ the uniform ϱ-prox-regularity of S is equivalent to the convexity of S.

The following propositions summarize some important consequences of uniform prox-regularity needed in the sequel.

Proposition 2.1 ([17]). *The following assertions hold.*

(1) $\partial^P d_S(x) = N_S^P(x) \cap \overline{\mathbb{B}}$.

(2) *Let $\varrho \in]0, \infty]$. If S is uniformly ϱ-prox-regular, then for all $x \in H$ with $d_S(x) < \varrho$ one has $\operatorname{proj}_S(x) \neq \emptyset$ and $\partial^P d_S(x) = \partial^C d_S(x)$, see L. Thibault in [18]. So, in such a case, the subdifferential $\partial d_S(x) := \partial^P d_S(x) = \partial^C d_S(x)$ is a closed convex set in H.*

(3) *If S is uniformly ϱ-prox-regular, then for all $x_i \in S$ and all $v_i \in N_S^P(x_i)$ with $\|v_i\| \leq \varrho$, $i = 1, 2$, one has*

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geq -\|x_1 - x_2\|^2.$$

As a consequence of (3), we get that for uniformly ϱ-prox-regular sets, the proximal and Clarke normal cones at all points $x \in S$ coincide, i.e., $N_S^P(x) = N_S^C(x)$. In such a case, we put $N_S(x) := N_S^P(x) = N_S^C(x)$.

Proposition 2.2 ([5]). *Let $\varrho \in]0, \infty]$ and Ω be an open subset in H and let $C \colon \Omega \longrightarrow 2^H$ be a Hausdorff-continuous set-valued mapping. Assume that C has uniformly ϱ-prox-regular values. Then, the set-valued mapping given by $(z, x) \longrightarrow \partial d_{C(z)}(x)$ from $\Omega \times H$ (endowed with the strong topology) to H (endowed with the weak topology) is upper semicontinuous, which is equivalent to the upper semicontinuity of the function $(z, x) \longrightarrow \sigma(\partial d_{C(z)}(x), p)$ for any $p \in H$. Here $\sigma(S, p)$ denotes the support function associated with S, i.e., $\sigma(S, p) = \sup\{\langle s, p \rangle \colon s \in S\}$.*

Lemma 2.3 ([19])**.** *Let Ω be a nonempty set in H. Assume that $F\colon [a,b]\times\Omega \longrightarrow 2^H$ is a set-valued mapping with nonempty closed values satisfying:*

- *for every $x\in\Omega$, $F(.,x)$ is measurable on $[a,b]$;*
- *for every $t\in[a,b]$, $F(t,.)$ is (Hausdorff) continuous on Ω.*

Then for any measurable function $x\colon [a,b]\longrightarrow\Omega$, the set-valued mapping $F(.,x)$ is measurable on $[a,b]$.

Lemma 2.4 ([19])**.** *Let $G\colon [a,b]\longrightarrow 2^H$ be a measurable set-valued mapping and $h\colon [a,b]\longrightarrow H$ a measurable function. Then for any positive measurable function $r\colon [a,b]\longrightarrow\mathbb{R}^+$, there exists a measurable selection g of G such that for almost all $t\in[a,b]$*

$$\|g(t)-h(t)\|\le d(h(t),G(t))+r(t).$$

3. Main result

In this section, we prove our main result.

Theorem 3.1. *Let H be a separable Hilbert space, $b>0$, $I=[0,b]$, $C\colon I\times H\longrightarrow 2^H$ be a set-valued mapping with nonempty closed values and $F\colon I\times C_r\longrightarrow 2^H$ be a set-valued mapping with nonempty closed values. Assume that the following hypotheses hold:*

(C_1) *for each $t\in I$, $C(t,.)$ is ϱ-prox-regular for some fixed $\varrho\in\,]0,\infty[$;*

(C_2) *there are positive constant λ and an absolutely continuous function $v\colon I\longrightarrow\mathbb{R}$ such that*

$$|d_{C(t_1,x_1)}(u_1)-d_{C(t_2,x_2)}(u_2)|\le|v(t_1)-v(t_2)|+\|x_1-x_2\|+\lambda\|u_1-u_2\|$$

for all $u_1,u_2,x_1,x_2\in H$ and $t_1,t_2\in I$;

(C_3) *for any $t\in I$, and any bounded set M in H with $\gamma(M)>0$, $\kappa>0$ one has*

$$\gamma(C(t,M)\cap\kappa\overline{\mathbb{B}})<\gamma(M),$$

where $\gamma=\alpha$ or $\gamma=\beta$ is either the Kuratowski or the Housdorff measure of noncompactness;

(F_1) *for each $\psi\in C_r$, $t\longrightarrow F(t,\psi)$ is a measurable;*

(F_2) *there is a function $m\in L^1(I,\mathbb{R}^+)$ such that for all $t\in I$, and for all $\psi_1,\psi_2\in C_r$*

$$h(F(t,\psi_1),F(t,\psi_2))\le m(t)(\|\psi_1-\psi_2\|_\infty);$$

(F_3) *there exist two functions $g,p\in L^1(I,\mathbb{R}^+)$ such that for all $t\in I$ and for all $\psi\in C_r$*

$$\|F(t,\psi)\|\le g(t)+p(t)\|\psi\|_\infty.$$

Then, for any $\varphi \in C_r$ *with* $\varphi(0) \in C(0, \varphi(0))$, *there exist* $T \in]0, b[$ *and a continuous function* $u\colon [-r, T] \longrightarrow H$ *that is absolutely continuous on* $[0, T]$ *and such that* u *is a solution of* (1) *and satisfies*

$$\|u'(t)\| \leq 2g(t) + 2p(t)(\|\varphi\|_\infty + r) + |v'(t)| \qquad \text{for almost all } t \in [0, T].$$

Proof: Let $T_1 > 0$ be such that

$$\int_0^{T_1} (2g(t) + 2p(t)(\|\varphi\|_\infty + r) + |v'(t)|)\, \mathrm{d}t < \inf\Big\{\frac{\varrho}{2}, \frac{r}{2}\Big\}. \tag{2}$$

The idea of such T_1 has been used in [11]. For $\varepsilon > 0$ set

$$\begin{aligned} \eta(\varepsilon) = \sup\Big\{\zeta \in]0, \varepsilon] \colon \Big|\int_{t_1}^{t_2} (2g(s) + 2p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\, \mathrm{d}s\Big| < \varepsilon, \\ \text{and } \|\varphi(t_1) - \varphi(t_2)\| < \varepsilon \ \text{ if } \ |t_1 - t_2| < \zeta\Big\}. \end{aligned} \tag{3}$$

Put

$$T = \min\Big\{T_1, \frac{1}{2}\eta\Big(\frac{\varepsilon}{2}\Big), b\Big\},$$

and

$$\mu = \|\varphi(0)\| + \int_0^T (2g(\beta) + 2p(\beta)(\|\varphi\|_\infty + r) + |v'(\beta)|)\, \mathrm{d}\beta.$$

Let n be a fixed natural number. Consider a sequence of subdivisions $(P_n)_{n\geq 1}$ of $[0, T]$:

$$P_n = \{0 = t_0^n < t_1^n < \cdots < t_i^n < \cdots < t_{2^n}^n = T\},$$

where $t_i^n = iT/2^n$, $0 < i < 2^n$.

In order to make it easier for the reader we will divide the proof in the following steps:

Step 1: For all $z \in L^1([0, T], H)$, there exist a sequence (x_i^n) of H, $0 \leq i \leq 2^n$, with $x_0^n = \varphi(0)$, a continuous function $u_n\colon [-r, T] \longrightarrow H$, and $f_n \in L^1([0, T], H)$

such that for all $i = 0, 1, \dots, 2^n - 1$, the following properties are satisfied:

$$
(4)\qquad \begin{cases}
\text{(i)} & u_n(t) = \varphi(t) \quad \text{for} \quad t \in [-r, 0];\\
\text{(ii)} & f_n(0) \in F(t, \tau(0)\varphi) \quad \text{and} \quad f_n(t) \in F(t, \tau(t_i^n)u_n)\\
& \qquad \text{for } t \in (t_i^n, t_{i+1}^n];\\
\text{(iii)} & u_n(t_{i+1}^n) = x_{i+1}^n \in \operatorname{proj}_{C(t_{i+1}^n, x_i^n)}\Big(x_i^n + \int\limits_{t_i^n}^{t_{i+1}^n} f_n(s)\,\mathrm{d}s\Big);\\
\text{(iv)} & \|u_n(t)\| \le \mu \quad \text{for} \quad t \in [t_i^n, t_{i+1}^n];\\
\text{(v)} & \|u_n(t) - u_n(s)\| \le \int\limits_s^t (2g(\beta) + 2p(\beta)(\|\varphi\|_\infty + r)\\
& \qquad + |v'(\beta)|)\,\mathrm{d}\beta \quad \text{for} \quad t_i^n \le s < t \le t_{i+1}^n;\\
\text{(vi)} & \|f_n(0) - z(0)\| \le d(z(0), F(0, \tau(0)\varphi) + \frac{1}{n^2},\\
& \|f_n(t) - z(t)\| \le d(z(t), F(t, \tau(t_i^n)u_n)) + \frac{1}{n^2}, \quad \text{and}\\
& \|f_n(t)\| \le g(t) + p(t)(\|\varphi\|_\infty + r) \text{ for } t \in (t_i^n, t_{i+1}^n];\\
\text{(vii)} & u_n'(t) - f_n(t) \in -N_{C(t_{i+1}^n, u_n(t_i^n))}(u_n(t_{i+1}^n))\\
& \qquad \text{for a.e.} \quad t \in (t_i^n, t_{i+1}^n];\\
\text{(viii)} & \|u_n'(t) - f_n(t)\| \le |e'(t)| = g(t) + p(t)(\|\varphi\|_\infty + r) + |v'(t)|\\
& \qquad \text{for a.e.} \quad t \in (t_i^n, t_{i+1}^n],
\end{cases}
$$

where $e\colon [0, T] \longrightarrow \mathbb{R}^+$ is defined by

$$
(5)\qquad e(t) := \int_0^t (g(s) + p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s \qquad \text{for all} \quad t \in [0, T].
$$

Set $u_n(s) = \varphi(s)$ for all $s \in [-r, 0]$. Put $x_0^n = \varphi(0) \in C(t_0^n, \varphi(0))$.

In view of Lemma 2.4, there exists a function $f_0^n \in L^1([0, t_1^n]), H)$ such that $f_0^n(t) \in F(t, \tau(0)u_n)$ and

$$
\|f_0^n(t) - z(t)\| \le d(z(t), F(t, \tau(0)u_n)) + \frac{1}{n^2} \qquad \text{for all} \quad t \in [0, t_1^n].
$$

Note that, by condition (F_3) for any $t \in [0, t_1^n]$, we have

$$
(6)\qquad \begin{aligned}
\|f_0^n(t)\| &\le g(t) + p(t)\|\tau(0)u_n\|_\infty\\
&= g(t) + p(t) \sup_{s \in [-r, 0]} \|\tau(0)u_n(s)\| = g(t) + p(t) \sup_{s \in [-r, 0]} \|u_n(s)\|\\
&= g(t) + p(t) \sup_{s \in [-r, 0]} \|\varphi(s)\| = g(t) + p(t)\|\varphi\|_\infty.
\end{aligned}
$$

Then, thanks to the condition (C_2) and (6), we have

$$
\begin{aligned}
(7)\quad d_{C(t_1^n,x_0^n)}\Big(x_0^n+\int_{t_0^n}^{t_1^n} f_0^n(s)\,\mathrm{d}s\Big) &\le d_{C(t_0^n,x_0^n)}\Big(x_0^n+\int_{t_0^n}^{t_1^n} f_0^n(s)\,\mathrm{d}s\Big)\\
&\quad+\Big|d_{C(t_1^n,x_0^n)}\Big(x_0^n+\int_{t_0^n}^{t_1^n} f_0^n(s)\,\mathrm{d}s\Big)-d_{C(t_0^n,x_0^n)}\Big(x_0^n+\int_{t_0^n}^{t_1^n} f_0^n(s)\,\mathrm{d}s\Big)\Big|\\
&\le \int_{t_0^n}^{t_1^n}\|f_0^n(s)\|\,\mathrm{d}s+|v(t_1^n)-v(t_0^n)|\\
(8)\qquad &\le \int_{t_0^n}^{t_1^n}(g(s)+p(s)\|\varphi\|_\infty+|v'(s)|)\,\mathrm{d}s.
\end{aligned}
$$

This inequality with (2) gives

$$d_{C(t_1^n,x_0^n)}\Big(x_0^n+\int_{t_0^n}^{t_1^n} f_0^n(s)\,\mathrm{d}s\Big)\le\frac{\varrho}{2}.$$

As C has uniformly ϱ-prox-regular values, by Proposition 3.2.1, we have

$$\mathrm{proj}_{C(t_1^n,x_0^n)}\Big(x_0^n+\int_{t_0^n}^{t_1^n} f_0^n(s)\,\mathrm{d}s\Big)\neq\emptyset.$$

Then thanks to that, one can choose a point x_1^n such that

$$x_1^n\in\mathrm{proj}_{C(t_1^n,x_0^n)}\Big(x_0^n+\int_{t_0^n}^{t_1^n} f_0^n(s)\,\mathrm{d}s\Big). \tag{9}$$

Note that, $x_1^n\in C(t_1^n,x_0^n)$ and by (8) we get

$$
\begin{aligned}
\Big\|x_1^n-\Big(x_0^n+\int_{t_0^n}^{t_1^n} f_0^n(s)\,\mathrm{d}s\Big)\Big\| &= d_{C(t_1^n,x_0^n)}\Big(x_0^n+\int_{t_0^n}^{t_1^n} f_0^n(s)\,\mathrm{d}s\Big)\\
&\le\int_{t_0^n}^{t_1^n}(g(s)+p(s)\|\varphi\|_\infty+|v'(s)|)\,\mathrm{d}s.
\end{aligned}\tag{10}
$$

This inequality gives

$$\left\| x_1^n - \left(x_0^n + \int_{t_0^n}^{t_1^n} f_0^n(s)\,\mathrm{d}s \right) \right\| = e(t_1^n) - e(t_0^n). \tag{11}$$

Remark that, by (6) and (10), we have

$$\begin{aligned} \|x_1^n\| &\le \left\| x_1^n - \left(x_0^n + \int_{t_0^n}^{t_1^n} f_0^n(s)\,\mathrm{d}s \right) \right\| + \|x_0^n\| + \int_{t_0^n}^{t_1^n} \|f_0^n(s)\|\,\mathrm{d}s \\ &\le \int_{t_0^n}^{t_1^n} (g(s) + p(s)\|\varphi\|_\infty + |v'(s)|)\,\mathrm{d}s + \|\varphi(0)\| \\ &\quad + \int_{t_0^n}^{t_1^n} (g(s) + p(s)\|\varphi\|_\infty)\,\mathrm{d}s \\ &\le \|\varphi(0)\| + \int_{t_0^n}^{t_1^n} (2g(s) + 2p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s. \end{aligned} \tag{12}$$

Note that, $x_1^n \in \mu\overline{\mathbb{B}}$.

Next, for all $t \in [t_0^n, t_1^n]$, we set $f_n(t) = f_0^n(t)$ and

$$u_n(t) := x_0^n + \frac{e(t) - e(t_0^n)}{e(t_1^n) - e(t_0^n)} \left(x_1^n - x_0^n - \int_{t_0^n}^{t_1^n} f_n(s)\,\mathrm{d}s \right) + \int_{t_0^n}^{t} f_n(s)\,\mathrm{d}s. \tag{13}$$

Obviously, f_n and u_n satisfy (i), (ii), (iii) and (vi) of (4) for $i = 0$. Let us claim that (v) is satisfied for $i = 0$. So let $t, s \in [t_0^n, t_1^n]$, $s < t$, from (5), (6), (11) and (13) one has

$$\begin{aligned} \|u_n(t) - u_n(s)\| &\le \frac{e(t) - e(s)}{e(t_1^n) - e(t_0^n)} \left\| x_1^n - x_0^n - \int_{t_0^n}^{t_1^n} f_n(s)\,\mathrm{d}s \right\| + \int_s^t \|f_n(\beta)\|\,\mathrm{d}\beta \\ &\le |e(t) - e(s)| + \int_s^t (g(\beta) + p(\beta)\|\varphi\|_\infty)\,\mathrm{d}\beta \end{aligned}$$

$$
\begin{aligned}
&\le \int_s^t (g(\beta)+p(\beta)(\|\varphi\|_\infty+r)+|v'(\beta)|)\,\mathrm{d}\beta \\
&\quad + \int_s^t (g(\beta)+p(\beta)(\|\varphi\|_\infty+r))\,\mathrm{d}\beta \\
&= \int_s^t (2g(\beta)+2p(\beta)(\|\varphi\|_\infty+r)+|v'(\beta)|)\,\mathrm{d}\beta.
\end{aligned}
\tag{14}
$$

This shows that (v) is satisfied for $i=0$.

Observe that from (14) for all $t \in [t_0^n, t_1^n]$

$$
\begin{aligned}
\|u_n(t)-\varphi(0)\| &= \|u_n(t)-u_n(t_0^n)\| \\
&\le \|u_n(t)-u_n(t_1^n)\| + \|u_n(t_1^n)-u_n(t_0^n)\| \\
&\le \int_t^{t_1^n} (2g(s)+2p(s)(\|\varphi\|_\infty+r)+|v'(s)|)\,\mathrm{d}s \\
&\quad + \int_{t_0^n}^{t_1^n} (2g(s)+2p(s)(\|\varphi\|_\infty+r)+|v'(s)|)\,\mathrm{d}s \\
&\le \int_{t_0^n}^{T} (2g(s)+2p(s)(\|\varphi\|_\infty+r)+|v'(s)|)\,\mathrm{d}s.
\end{aligned}
\tag{15}
$$

This inequality leads to $u_n(t) \in \mu\overline{\mathbb{B}}$ for all $t \in [t_0^n, t_1^n]$. So, (iv) is satisfied for $i=0$.

Consequently, (15) with (2) for all $t \in [t_0^n, t_1^n]$ gives us

$$
\|u_n(t)-\varphi(0)\| \le \frac{r}{2}. \tag{16}
$$

Remark that from (13) for $t \in]t_0^n, t_1^n[$ we get

$$
u_n'(t) = \frac{e'(t)}{e(t_1^n)-e(t_0^n)}\left(x_1^n - x_0^n - \int_{t_0^n}^{t_1^n} f_n(s)\,\mathrm{d}s\right) + f_n(t). \tag{17}
$$

This equality with (5) and (11) gives us

$$
\|u_n'(t)-f_n(t)\| \le |e'(t)| = g(t)+p(t)(\|\varphi\|_\infty+r)+|v'(t)|
$$

for almost all $t \in]t_0^n, t_1^n[$. Moreover, from (9) we have

$$x_1^n - x_0^n - \int_{t_0^n}^{t_1^n} f_n(s)\,\mathrm{d}s \in -N_{C(t_1^n, x_0^n)}\,(x_1^n).$$

This relation with (17) gives us

$$u_n'(t) - f_n(t) \in -N_{C(t_1^n, u_n(t_0^n))}(u_n(t_1^n)) \qquad \text{for a.e. } \ t \in [t_0^n, t_1^n].$$

Then (vii) and (viii) are satisfied for $i = 0$.

In order to define u_n and f_n on $t \in (t_1^n, t_2^n]$, we note that, in view of Lemma 3.2.4, there exists a function $f_1^n \in L^1((t_1^n, t_2^n], H)$ such that $f_1^n(t) \in F(t, \tau(t_1^n)u_n)$ and

$$\|f_1^n(t) - z(t)\| \le d(z(t), F(t, \tau(t_1^n)u_n)) + \frac{1}{n^2} \qquad \text{for all } \ t \in (t_1^n, t_2^n].$$

Consequently, by condition (F_3) for any $t \in (t_1^n, t_2^n]$ we get

$$\begin{aligned} \|f_1^n(t)\| &\le g(t) + p(t)\|\tau(t_1^n)u_n\|_\infty = g(t) + p(t) \sup_{s\in[-r,0]} \|\tau(t_1^n)u_n(s)\| \\ &= g(t) + p(t) \sup_{s\in[-r,0]} \|u_n(t_1^n + s)\| \\ &\le g(t) + p(t)\Big(\sup_{s\in[-r,0]} \|u_n(t_1^n + s) - \varphi(0)\| + \|\varphi\|_\infty\Big). \end{aligned} \tag{18}$$

Now, we have to estimate $\|u_n(t_1^n + s) - \varphi(0)\|$ for each $s \in [-r, 0]$. We have two cases

(1) If $-t_1^n \le s \le 0$, then $t_1^n + s \in [0, t_1^n]$. Thus, by (16) we get

$$\|u_n(t_1^n + s) - \varphi(0)\| \le \frac{r}{2}. \tag{19}$$

(2) If $-r \le s \le -t_1^n$, then $t_1^n + s \in [t_1^n - r, 0] \subset [-r, 0]$. Therefore, by the fact that $|t_1^n + s| \le T < \eta(r/2)$ and (3), we have

$$\|u_n(t_1^n + s) - \varphi(0)\| = \|\varphi(t_1^n + s) - \varphi(0)\| \le \frac{r}{2}. \tag{20}$$

Then, by (18), (19) and (20) for all $t \in (t_1^n, t_2^n]$ one obtains

$$\|f_1^n(t)\| \le g(t) + p(t)(\|\varphi\|_\infty + r). \tag{21}$$

By condition (C$_2$), (10) and (21), we get

$$
\begin{aligned}
& d_{C(t_2^n, x_1^n)}\left(x_1^n + \int_{t_1^n}^{t_2^n} f_1^n(s)\,\mathrm{d}s\right) \\
& \le d_{C(t_1^n, x_0^n)}\left(x_1^n + \int_{t_1^n}^{t_2^n} f_1^n(s)\,\mathrm{d}s\right) + |v(t_2^n) - v(t_1^n)| + \|x_1^n - x_0^n\| \qquad (22)\\
& \le \int_{t_1^n}^{t_2^n} (\|f_1^n(s)\| + |v'(s)|)\,\mathrm{d}s + \left\| x_1^n - x_0^n - \int_{t_1^n}^{t_0^n} f_0^n(s)\,\mathrm{d}s \right\| \\
& \quad + \int_{t_0^n}^{t_1^n} \|f_0^n(s)\|\,\mathrm{d}s \\
& \le \int_{t_1^n}^{t_2^n} (g(s) + p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s \\
& \quad + \int_{t_0^n}^{t_1^n} (g(s) + p(s)\|\varphi\|_\infty + |v'(s)|)\,\mathrm{d}s + \int_{t_0^n}^{t_1^n} (g(s) + p(s)\|\varphi\|_\infty)\,\mathrm{d}s \\
& \le \int_{t_1^n}^{t_2^n} (g(s) + p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s \\
& \quad + \int_{t_0^n}^{t_1^n} (2g(s) + 2p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s. \qquad (23)
\end{aligned}
$$

This inequality with (2) gives us

$$d_{C(t_2^n, x_1^n)}\left(x_1^n + \int_{t_1^n}^{t_2^n} f_1^n(s)\,\mathrm{d}s\right) \le \int_0^T (2g(s) + 2p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s \le \frac{\varrho}{2}.$$

As C has uniformly ϱ-prox-regular values, by Proposition 3.2.1, we have

$$\operatorname{proj}_{C(t_2^n, x_1^n)}\left(x_1^n + \int_{t_1^n}^{t_2^n} f_1^n(s)\,\mathrm{d}s\right) \ne \emptyset.$$

Then thanks to that, one can choose a point x_2^n such that

$$x_2^n \in \operatorname{proj}_{C(t_2^n, x_1^n)}\left(x_1^n + \int_{t_1^n}^{t_2^n} f_1^n(s)\,\mathrm{d}s\right). \tag{24}$$

Note that, $x_2^n \in C(t_2^n, x_1^n)$ and

$$\left\| x_2^n - \left(x_1^n + \int_{t_1^n}^{t_2^n} f_1^n(s)\,\mathrm{d}s\right)\right\| = d_{C(t_2^n, x_1^n)}\left(x_1^n + \int_{t_1^n}^{t_2^n} f_1^n(s)\,\mathrm{d}s\right).$$

This equality with (5) and (23) yields

$$\begin{aligned} \left\| x_2^n - x_1^n - \int_{t_1^n}^{t_2^n} f_1^n(s)\,\mathrm{d}s\right\| &\le \int_{t_1^n}^{t_2^n} (g(s) + p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s \\ &= e(t_2^n) - e(t_1^n). \end{aligned} \tag{25}$$

Remark that, by (12), (21) and (25), we have

$$\begin{aligned} \|x_2^n\| &\le \left\| x_2^n - x_1^n - \int_{t_1^n}^{t_2^n} f_1^n(s)\,\mathrm{d}s\right\| + \|x_1^n\| + \int_{t_1^n}^{t_2^n} \|f_1^n(s)\|\,\mathrm{d}s \\ &\le e(t_2^n) - e(t_1^n) + \|\varphi(0)\| + \int_{t_0^n}^{t_1^n} (2g(s) + 2p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s \\ &\quad + \int_{t_1^n}^{t_2^n} (g(s) + p(s)\|\varphi\|_\infty + r)\,\mathrm{d}s \\ &\le \int_{t_1^n}^{t_2^n} (2g(s) + 2p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s + \|\varphi(0)\| \\ &\quad + \int_{t_0^n}^{t_1^n} (2g(s) + 2p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s \\ &= \|\varphi(0)\| + \int_{t_0^n}^{t_2^n} (2g(s) + 2p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s. \end{aligned}$$

It follows that $x_2^n \in \mu\overline{\mathbb{B}}$.

Next, set for $t \in (t_1^n, t_2^n]$, $f_n(t) = f_1^n(t)$ and

$$u_n(t) = x_1^n + \frac{e(t) - e(t_1^n)}{e(t_2^n) - e(t_1^n)}\left(x_2^n - x_1^n - \int_{t_1^n}^{t_2^n} f_n(s)\,\mathrm{d}s\right) + \int_{t_1^n}^{t} f_n(s)\,\mathrm{d}s. \tag{26}$$

Obviously, f_n and u_n satisfy (i), (ii), (iii) and (vi) of (4) for $i = 1$. Let us prove that (v) is satisfied for $i = 1$. So let $t, s \in (t_1^n, t_2^n]$, $s < t$, by (5), (21), (25) and (26) one has

$$\begin{aligned}
\|u_n(t) - u_n(s)\| &\le \frac{e(t) - e(s)}{e(t_2^n) - e(t_1^n)}\left\|x_2^n - x_1^n - \int_{t_1^n}^{t_2^n} f_n(s)\,\mathrm{d}s\right\| \\
&\quad + \int_s^t \|f_n(\beta)\|\,\mathrm{d}\beta \\
&\le |e(t) - e(s)| + \int_s^t (g(\beta) + p(\beta)(\|\varphi\|_\infty + r)\,\mathrm{d}\beta \\
&\le \int_s^t (g(\beta) + p(\beta)(\|\varphi\|_\infty + r) + |v'(\beta)|)\,\mathrm{d}\beta \\
&\quad + \int_s^t (g(\beta) + p(\beta)(\|\varphi\|_\infty + r))\,\mathrm{d}\beta \\
&= \int_s^t (2g(\beta) + 2p(\beta)(\|\varphi\|_\infty + r) + |v'(\beta)|)\,\mathrm{d}\beta.
\end{aligned} \tag{27}$$

This shows that (v) is true for $i = 1$.

Next, by (13), (14), (26) and (27) for all $t \in (t_1^n, t_2^n]$

$$\begin{aligned}
\|u_n(t) - \varphi(0)\| &= \|u_n(t) - u_n(t_0^n)\| \\
&\le \|u_n(t) - u_n(t_2^n)\| + \|u_n(t_2^n) - u_n(t_1^n)\| \\
&\quad + \|u_n(t_1^n) - u_n(t_0^n)\| \\
&\le \int_t^{t_2^n} (2g(s) + 2p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s \\
&\quad + \int_{t_1^n}^{t_2^n} (2g(s) + 2p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s
\end{aligned} \tag{28}$$

$$+\int_{t_0^n}^{t_1^n}(2g(s)+2p(s)(\|\varphi\|_\infty+r)+|v'(s)|)\,\mathrm{d}s$$
$$\leq \int_{t_0^n}^{T}(2g(s)+2p(s)(\|\varphi\|_\infty+r)+|v'(s)|)\,\mathrm{d}s.$$

for all $t\in(t_1^n,t_2^n]$. Then (iv) is satisfied for $i=1$.

Also, the inequality (28) with (2) for all $t\in(t_1^n,t_2^n]$ gives us

$$\|u_n(t)-\varphi(0)\|\leq\frac{r}{2}.$$

Moreover, from (26) for $t\in\,]t_1^n,t_2^n[$ we get

$$u_n'(t)=\frac{e'(t)}{e(t_2^n)-e(t_1^n)}\left(x_2^n-x_1^n-\int_{t_1^n}^{t_2^n}f_n(s)\,\mathrm{d}s\right)+f_n(t). \tag{29}$$

This equality with (5) and (25), gives us

$$\begin{aligned}\|u_n'(t)-f_n(t)\|&\leq|e'(t)|\\&=g(t)+p(t)(\|\varphi\|_\infty+r)+|v'(t)|\quad\text{for a.e. } t\in\,]t_1^n,t_2^n[\,.\end{aligned}$$

Also, from (24) we have

$$x_2^n-x_1^n-\int_{t_1^n}^{t_2^n}f_n(s)\,\mathrm{d}s\in-N_{C(t_2^n,x_1^n)}\,(x_2^n).$$

This relation with (29) leads to

$$u_n'(t)-f_n(t)\in-N_{C(t_2^n,u_n(t_1^n))}\,(u_n(t_2^n))\qquad\text{for a.e. } t\in(t_1^n,t_2^n].$$

Thus (vii) and (viii) are true for $i=1$.

We reiterate this process for constructing sequence (x_i^n), $0\leq i\leq 2^n$, with $x_0^n=\varphi(0)$, a continuous function $u_n\colon[-r,T]\longrightarrow H$ and $f_n\in L^1([0,T],H)$ such that (4) is satisfied.

Now, for each positive integer n we define functions θ_n, $\delta_n\colon[0,T]\longrightarrow[0,T]$ and by setting

$$\theta_n(0)=t_1^n,\qquad\delta_n(0)=0,\qquad\theta_n(t)=t_{i+1}^n,\qquad\delta_n(t)=t_i^n$$

for all $t\in(t_i^n,t_{i+1}^n)$, $i=0,1,\dots,2^n-1$. Note that,

$$\lim_{n\to\infty}\theta_n(t)=\lim_{n\to\infty}\delta_n(t)=t\qquad\text{for all } t\in[0,T]. \tag{30}$$

Next, let t be any fixed point in $[0,T]$. Then, there is i, $0 \le i \le 2^n - 1$, such that $t \in (t_i^n, t_{i+1}^n]$. From (4) (v), we get

$$
\begin{aligned}
\|u_n(t) - \varphi(0)\| &= \|u_n(t) - u_n(t_0^n)\| \\
&\le \|u_n(t) - u_n(t_{i+1}^n)\| + \|u_n(t_{i+1}^n) - u_n(t_i^n)\| + \dots \\
&\quad + \|u_n(t_1^n) - u_n(t_0^n)\| \\
&\le \int_t^{t_{i+1}^n} (2g(s) + 2p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s \\
&\quad + \int_{t_i^n}^{t_{i+1}^n} (2g(s) + 2p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s + \dots \\
&\quad + \int_{t_0^n}^{t_1^n} (2g(s) + 2p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s \\
&\le \int_{t_0^n}^{T} (2g(s) + 2p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s.
\end{aligned} \tag{31}
$$

This inequality gives us

$$\|u_n(t)\| \le \mu \qquad \text{for all } t \in [0,T].$$

Moreover, from (31) with (2) we obtain

$$\|u_n(t) - \varphi(0)\| \le \frac{r}{2} \qquad \text{for all } t \in [0,T].$$

Observe that by (4) (ii) and (vi), we get

$$f_n(t) \in F(t, \tau(\delta_n(t))u_n),$$

and

$$\|f_n(t) - z(t)\| \le d(z(t), F(t, \tau(\delta_n(t))u_n)) + \frac{1}{n^2}$$

for all $t \in [0,T]$. Also, by (4) (vii) we have for almost every $t \in [0,T]$

$$u_n'(t) - f_n(t) \in -N_{C(\theta_n(t), u(\delta_n(t)))}\big(u_n(\theta_n(t))\big).$$

Step 2: The sequence (u_n), $n \ge 1$, is equicontinuous. Let n be a fixed integer and $t, s \in [0,T]$, $s < t$. Then, there are i, j such that $s \in [t_i^n, t_{i+1}^n]$ and $t \in [t_j^n, t_{j+1}^n]$, $0 \le i \le j \le 2^n - 1$.

Thus, by (4) (v) we get

$$
\begin{aligned}
\|u_n(t)-u_n(s)\| &= \|u_n(t)-u_n(t_j^n)\| \\
&\quad + \|u_n(t_j^n)-u_n(t_{j-1}^n)\| + \dots + \|u_n(t_{i+1}^n)-u_n(s)\| \\
&\le \int_{t_j^n}^{t} (2g(\beta)+2p(\beta)(\|\varphi\|_\infty + r) + |v'(\beta)|)\,\mathrm{d}\beta \\
&\quad + \int_{t_{j-1}^n}^{t_j^n} (2g(\beta)+2p(\beta)(\|\varphi\|_\infty + r) + |v'(\beta)|)\,\mathrm{d}\beta + \dots \\
&\quad + \int_{s}^{t_{i+1}^n} (2g(\beta)+2p(\beta)(\|\varphi\|_\infty + r) + |v'(\beta)|)\,\mathrm{d}\beta \\
&= \int_{s}^{t} (2g(\beta)+2p(\beta)(\|\varphi\|_\infty + r) + |v'(\beta)|)\,\mathrm{d}\beta.
\end{aligned} \tag{32}
$$

This shows that (u_n) is equicontinuous.

Step 3: The sequence (u_n) converges uniformly to a continuous function $u\colon [-r,T] \longrightarrow H$, with $u(t)=\varphi(t)$ for $t\in[-r,0]$.

In view of step 2 and Arzelà–Ascoli's theorem we have to show that the set $A(t)=\{u_n(t)\colon n\ge 1\}$ is relatively compact in H for all $t\in[0,T]$. Let t be a fixed point in $[0,T]$. By construction, for any $n\ge 1$ we have

$$
u_n(\theta_n(t)) \in C\big(\theta_n(t), u_n(\delta_n(t))\big) \cap \mu\overline{\mathbb{B}}. \tag{33}
$$

From condition (C_2), (32) and (33), we get

$$
\begin{aligned}
u_n(t) &\in C\big(\theta_n(t), u_n(\delta_n(t))\big) + \|u_n(\theta_n(t)) - u_n(t)\|\,\overline{\mathbb{B}} \\
&\subseteq C(t,u_n(t)) + (|v(\theta_n(t)) - v(t)| + \|u_n(\delta_n(t)) - u_n(t)\|)\,\overline{\mathbb{B}} \\
&\quad + \|u_n(\theta_n(t)) - u_n(t)\|\,\overline{\mathbb{B}} \\
&\subseteq C(t,u_n(t)) + \Bigg[\int_{t}^{\theta_n(t)} |v'(s)|\,\mathrm{d}s \\
&\quad + \int_{\delta_n(t)}^{t} (2g(s)+2p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s \\
&\quad + \int_{t}^{\theta_n(t)} (2g(s)+2p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s\Bigg]\,\overline{\mathbb{B}}.
\end{aligned} \tag{34}
$$

Now, for any $n \geq 1$ let

$$
\begin{aligned}
R_n = &\int_t^{\theta_n(t)} |v'(s)| \,\mathrm{d}s \\
&+ \int_t^{\delta_n(t)} (2g(s) + 2p(s)(\|\varphi\|_\infty + r) + |v'(s)|) \,\mathrm{d}s \\
&+ \int_t^{\theta_n(t)} (2g(s) + 2p(s)(\|\varphi\|_\infty + r) + |v'(s)|) \,\mathrm{d}s.
\end{aligned} \tag{35}
$$

Therefore, from (4) (iv), (34) and (35) for any $n \geq 1$ we get

$$u_n(t) \in C(t, u_n(t)) \cap \mu\,\overline{\mathbb{B}} + R_n\,\overline{\mathbb{B}}.$$

Assume by contradiction that there is $t_0 \in [0,T]$ such that $A(t_0)$ is not relatively compact in H. So, $\gamma(A(t_0)) > 0$. Using condition (C$_3$) and the fact that $A(t_0)$ is bounded, we get

$$\gamma\big(C(t, A(t_0))\big) \cap \mu\,\overline{\mathbb{B}} < \gamma(A(t_0)).$$

Then, there is $\overline{\delta} > 0$ such that

$$\gamma(A(t_0)) - \gamma\big(C(t, A(t_0))\big) \cap \mu\,\overline{\mathbb{B}} > 2\overline{\delta}.$$

Note that, by (30) then $\lim\limits_{n\to\infty} R_n(t) = 0$. So we can find a natural number n_0 such that $2R_{n_0} < \overline{\delta}$ for $n \geq n_0$.

Fix now $n_0 \in \mathbb{N}$ such that $2R_n < \overline{\delta}$ for $n \geq n_0$. Then the properties of γ imply

$$
\begin{aligned}
\gamma(A(t_0)) &= \gamma\{u_n(t_0)\colon n \geq n_0\} \leq \gamma(C(t, A(t_0)) \cap \mu\,\overline{\mathbb{B}} + \gamma(R_{n_0}\,\overline{\mathbb{B}}) \\
&< \gamma(A(t_0)) - 2\overline{\delta} + 2R_{n_0} < \gamma(A(t_0)) - 2\overline{\delta} + \overline{\delta} = \gamma(A(t_0)) - \overline{\delta},
\end{aligned}
$$

which is a contradiction. Therefore, the set $A(t)$ is relatively compact in H for all $t \in [0,T]$. Thus, by Arzelà–Ascoli's theorem, we can select a subsequence of u_n, again denoted by u_n, which converges uniformly to an absolutely continuous function u on $[0,T]$. We extend the definition of u on $[-r,T]$ by setting $u = \varphi$ on $[-r,0]$. Then (u_n) converges uniformly to u on $[-r,T]$.

Step 4: For $t \in [0,T]$, we have $u(t) \in C(t, u(t))$ and

$$\lim_{n\to\infty} h\big(C(\theta_n(t), u_n(t)), C(t, u(t))\big) = 0. \tag{36}$$

Thanks to the condition (C_2), (32) and (33) for all $t \in [0,T]$, we have

$$
\begin{aligned}
\lim_{n\to\infty} d_{C(t,u(t))}(u_n(t)) \le \lim_{n\to\infty} & d_{C(\theta_n(t),u_n(\delta_n(t)))}\big(u_n(\theta_n(t))\big) \\
& + \lim_{n\to\infty}\left|d_{C(t,u(t))}(u_n(t)) - d_{C(\theta_n(t),u_n(\delta_n(t)))}\big(u_n(\theta_n(t))\big)\right| \\
\le & \lim_{n\to\infty}[|v(\theta_n(t)) - v(t)| + \|u_n(\delta_n(t)) - u_n(t)\| \\
& + \lambda\|u_n(\theta_n(t)) - u_n(t)\|] \\
\le & \lim_{n\to\infty} \int_t^{\delta_n(t)} (2g(s) + 2p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s \\
& + \lim_{n\to\infty} \lambda \int_t^{\theta_n(t)} (2g(s) + 2p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s \\
= & \ 0.
\end{aligned}
$$

Using this inequality with the uniform convergence of u_n to u, and the closedness of the set $C(t,u(t))$ we conclude that $u(t) \in C(t,u(t))$ for all $t \in [0,T]$.

Note that, the relation (36) is obvious from (C_2) and the fact that

$$
\lim_{n\to\infty} \theta_n(t) = t \qquad \text{and} \qquad \lim_{n\to\infty} u_n(t) = u(t).
$$

Step 5: For any $t \in [0,T]$, the sequence $\tau(\delta_n(t)u_n)$ converges to $\tau(t)u$ in C_r.

Let us denote the modulus continuity of a function ψ defined on an interval $[0,T]$ of $\mathbb{R}$ by

$$
w(\psi,[0,T],\eta) := \sup_{|s-t|<\eta} \{\|\psi(t) - \psi(s)\|\colon\ s,t \in [0,T]\}.
$$

Let $\varepsilon > 0$ and let $t,t' \in [0,T]$, assume that $0 \le t' - t < \eta(\varepsilon/2)$. By (3) and (4) (v), we have

$$
\|u_n(t) - u_n(t')\| \le \int_t^{t'} (2g(s) + 2p(s)(\|\varphi\|_\infty + r) + |v'(s)|)\,\mathrm{d}s \le \frac{\varepsilon}{2}.
$$

Hence

$$
w\Big(u_n,[0,T],\eta\Big(\frac{\varepsilon}{2}\Big)\Big) = \sup_{|t-t'|<\eta(\varepsilon/2)} \{\|u_n(t) - u_n(t')\|\colon\ t,t' \in [0,T]\} \le \frac{\varepsilon}{2}.
$$

Also for $t,t' \in [-r,0]$ such that $|t'-t| < \eta(\varepsilon/2)$, we have by (3)

$$
\|\varphi(t) - \varphi(t')\| < \frac{\varepsilon}{2}.
$$

Then

$$w(\varphi, [-r, 0], \eta(\varepsilon)) \leq \frac{\varepsilon}{2}.$$

Now, let $t \in [0, T]$. Since $\delta_n(t) \longrightarrow t$ as $n \longrightarrow \infty$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $|\delta_n(t) - t| < \eta(\varepsilon/2)$. Then, for all $n \geq n_0$

$$\begin{aligned}
\|\tau(\delta_n(t))u_n - \tau(t)u\|_\infty &= \sup_{-r \leq s \leq 0} \|u_n(\delta_n(t) + s) - u_n(t + s)\| \\
&= w\Big(u_n, [-r, T], \eta\Big(\frac{\varepsilon}{2}\Big)\Big) \\
&\leq w\Big(\varphi, [-r, 0], \eta\Big(\frac{\varepsilon}{2}\Big)\Big) + w\Big(u_n, [0, T], \eta\Big(\frac{\varepsilon}{2}\Big)\Big) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

hence $\|\tau(\delta_n(t))u_n - \tau(t)u\|_\infty$ converges to 0 as $n \longrightarrow \infty$. Therefore, since the uniform convergence of u_n to u on $[-r, T]$ implies that $\tau(t)u_n$ converges to $\tau(t)u$ uniformly on $[-r, 0]$, we deduce that

$$\lim_{n \to \infty} \|\tau(\delta_n(t))u_n - \tau(t)u\| = 0. \tag{37}$$

Step 6: The sequence (f_n) converges pointwise to a function $f \in L^1(I, H)$ satisfying $f(t) \in F(t, \tau(t)u)$, $t \in [0, T]$.

Let $t \in [0, T]$ be fixed. In view of (4) (vi) and condition (F_2) we obtain for $n \geq 1$

$$\begin{aligned}
\|f_{n+1}(t) - f_n(t)\| &\leq d(f_n(t), F(t, \tau(\delta_{n+1}(t))u_{n+1})) + \frac{1}{(n+1)^2} \\
&\leq h(F(t, \tau(\delta_n(t))u_n), F(t, \tau(\delta_{n+1}(t))u_{n+1})) + \frac{1}{(n+1)^2} \qquad (38) \\
&\leq m(t)(\|\tau(\delta_n(t))u_n - \tau(\delta_{n+1}(t))u_{n+1}\|_\infty) + \frac{1}{(n+1)^2}.
\end{aligned}$$

Thus for any two natural numbers p, q, $p < q$, it follows that

$$\begin{aligned}
&\|f_q(t) - f_p(t)\| \\
&\quad \leq \|f_{p+1}(t) - f_p(t)\| + \|f_{p+2}(t) - f_p(t)\| + \cdots + \|f_q(t) - f_{q-1}(t)\| \\
&\quad \leq m(t)\big[\|\tau(\delta_p(t))u_p - \tau(\delta_{p+1}(t))u_{p+1}\|_\infty + \cdots \\
&\qquad + \|\tau(\delta_{q-1}(t))u_{q-1} - \tau(\delta_q(t))u_q\|_\infty\big] + \frac{1}{p^2} + \frac{1}{(p+1)^2} + \cdots + \frac{1}{q^2} \\
&\quad \leq m(t)\big[\|\tau(\delta_p(t))u_p - \tau(\delta_{p+1}(t))u_{p+1}\|_\infty + \cdots \\
&\qquad + \|\tau(\delta_{q-1}(t))u_{q-1} - \tau(\delta_q(t))u_q\|_\infty\big] + \frac{q-p}{p^2}.
\end{aligned}$$

From (37), we get for $t \in I$,

$$\lim_{p,q \to \infty} \|f_q(t) - f_p(t)\| = 0,$$

which means that the sequence $(f_n(t))$ is Cauchy in H for any $t \in I$, then it converges pointwise to a function $f \in L^1(I, H)$. Moreover, by (4) (vi) and condition (F_2) we get

$$\begin{aligned} d(f(t), F(t, \tau(t)u)) &\leq \|f(t) - f_n(t)\| + d(f_n(t), F(t, \tau(t)u)) \\ &\leq \|f(t) - f_n(t)\| + h(F(t, \tau(t)u), F(t, \tau(\delta_n(t))u_n)) \\ &\leq \|f(t) - f_n(t)\| + m(t)(\|\tau(t)u - \tau(\delta_n(t))u_n\|). \end{aligned}$$

Again, by (37), the right hand side of this inequality tends to zero when $n \longrightarrow \infty$. Hence, $f(t) \in F(t, \tau(t)u)$, $t \in [0, T]$.

Step 7: For almost all $t \in [0, T]$, $u'(t) - f(t) \in -N_{C(t,u(t))}(u(t))$.

Note that by (4) (vi) and (viii) for almost all $t \in [0, T]$ we have

$$\|u_n'(t)\| \leq 2g(t) + 2p(t)(\|\varphi\|_\infty + r) + |v'(t)|.$$

Since H is a reflexive, the sequence (u_n') converges weakly to a function $v \in L^1([0, T], H)$. Because

$$u_n(t) = u_n(0) + \int_0^t u_n'(s)\, \mathrm{d}s$$

we get $v = u'$ a.e. So, the sequence $(u_n' - f_n)$ converges weakly to $(u' - f)$ in $L^1([0, T], H)$ and the Mazur's lemma gives us

$$u'(t) - f(t) \in \bigcap_n \overline{Co}\{u_m'(t) - f_m(t) \colon m \geq n\} \qquad \text{for a.e. } t \in [0, T].$$

Fix any t such that the preceding relation is satisfied and consider $x \in H$. The last relation above yields

$$\langle u'(t) - f(t), x\rangle \leq \inf_n \sup_{m \geq n} \langle u_m'(t) - f_m(t), x\rangle.$$

By (4) (viii), one has

$$u_n'(t) - f_n(t) \in -N_{C(\theta_n(t), u_n(\delta_n(t)))}\big(u_n(\theta_n(t))\big) \cap e'(t)\,\overline{\mathbb{B}}$$

for almost all $t \in [0, T]$. Hence, by Proposition 3.2.1, we get

$$u_n'(t) - f_n(t) \in -e'(t)\partial d_{C(\theta_n(t), u_n(\delta_n(t)))}\big(u_n(\theta_n(t))\big).$$

In view of Proposition 3.2.2, for $x \in H$ and a.e. $t \in [0,T]$ one obtains

$$\begin{aligned}\langle u'(t) - f\ (t), x\rangle &\leq \lim_{n\to\infty} \sup \langle u'_m(t) - f_m(t), x\rangle \\ &\leq e'(t) \lim_{n\to\infty} \sup \sigma(x, -\partial d_{C(\theta_n(t), u_n(\delta_n(t)))}\big(u_n(\theta_n(t))\big), \\ &\leq e'(t)\sigma\big(x, -\partial d_{C(t,u(t))}(u(t))\big).\end{aligned}$$

So, the convexity and the closedness of the set $\partial d_{C(t,u(t))}(u(t))$ ensure

$$u'_n(t) - f_n(t) \in -e'(t)\partial d_{C(t,u(t))}(u(t)) \subset -N_{C(t,u(t))}(u(t)).$$

Finally, by Steps 6 and 7 we have for almost all $t \in [0,T]$

$$u'(t) \in -N_{C(t,u(t))}(u(t)) + F(t, \tau(t)u(t))$$

and the proof is complete. □

4. Concluding remarks

In this paper, existence problem of solution of the sweeping process in Hilbert space with nonconvex, noncompact perturbation has been considered. Some sufficient conditions have been obtained. The importance of this work is that the values of the sweeping process are nonconvex, noncompact, the perturbation is not necessarily compact, and the space is Hilbert.

An interesting extension of our studies would be to extend Theorem 3.1 to Banach space setting.

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