Geodesic graphs in Randers g.o. spaces

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Abstract. The concept of geodesic graph is generalized from Riemannian geometry to Finsler geometry, in particular to homogeneous Randers g.o. manifolds. On modified H-type groups which admit a Riemannian g.o. metric, invariant Randers g.o. metrics are determined and geodesic graphs in these Finsler g.o. manifolds are constructed. New structures of geodesic graphs are observed.

Keywords: Finsler space; Randers space; homogeneous geodesic; geodesic graph; g.o. space

Classification: 53C22, 53C60, 53C30

1. Introduction

A *Minkowski norm* on the vector space $\mathbb{V}$ is a nonnegative function $F\colon \mathbb{V} \to \mathbb{R}$ which is smooth on $\mathbb{V} \setminus \{0\}$, positively homogeneous ($F(\lambda y) = \lambda F(y)$ for any $\lambda > 0$) and whose Hessian $g_{ij} = (\frac{1}{2}F^2)_{y^i y^j}$ is positively definite on $\mathbb{V} \setminus \{0\}$. Here (y^i) are the components of a vector $y \in \mathbb{V}$ with respect to a fixed basis B of $\mathbb{V}$ and putting y^i to a subscript means the partial derivative. Then the pair $(\mathbb{V}, F)$ is called a *Minkowski space*. The tensor g_y with components $g_{ij}(y)$ is the *fundamental tensor*. A Finsler metric on the smooth manifold M is a function F on TM which is smooth on $TM \setminus \{0\}$ and whose restriction to any tangent space T_xM is a Minkowski norm. Then the pair (M, F) is called the *Finsler manifold*. On a Finsler manifold, functions g_{ij} depend smoothly on $x \in M$ and on $o \neq y \in T_xM$.

Special Minkowski norms are the Randers norms. They are determined by a symmetric positively definite bilinear form α and a vector V such that $\alpha(V,V) < 1$, or, equivalently, its α-equivalent 1-form β related with V by the formula

$$\beta(U) = \alpha(V, U) \qquad \forall\, U \in \mathbb{V}. \tag{1}$$

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The Randers norm F is defined by the formula

$$F(U)=\sqrt{\alpha(U,U)}+\beta(U) \qquad \forall U\in\mathbb{V}. \tag{2}$$

If a Finsler metric F on M restricted to any tangent space T_pM is a Randers norm, it is called a *Randers metric*. Obviously, the Randers metric F is determined by a Riemannian metric α and a smooth 1-form β and formula (2) holds on each tangent space T_pM. We remark that, in the literature, the letter α is sometimes used for the norm induced by the 2-form α and then formula (2) above is without the square root. We choose the notation above because for $\beta=0$, F is the Riemannian norm and components g_{ij} of the fundamental tensor are just the components of the Riemannian metric α.

Let M be a Finsler manifold (M,F). If there is a connected Lie group G which acts transitively on M as a group of isometries, then M is called a *homogeneous manifold*. The following theorem gives the relation between the isometry group of a Randers manifold and the isometry group of the corresponding underlying Riemannian manifold. We shall use this theorem later.

Theorem 1 ([4]). *Let (M,F) be a Randers manifold with the Finsler function $F=\sqrt{\alpha}+\beta$. Then the group of isometries of (M,F) is a closed subgroup of the group of isometries of the Riemannian manifold (M,α).*

Homogeneous manifold M can be naturally identified with the *homogeneous space* G/H, where H is the isotropy group of the origin $p\in M$. A homogeneous Finsler space $(G/H,F)$ is always a *reductive homogeneous space*: We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively and consider the adjoint representation $\mathrm{Ad}\colon H\times\mathfrak{g}\to\mathfrak{g}$ of H on $\mathfrak{g}$. There exists a *reductive decomposition* of the form $\mathfrak{g}=\mathfrak{m}+\mathfrak{h}$ where $\mathfrak{m}\subset\mathfrak{g}$ is a vector subspace such that $\mathrm{Ad}(H)(\mathfrak{m})\subset\mathfrak{m}$. For a fixed reductive decomposition $\mathfrak{g}=\mathfrak{m}+\mathfrak{h}$ there is the natural identification of $\mathfrak{m}\subset\mathfrak{g}=T_eG$ with the tangent space T_pM via the projection $\pi\colon G\to G/H=M$. Using this natural identification, from the Minkowski norm and its fundamental tensor on T_pM, we obtain the $\mathrm{Ad}(H)$-invariant Minkowski norm and the $\mathrm{Ad}(H)$-invariant fundamental tensor on $\mathfrak{m}$ and we denote these again by F and g. In particular, for the invariant Randers metrics we shall use the following theorem.

Theorem 2 ([4]). *Let $(G/H,\alpha)$ be a Riemannian homogeneous space with the reductive decomposition $\mathfrak{g}=\mathfrak{m}+\mathfrak{h}$. Then there is a one-to-one correspondence between G-invariant Randers metrics on G/H whose underlying Riemannian metric is α and the set*

$$\mathcal{V}=\{V\in\mathfrak{m}\colon\ \alpha(V)<1,\ \mathrm{Ad}(H)(V)=V\}.$$

We further recall that the *slit tangent bundle* TM_0 is defined as $TM_0=TM\setminus\{0\}$. Using the restriction of the natural projection $\pi\colon TM\to M$ to TM_0, we naturally construct the pullback vector bundle π^*TM over TM_0. The *Chern connection* is the unique linear connection on the vector bundle π^*TM which is torsion free and almost g-compatible, see some monograph, for example [2] by D. Bao, S.-S. Chern and Z. Shen or [4] by S. Deng for details. Using the Chern connection, the derivative along a curve $\gamma(t)$ can be defined. A regular smooth curve γ with tangent vector field T is a *geodesic* if $D_T\big(\frac{T}{F(T)}\big) = 0$. In particular, a geodesic of constant speed satisfies $D_T T = 0$.

A geodesic $\gamma(s)$ through the point p is *homogeneous* if it is an orbit of a one-parameter group of isometries. More explicitly, if there exists a nonzero vector $X \in \mathfrak{g}$ such that $\gamma(t) = \exp(tX)(p)$ for all $t \in \mathbb{R}$. The vector X is called a *geodesic vector.*

Definition 3. A homogeneous space $(G/H, F)$ is called a Finsler g.o. space, if each geodesic of $(G/H, F)$ (with respect to the Chern connection) is an orbit of a one-parameter subgroup $\{\exp(tZ)\}$, $Z \in \mathfrak{g}$, of the group of isometries G.

We remark that a homogeneous manifold (M, F) may admit more presentations as a homogeneous space in the form G/H, corresponding to various transitive isometry groups. In a g.o. space G/H, we investigate some sets of geodesic vectors which generate all geodesics through a fixed point. Those sets which are reasonable in a good sense are called geodesic graphs. The first concept originated from the work [17] by J. Szenthe.

Definition 4. Let $(G/H, F)$ be a g.o. space and $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ an $\mathrm{Ad}(H)$-invariant decomposition of the Lie algebra $\mathfrak{g}$. A geodesic graph is an $\mathrm{Ad}(H)$-equivariant map $\xi\colon \mathfrak{m} \to \mathfrak{h}$ such that $X + \xi(X)$ is a geodesic vector for each $o \neq X \in \mathfrak{m}$.

It often happens that the vector $\xi(X)$ is uniquely determined. Then the map ξ is $\mathrm{Ad}(H)$-equivariant and we are interested in the algebraic structure of the mapping ξ. Sometimes, there are more choices for the vector $\xi(X)$. Then we want to choose it in a way that the algebraic structure of the mapping ξ is as simple as possible.

Theory of Riemannian geodesic graphs (canonical and general) was developed and examples of geodesic graphs on compact and noncompact g.o. manifolds and also on g.o. nilmanifolds in dimensions 5, 6 and 7 were described by O. Kowalski and S. Nikčević in [10]. Geodesic graph is either linear, which is equivalent with the natural reductivity of the space G/H, or its components are rational functions $\xi_i = P_i/P$, where P_i and P are homogeneous polynomials and $\deg(P_i) = \deg(P) + 1$. The *degree of the geodesic graph* ξ is defined as $\deg(\xi) = \deg(P)$. The special situation of geodesic graph of degree 0 corresponds to the linear geodesic graph. If the geodesic graph is unique, its degree is also the *degree of the g.o. space* G/H. If there are more geodesic graphs in G/H, the degree of the g.o. space is the minimum of the degrees of these geodesic graphs.

The degree of the mentioned examples in [10] is 0 (linear geodesic graph) or 2. Further geodesic graphs of degree 0 or 2 on H-type groups were described by the author in [5]. Geodesic graph of degree 4 on the flag manifold $SO(7)/U(3)$ was constructed by the author in [6]. In [8], the author with O. Kowalski constructed the canonical geodesic graph of degree 6 and a general geodesic graph of degree 3 on the H-type group of dimension 13 with 5-dimensional center.

We recall that in dimension less than or equal to 5, all Riemannian g.o. manifolds (M, g) are naturally reductive, hence they admit a presentation $M = G/H$ in which a linear geodesic graph exists. Equivalently, they admit a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ such that all vectors from $\mathfrak{m}$ are geodesic. In dimension 6, all g.o. manifolds which are not naturally reductive were classified by O. Kowalski and L. Vanhecke in [11]. Interesting compact Riemannian g.o. manifolds are, for example, the two series of flag manifolds described by D. Alekseevsky, A. Arvanitoyeorgos in [1]. Interesting Riemannian g.o. nilmanifolds are the modified H-type groups, see Section 3 for details. For a more detailed exposition about geodesic graphs in Riemannian g.o. manifolds, some related topics and further references, we refer the reader to the recent survey paper [7] by the author. Another structural approach to Riemannian g.o. manifolds using the Lie theory can be found in the recent papers [9] and [14] by C. S. Gordon and Yu. G. Nikonorov.

In [18], Z. Yan and S. Deng studied Finsler g.o. spaces and their relation with Riemannian g.o. spaces. Some nilpotent examples of reversible non-Berwaldian Finsler g.o. spaces and examples of invariant Randers g.o. metrics on spheres S^{2n+1} were constructed in this paper. For the study of the Randers case, the navigation data and G-invariant Killing vector fields were used. In the paper [15] by D. Latifi and M. Parhizkar, geodesic vectors of some Randers metrics on nilmanifolds were analyzed, but it was not done properly in detail.

In the present paper, we determine all invariant Randers g.o. metrics on modified H-type groups which admit a Riemannian g.o. metric. We remark that Randers metrics are not reversible. We construct Finslerian geodesic graphs on these homogeneous Randers g.o. manifolds. In all these examples, geodesic graph is unique. We focus on the difference of Finslerian geodesic graph of a Randers metric and Riemannian geodesic graph of underlying Riemannian metric.

The simplest geodesic graph of the Randers metric is a cone. This is the situation when the underlying Riemannian metric α is naturally reductive with respect to a group G and the Randers metric $F = \sqrt{\alpha} + \beta$ admits G as a group of isometries. This situation occurs for H-type groups with $\dim(\mathfrak{z}) = 1$. If $\dim(\mathfrak{z}) = 2$, geodesic graph of the Randers metrics with the same group of isometries as the underlying Riemannian metrics arise as the Riemannian geodesic graph with a deformation term. Interesting situation occurs for H-type groups with $\dim(\mathfrak{z}) = 3$ whose invariant Riemannian metrics are naturally reductive and they admit a linear Riemannian geodesic graph with respect to a group G. With respect to this group of transformations, there are no invariant Randers metrics. However, there exist invariant Randers metrics with respect to different group $\overline{G}$ and the degree of the Finslerian geodesic graph with respect to $\overline{G}$ is equal to 2.

2. Geodesic lemma for Randers metrics

Geodesic vectors are characterized by the following *geodesic lemma*, proved by D. Latifi.

Lemma 5 ([12]). *Let $(G/H, F)$ be a homogeneous Finsler space with a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$. A nonzero vector $Y \in \mathfrak{g}$ is geodesic vector if and only if it holds*

$$g_{Y_{\mathfrak{m}}}(Y_{\mathfrak{m}}, [Y, U]_{\mathfrak{m}}) = 0 \qquad \forall\, U \in \mathfrak{m}, \tag{3}$$

where the subscript $\mathfrak{m}$ indicates the projection of vector from $\mathfrak{g}$ to $\mathfrak{m}$.

We shall now adapt formula (3) for the Randers space and obtain a formula in terms of the Riemannian metric α and the 1-form β. Let us remark that the formula (5) below appeared in previous works [15] or [18]. However, in [15], it refers to verification by the direct calculations and in [18], it is proved using the navigation data, which is an alternative description of the Randers metric. For the convenience of the reader and to remain self-contained, we include a complete proof in our notation here.

Proposition 6. *Let $F = \sqrt{\alpha} + \beta$ be a homogeneous Randers metric on G/H, let $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ a reductive decomposition and $V \in \mathfrak{m}$ be the vector α-equivalent with β. The fundamental tensor g satisfies the formula*

$$g_y(y, v) = F(y) \cdot \alpha\Big(\frac{y}{\sqrt{\alpha(y, y)}} + V, v\Big) \qquad \forall\, y, v \in \mathfrak{m}. \tag{4}$$

Proof: For any vector $w = y + su + tv \in \mathfrak{m}$, for arbitrary $s, t \in \mathbb{R}$, it holds

$$\begin{aligned} F^2(w) = \big(\sqrt{\alpha(w, w)} + \beta(w)\big)^2 &= \alpha(w, w) + 2\sqrt{\alpha(w, w)}\beta(w) + \beta^2(w) \\ &= \alpha(w, w) + 2\sqrt{\alpha(w, w)}\alpha(V, w) + \alpha^2(V, w), \end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}F^2(w)}{\mathrm{d}s} &= 2\alpha(u,w) + 2\frac{\alpha(u,w)\alpha(V,w)}{\sqrt{\alpha(w,w)}}\\ &\quad + 2\sqrt{\alpha(w,w)}\alpha(V,u) + 2\alpha(V,w)\alpha(V,u),\\ \frac{\mathrm{d}^2F^2(w)}{\mathrm{d}s\mathrm{d}t} &= 2\alpha(u,v) + 2\frac{\alpha(u,v)\alpha(V,w)}{\sqrt{\alpha(w,w)}} + 2\frac{\alpha(u,w)\alpha(V,v)}{\sqrt{\alpha(w,w)}}\\ &\quad - 2\frac{\alpha(u,w)\alpha(V,w)\alpha(v,w)}{\sqrt{\alpha(w,w)}^3} + 2\frac{\alpha(v,w)\alpha(V,u)}{\sqrt{\alpha(w,w)}}\\ &\quad + 2\alpha(V,v)\alpha(V,u),\end{aligned}$$

and finally

$$\begin{aligned}g_y(u,v) = \frac{1}{2}\frac{\mathrm{d}F^2(y+su+tv)}{\mathrm{d}s\mathrm{d}t}\Big|_{s=t=0} &= \alpha(u,v) - \frac{\alpha(u,y)\alpha(v,y)\alpha(V,y)}{\sqrt{\alpha(y,y)}^3}\\ + \frac{\alpha(u,v)\alpha(V,y)}{\sqrt{\alpha(y,y)}} + \frac{\alpha(u,y)\alpha(V,v)}{\sqrt{\alpha(y,y)}} &+ \frac{\alpha(v,y)\alpha(V,u)}{\sqrt{\alpha(y,y)}} + \alpha(V,v)\alpha(V,u).\end{aligned}$$

In the special case $u = y$, we obtain

$$\begin{aligned}g_y(y,v) &= \alpha(y,v) - \frac{\alpha(y,y)\alpha(v,y)\alpha(V,y)}{\sqrt{\alpha(y,y)}^3} + \frac{\alpha(y,v)\alpha(V,y)}{\sqrt{\alpha(y,y)}}\\ &\quad + \frac{\alpha(y,y)\alpha(V,v)}{\sqrt{\alpha(y,y)}} + \frac{\alpha(v,y)\alpha(V,y)}{\sqrt{\alpha(y,y)}} + \alpha(V,v)\alpha(V,y)\\ &= \alpha(y,v) + \sqrt{\alpha(y,y)}\alpha(V,v) + \frac{\alpha(V,y)\alpha(y,v)}{\sqrt{\alpha(y,y)}} + \alpha(V,y)\alpha(V,v)\\ &= \big(\sqrt{\alpha(y,y)} + \alpha(V,y)\big)\cdot\alpha\Big(\frac{y}{\sqrt{\alpha(y,y)}} + V, v\Big)\\ &= F(y)\cdot\alpha\Big(\frac{y}{\sqrt{\alpha(y,y)}} + V, v\Big).\end{aligned}$$

□

In the corollaries which follow, notation $\xi(X)$ refers to geodesic graph $\xi\colon \mathfrak{m} \to \mathfrak{h}$ introduced in Definition 4.

Corollary 7. *Let $F = \sqrt{\alpha} + \beta$ be a homogeneous Randers metric on G/H, let $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ be a reductive decomposition and $V \in \mathfrak{m}$ be the vector α-equivalent with β. The vector $X + \xi(X)$, where $X \in \mathfrak{m}$ and $\xi(X) \in \mathfrak{h}$, is geodesic vector if and only if*

$$\alpha\big(X + \sqrt{\alpha(X,X)}\cdot V, [X + \xi(X), U]_{\mathfrak{m}}\big) = 0 \qquad \forall U \in \mathfrak{m}. \tag{5}$$

Proof: From Lemma 5 and using Proposition 6, we obtain that the vector $Y = X + \xi(X)$ is geodesic if and only if for any vector $U \in \mathfrak{m}$ it holds

$$0 = g_X\big(X, [X+\xi(X), U]_{\mathfrak{m}}\big) = F(X) \cdot \alpha\Big(\frac{X}{\sqrt{\alpha(X,X)}} + V, [X+\xi(X), U]_{\mathfrak{m}}\Big)$$

and the statement follows. □

Corollary 8. *Let $F = \sqrt{\alpha} + \beta$ be a homogeneous Randers metric on G/H, such that α be the naturally reductive Riemannian metric. Let $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ be the naturally reductive decomposition and $V \in \mathfrak{m}$ be the vector α-equivalent with β. The vector $X + \xi(X)$, where $X \in \mathfrak{m}$ and $\xi(X) \in \mathfrak{h}$, is geodesic vector if and only if*

$$-\alpha\big(X, [\xi(X), U]\big) = \sqrt{\alpha(X,X)} \cdot \alpha\big(V, [X,U]_{\mathfrak{m}}\big) \qquad \forall\, U \in \mathfrak{m}. \tag{6}$$

Proof: We use formula (5) and continue. For any vector $U \in \mathfrak{m}$ it holds

$$\begin{aligned}\alpha\big(X + \sqrt{\alpha(X,X)} \cdot V, [X+\xi(X), U]_{\mathfrak{m}}\big) &= \alpha\big(X, [X,U]_{\mathfrak{m}}\big) + \alpha\big(X, [\xi(X), U]_{\mathfrak{m}}\big)\\ &+ \sqrt{\alpha(X,X)} \cdot \alpha\big(V, [X,U]_{\mathfrak{m}}\big) + \sqrt{\alpha(X,X)} \cdot \alpha\big(V, [\xi(X), U]_{\mathfrak{m}}\big).\end{aligned}$$

Here the first summand is zero, because α is naturally reductive. Next, $\xi(X) \in \mathfrak{h}$ and V is $\mathrm{Ad}(H)$-invariant, hence $[\xi(X), V] = 0$. Because α is $\mathrm{Ad}(H)$-invariant, the last term is also zero. Finally, $\xi(X) \in \mathfrak{h}$ and the vector space $\mathfrak{m}$ is $\mathrm{Ad}(H)$-invariant. Hence, in the second term, $[\xi(X), U] \in \mathfrak{m}$ and the statement follows. □

3. Geodesic graphs in modified H-type groups

Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra with an inner product $\langle\,,\rangle$. Let $\mathfrak{z}$ be the center of $\mathfrak{n}$ and let $\mathfrak{v}$ be its orthogonal complement. For each vector $Z \in \mathfrak{z}$, define the operator $J_Z \colon \mathfrak{v} \to \mathfrak{v}$ by the formula

$$\langle J_Z(X), Y\rangle = \langle Z, [X,Y]\rangle \qquad \forall\, X, Y \in \mathfrak{v}.$$

The algebra n is called a *modified H-type algebra* if for each $o \neq Z \in \mathfrak{z}$ the operator J_Z satisfies the identity

$$(J_Z)^2 = \lambda(Z) \cdot \mathrm{id}_{\mathfrak{v}}$$

for some $\lambda(Z) < 0$. A connected and simply connected Lie group whose Lie algebra is a modified H-type algebra is diffeomorphic to $\mathbb{R}^n$ and it is called a *modified H-type group*. It is endowed with a left-invariant Riemannian metric. The special case of a *generalized Heisenberg algebra (H-type algebra)* and corresponding *generalized Heisenberg group (H-type group)* is obtained for

$$\lambda(Z) = -\langle Z, Z\rangle.$$

It was proved by J. Lauret in [13], that modified H-type algebras are just the pairs $(\mathfrak{n}, \langle\, ,\rangle_S)$, where $(\mathfrak{n}, \langle\, ,\rangle)$ is an H-type algebra and S is a positive definite symmetric transformation of $\mathfrak{z}$ which determines the inner product $\langle\, ,\rangle_S$ by the formula

$$\langle X+U, Y+V\rangle_S = \langle X, Y\rangle + \langle S(U), V\rangle \qquad \forall\, X, Y \in \mathfrak{v}, \quad \forall\, U, V \in \mathfrak{z}.$$

H-type algebras are completely classified, see for example the book [3] by J. Berndt, F. Tricerri and L. Vanhecke. For each dimension of the center $\mathfrak{z}$, there is a series of H-type algebras. Each algebra of the series contains the center $\mathfrak{z}$ and the complement $\mathfrak{v}$ which decomposes into irreducible $\mathfrak{z}$-modules (the operators J_Z make $\mathfrak{v}$ a $\mathfrak{z}$-module). Irreducible $\mathfrak{z}$-modules are all equivalent if $\dim(\mathfrak{z}) \neq 3(\mod 4)$, otherwise there exist two nonequivalent irreducible modules of the same dimension (called nonisotypic modules). From the classification by C. Riehm in [16], the following result follows:

An H-type group is a Riemannian g.o. manifold if and only if

- ◦ $\dim(\mathfrak{z}) \in \{1, 2, 3\}$ or
- ◦ $\dim(\mathfrak{z}) \in \{5, 6, 7\}$ and $\dim(\mathfrak{v}) = 8$, or
- ◦ $\dim(\mathfrak{z}) = 7$ and $\dim(\mathfrak{v}) \in \{16, 24\}$ and isotypic modules in $\mathfrak{v}$.

An H-type group is naturally reductive if and only if

- ◦ $\dim(\mathfrak{z}) = 1$ or
- ◦ $\dim(\mathfrak{z}) = 3$ and isotypic modules in $\mathfrak{v}$.

The refinement of this classification for modified H-type groups was obtained by J. Lauret in [13]: A modified H-type group with $\dim(\mathfrak{z}) \leq 3$ is a Riemannian g.o. manifold if and only if the corresponding H-type group is a Riemannian g.o. manifold. If $\dim(\mathfrak{z}) \geq 5$, only some of the modified H-type groups (corresponding to the above g.o. H-type groups) are g.o. manifolds, see [13] for details. Further, a modified H-type group is naturally reductive if and only if

- ◦ $\dim(\mathfrak{z}) = 1$ or
- ◦ $\dim(\mathfrak{z}) = 3$, $S = k \cdot \mathrm{Id}_{\mathfrak{z}}$ and isotypic modules in $\mathfrak{v}$.

In the following part, we determine all invariant Randers g.o. metrics on modified H-type groups with $\dim(\mathfrak{z}) \leq 3$, which admit Riemannian g.o. metrics, and we construct geodesic graphs on these Randers g.o. manifolds. For each of these groups N, its presentation as a Randers g.o. space $N = G/H$ is unique and geodesic graph is also unique. We also show that modified H-type groups with

$\dim(\mathfrak{z}) = 5$ which are Riemannian g.o. manifolds do not admit invariant Randers g.o. metrics.

3.1 $\dim(\mathfrak{z}) = 1$. We start with the 5-dimensional modified H-type group with 1-dimensional center and two 2-dimensional $\mathfrak{z}$-modules $\mathfrak{v}_i$. This example can be easily simplified into the simplest possible example—the 3-dimensional Heisenberg group. General behaviour of all modified H-type groups with $\dim(\mathfrak{z})$=1 can be also easily determined from this example.

Let us consider the Lie algebra $\mathfrak{n}$ with the scalar product α determined by the orthonormal basis $\{E_1, \dots, E_4, Z\}$ and generated by the nontrivial relations

$$[E_1, E_2] = Z, \quad [E_3, E_4] = \mu Z, \qquad \mu > 0.$$

We obtain the 1-parameter family of modified H-type algebras and corresponding modified H-type groups with the Riemannian metric induced by the above scalar product, in the sense of J. Lauret, see [13]. Let us further denote by A_{ij} the elementary operators on $\mathfrak{n}$ with the action generated by the relations

$$A_{ij}(E_i) = E_j, \quad A_{ij}(E_j) = -E_i, \quad A_{ij}(E_k) = 0, \qquad i \neq k \neq j.$$

It is easy to verify that the operators $D_1 = A_{12}$ and $D_2 = A_{34}$ act as derivations on $\mathfrak{n}$. Because these two operators commute, it holds $\mathfrak{h} = \mathrm{span}(D_1, D_2) \simeq \mathfrak{so}(2) \times \mathfrak{so}(2)$. We put $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$ and we express each group N in the form $N = G/H$, where $H = \mathrm{SO}(2) \times \mathrm{SO}(2)$ and $G = N \rtimes H$.

The Lie groups N with the above Riemannian metrics α are known to be naturally reductive, however, the decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$ is not the naturally reductive one. We define $\widetilde{Z} = Z + A_{12} + \mu A_{34}$ and we put $\mathfrak{n}' = \mathrm{span}(E_1, \dots, E_4, \widetilde{Z})$. The nontrivial Lie brackets of the new elements are

$$[E_1, E_2]_{\mathfrak{n}'} = \widetilde{Z}, \qquad [E_3, E_4]_{\mathfrak{n}'} = \mu \widetilde{Z}$$

and

$$\begin{aligned} [E_1, \widetilde{Z}]_{\mathfrak{n}'} &= -E_2, & [E_2, \widetilde{Z}]_{\mathfrak{n}'} &= E_1, \\ [E_3, \widetilde{Z}]_{\mathfrak{n}'} &= -\mu E_4, & [E_4, \widetilde{Z}]_{\mathfrak{n}'} &= \mu E_3. \end{aligned}$$

Now $\mathfrak{g} = \mathfrak{n}' + \mathfrak{h}$ is the naturally reductive decomposition.

We now introduce a family of invariant Randers metrics on each group N. Consider a vector $V = v\widetilde{Z} \in \mathfrak{n}$ for $0 < v < 1$, its α-equivalent 1-form β on $\mathfrak{n}$ and the Randers norm F defined on $\mathfrak{n}$ by formula (2). Because the vector V is invariant with respect to the group H, this norm F gives rise to the invariant Randers metric on $N = G/H$.

Proposition 9. *The modified H-type group N with* $\dim(\mathfrak{z}) = 1$ *and* $\dim(\mathfrak{v}) = 4$ *admits a 1-parameter family of invariant Randers g.o. metrics with respect to the isometry group $G = N \rtimes \mathrm{SO}(2)^2$ whose isotropy representation on the $\mathfrak{z}$-module $\mathfrak{v}$ is given by the operators A_{12}, A_{34}.*

Proof: We are going to construct the canonical geodesic graph in the above reductive decomposition $\mathfrak{g} = \mathfrak{n}' + \mathfrak{h}$. We put $Y = X + \xi(X)$, where $X = x_1 E_1 + \dots + x_4 E_4 + x_5 \widetilde{Z} \in \mathfrak{n}'$ and $\xi(X) = \xi_1 D_1 + \xi_2 D_2 \in \mathfrak{h}$; ξ_1, ξ_2 to be determined. We write down the Lie brackets

$$
\begin{aligned}
[X + \xi(X), E_1]_{\mathfrak{n}'} &= -x_2 \widetilde{Z} + x_5 E_2 + \xi_1 E_2,\\
[X + \xi(X), E_2]_{\mathfrak{n}'} &= x_1 \widetilde{Z} - x_5 E_1 - \xi_1 E_1,\\
[X + \xi(X), E_3]_{\mathfrak{n}'} &= -x_4 \mu \widetilde{Z} + x_5 \mu E_4 + \xi_2 E_4,\\
[X + \xi(X), E_4]_{\mathfrak{n}'} &= x_3 \mu \widetilde{Z} - x_5 \mu E_3 - \xi_2 E_3.
\end{aligned}
$$

Now we use these expressions in formula (6) from Corollary 8, where we put $V = v\widetilde{Z}$ and we substitute, step by step, $U = E_1, \dots, E_4, \widetilde{Z}$. We obtain the system of equations

$$
\begin{aligned}
-x_2 \xi_1 &= -\|x\| \cdot v x_2,\\
x_1 \xi_1 &= \|x\| \cdot v x_1,\\
-x_4 \xi_2 &= -\|x\| \cdot v \mu x_4,\\
x_3 \xi_2 &= \|x\| \cdot v \mu x_3.
\end{aligned}
$$

Here $\|x\|$ stands for $\sqrt{x_1^2 + \dots + x_5^2}$. The last equation, for $U = \widetilde{Z}$ is satisfied identically. We want to determine ξ_1 and ξ_2 depending on $x_1, \dots, x_5$. We obtain easily the unique solution

$$\xi_1 = v\|x\|, \qquad \xi_2 = v\mu\|x\| \tag{7}$$

and geodesic vectors are $X + \xi_1 D_1 + \xi_2 D_2$. These formulas determine the geodesic graph. □

Geodesic vectors $X + \xi(X)$ determined by the formula (7) above form a cone in $\mathfrak{g}$. Geodesic graphs with similar components were not observed in Riemannian geometry. If we consider $V = 0$, the Randers norm F becomes Riemannian and the geodesic graph given by formulas (7) becomes a zero map. Geodesic graph given by formulas (7) seems to be the simplest possibility for the Randers g.o. space, which motivates the following definition (see the discussion following Definition 4 for the motivation).

Definition 10. A geodesic graph whose components have no rational expressions and contain just multiples of terms $\|x\|$ is called Randers geodesic graph of degree 0.

The example above (with two 2-dimensional $\mathfrak{z}$-modules $\mathfrak{v}_i$) can be easily generalized to the Lie algebra $\mathfrak{n}$ with arbitrary number of $\mathfrak{z}$-modules $\mathfrak{v}_i$. Components of the unique geodesic graph to the respective operators A_{ij} are then just a multiple of components given by the formula (7), see [5] for the similar generalization with Riemannian geodesic graphs. We obtain the following conclusion.

Corollary 11. *Each modified H-type group N with $\dim(\mathfrak{z}) = 1$ and $\dim(\mathfrak{v}) = 2k$ admits a 1-parameter family of invariant Randers metrics with respect to the isometry group $G = N \rtimes \mathrm{SO}(2)^k$ whose isotropy representation on each irreducible $\mathfrak{z}$-module $\mathfrak{v}_i$ is given by the operator A_{12}. Corresponding homogenous spaces $N = G/H$ are Randers g.o. spaces of degree 0.*

3.2 $\dim(\mathfrak{z}) = 2$. Let us consider the Lie algebra $\mathfrak{n}$ with the scalar product α determined by the orthonormal basis $\{E_1, \dots, E_4, Z_1, Z_2\}$ and generated by the nontrivial relations

$$\begin{aligned} [E_1, E_2] &= 0, & [E_2, E_3] &= bZ_1 + cZ_2, \\ [E_1, E_3] &= aZ_1, & [E_2, E_4] &= -aZ_1, \\ [E_1, E_4] &= bZ_1 + cZ_2, & [E_3, E_4] &= 0 \end{aligned}$$

for arbitrary parameters $a, b, c \in \mathbb{R}$. We have the 3-parameter family of modified H-type algebras in the sense of J. Lauret, see [13]. Some of these modified H-type algebras are isometric, because in [13], the modified H-type metrics in this case form a 2-parameter family. However, we keep this notation from [10] to keep the possibility to compare Randers geodesic graphs with the Riemannian formulas in [10]. We denote by N the modified H-type groups corresponding to Lie algebras $\mathfrak{n}$. The skew-symmetric derivations on $\mathfrak{n}$ are

$$\begin{aligned} D_1 &= A_{12} - A_{34}, \\ D_2 &= A_{13} + A_{24}, \\ D_3 &= A_{14} - A_{23}. \end{aligned}$$

If $a^2 = c^2$ and $b = 0$, then also the operator

$$D_4 = 2B_{12} + A_{12} + A_{34}$$

is the derivation on $\mathfrak{n}$. There are no invariant vectors in $\mathfrak{n}$ with respect to the operator D_4 and consequently no invariant Randers metrics with respect to any isotropy group whose Lie algebra contains this operator. Hence put

$\mathfrak{h} = \mathrm{span}(D_1, D_2, D_3)$ for all groups N. If we write down the commutator relations for these operators, we easily verify that $\mathfrak{h} \simeq \mathfrak{su}(2)$. We put $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$ and we consider the homogeneous space $N = G/H$, where $H = \mathrm{SU}(2)$ and $G = N \rtimes H$.

We now introduce a family of invariant Randers metrics on each group N. Any vector $V = v_1 Z_1 + v_2 Z_2 \in \mathfrak{n}$ is invariant with respect to the group H. If $v_1^2 + v_2^2 < 1$, it gives rise to the invariant Randers norm F on $\mathfrak{n}$ determined by the 1-form β which is α-equivalent to V and formula (2) on $\mathfrak{n}$. Consequently, this Randers norm F determines invariant Randers metric on $N = G/H$.

Proposition 12. *Each modified H-type group with* $\dim(\mathfrak{z}) = 2$ *and* $\dim(\mathfrak{v}) = 4$ *admits a 2-parameter family of invariant Randers g.o. metrics with respect to the isometry group* $G = N \rtimes \mathrm{SU}(2)$ *whose isotropy representation on the* $\mathfrak{z}$*-module* $\mathfrak{v}$ *is given by the operators* D_1, D_2, D_3.

Proof: Again, we construct the geodesic graph. The homogeneous space $N = G/H$ with the Riemannian metric induced by α is not naturally reductive and we shall work in the reductive decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$. We put again $Y = X + \xi(X)$, where $X = x_1 E_1 + \cdots + x_4 E_4 + x_5 Z_1 + x_6 Z_2 \in \mathfrak{n}$ and $\xi(X) = \xi_1 D_1 + \xi_2 D_2 + \xi_3 D_3 \in \mathfrak{h}$. First, we write down the Lie brackets

$$\begin{aligned}
[X + \xi(X), E_1] &= -x_3 a Z_1 - x_4 (b Z_1 + c Z_2) + \xi_1 E_2 + \xi_2 E_3 + \xi_3 E_4, \\
[X + \xi(X), E_2] &= -x_3 (b Z_1 + c Z_2) + x_4 a Z_1 - \xi_1 E_1 + \xi_2 E_4 - \xi_3 E_3, \\
[X + \xi(X), E_3] &= x_1 a Z_1 + x_2 (b Z_1 + c Z_2) - xi_1 E_4 - \xi_2 E_1 + \xi_3 E_2, \\
[X + \xi(X), E_2] &= \ x_1 (b Z_1 + c Z_2) - x_2 a Z_1 + \xi_1 E_3 - \xi_2 E_2 - \xi_3 E_1.
\end{aligned}$$

Now we use these expressions in formula (5) from Corollary 7, where we put $V = v_1 Z_1 + v_2 Z_2$ and we substitute, step by step, $U = E_1, \dots, E_4, Z_1, Z_2$. We obtain the system of equations

$$\begin{aligned}
-\xi_1 x_2 - \xi_2 x_3 - \xi_3 x_4 &= -(x_3 a + x_4 b)(x_5 + v_1 \|x\|) - x_4 c (x_6 + v_2 \|x\|), \\
\xi_1 x_1 - \xi_2 x_4 + \xi_3 x_3 &= (x_4 a - x_3 b)(x_5 + v_1 \|x\|) - x_3 c (x_6 + v_2 \|x\|), \\
\xi_1 x_4 + \xi_2 x_1 - \xi_3 x_2 &= (x_1 a + x_2 b)(x_5 + v_1 \|x\|) + x_2 c (x_6 + v_2 \|x\|), \\
-\xi_1 x_3 + \xi_2 x_2 + \xi_3 x_1 &= (-x_2 a + x_1 b)(x_5 + v_1 \|x\|) + x_1 c (x_6 + v_2 \|x\|).
\end{aligned}$$

Here $\|x\|$ stands for $\sqrt{x_1^2 + \cdots + x_6^2}$. The last two equations for $U = Z_i$ are satisfied identically. This is the system of linear equations for variables ξ_i depending on parameters. The rank of this system is equal to 3 and using the Cramer's rule, we obtain the unique solution

$$\xi_1 = \Big(2a(x_5 + v_1\|x\|)(x_1x_4 + x_2x_3) + 2\big[b(x_5 + v_1\|x\|) + c(x_6 + v_2\|x\|)\big]$$
$$\times\, (x_2x_4 - x_1x_3)\Big)\frac{1}{(x_1^2 + \cdots + x_4^2)},$$
$$\xi_2 = \Big(a(x_5 + v_1\|x\|)(x_1^2 - x_2^2 + x_3^2 - x_4^2) + 2\big[b(x_5 + v_1\|x\|) + c(x_6 + v_2\|x\|)\big]$$
$$\times\, (x_1x_2 + x_3x_4)\Big)\frac{1}{(x_1^2 + \cdots + x_4^2)},$$
$$\xi_3 = \Big(2a(x_5 + v_1\|x\|)(x_3x_4 - x_1x_2) + \big[b(x_5 + v_1\|x\|) + c(x_6 + v_2\|x\|)\big]$$
$$\times\, (x_1^2 - x_2^2 - x_3^2 + x_4^2)\Big)\frac{1}{(x_1^2 + \cdots + x_4^2)}.$$

These formulas determine the geodesic graph. For $v_1 = v_2 = 0$, we obtain the Riemannian geodesic graph from [10]. □

Again, the structure of components of the above geodesic graph motivates the following definition.

Definition 13. A geodesic graph with homogeneous polynomial of degree d in denominator and containing terms $\|x\|$ in numerators is called Randers geodesic graph of degree d.

The construction can be also generalized to the arbitrary modified H-type algebra $\mathfrak{n}$ with $\dim(\mathfrak{z}) = 2$ and k $\mathfrak{z}$-modules $\mathfrak{v}_i$. The components of geodesic graph to each corresponding triplet of the operators $D_1, \dots, D_3$ have the same expressions as the components ξ_i above, see [5] for similar construction with Riemannian metrics. We obtain the following conclusion.

Corollary 14. *Each modified H-type group with* $\dim(\mathfrak{z}) = 2$ *and* $\dim(\mathfrak{v}) = 4k$ *admits a 2-parameter family of invariant Randers metrics with respect to the isometry group* $G = N \rtimes \mathrm{SU}(2)^k$ *whose isotropy representation on each* $\mathfrak{z}$*-module* $\mathfrak{v}_i$ *is given by the operators* D_1, D_2, D_3. *Corresponding homogenous spaces* G/H *are Randers g.o. spaces of degree* 2.

3.3 $\dim(\mathfrak{z}) = 3$. To avoid formulas with many parameters, we now consider just the 7-dimensional H-type group. Its invariant Riemannian metrics are naturally reductive. Other 7-dimensional modified H-type groups behave similarly. The Lie algebra $\mathfrak{n}$ is generated by the relations

$$[E_1, E_2] = Z_1, \qquad [E_2, E_3] = Z_3,$$
$$[E_1, E_3] = Z_2, \qquad [E_2, E_4] = -Z_2,$$
$$[E_1, E_4] = Z_3, \qquad [E_3, E_4] = Z_1$$

and $\{E_1, \dots, E_4, Z_1, Z_2, Z_3\}$ is the orthonormal basis with respect to the scalar product α. The skew-symmetric derivations on $\mathfrak{n}$ are

$$\begin{aligned} D_1 &= A_{12} - A_{34}, & D_4 &= 2B_{23} + A_{12} + A_{34},\\ D_2 &= A_{13} + A_{24}, & D_5 &= 2B_{13} - A_{13} + A_{24},\\ D_3 &= A_{14} - A_{23}, & D_6 &= 2B_{12} + A_{14} + A_{23}. \end{aligned}$$

The Lie algebra $\mathfrak{h} = \mathrm{span}(D_1, \dots, D_6)$ is isomorphic to $\mathfrak{su}(2) \times \mathfrak{su}(2)$. We can put $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$ and we have $H = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $G = N \rtimes H$. The homogeneous space $N = G/H$ with the Riemannian metric determined by α is naturally reductive. The group N admits another naturally reductive presentation $N = G'/H'$, where the corresponding isotropy algebra $\mathfrak{h}'$ is generated by the operators D_4, D_5, D_6, see [5] for details and for Riemannian geodesic graphs. However, with respect to any of these groups G or G', there are no invariant Randers metrics, because there are no invariant vectors $V \in \mathfrak{n}$ with respect to the operators D_4, D_5, D_6.

To obtain some invariant Randers metrics, we have to choose $\bar{\mathfrak{h}} = \mathrm{span}(D_1, D_2, D_3) \simeq \mathfrak{su}(2)$ and we put $\bar{\mathfrak{g}} = \mathfrak{n} + \bar{\mathfrak{h}}$. We consider the homogeneous space $N = \overline{G}/\overline{H}$, where $\overline{H} = \mathrm{SU}(2)$ and $\overline{G} = N \rtimes \overline{H}$. This homogeneous space with the Riemannian metric determined by α is not naturally reductive, but any vector $V = v_1 Z_1 + v_2 Z_2 + v_3 Z_3$ is invariant with respect to the isotropy representation of $\overline{H}$. If $v_1^2 + v_2^2 + v_3^2 < 1$, it gives rise to the invariant Randers g.o. metric on N.

Proposition 15. *The H-type group with* $\dim(\mathfrak{z}) = 3$, $\dim(\mathfrak{v}) = 4$ *admits a 3-parameter family of invariant Randers g.o. metrics with respect to the isometry group* $\overline{G} = N \rtimes \mathrm{SU}(2)$ *whose isotropy representation on the* $\mathfrak{z}$*-module* $\mathfrak{v}$ *is given by the operators* D_1, D_2, D_3.

Proof: We determine the geodesic graph in $\overline{G}/\overline{H}$. We put again $Y = X + \xi(X)$, where $X = x_1 E_1 + \dots + x_4 E_4 + x_5 Z_1 + x_6 Z_2 + x_7 Z_3 \in \mathfrak{n}$ and $\xi(X) = \xi_1 D_1 + \xi_2 D_2 + \xi_3 D_3 \in \bar{\mathfrak{h}}$. We write down the Lie brackets

$$\begin{aligned} [X + \xi(X), E_1] &= -x_2 Z_1 - x_3 Z_2 - x_4 Z_3 + \xi_1 E_2 + \xi_2 E_3 + \xi_3 E_4,\\ [X + \xi(X), E_2] &= +x_1 Z_1 + x_4 Z_2 - x_3 Z_3 - \xi_1 E_1 + \xi_2 E_4 - \xi_3 E_3,\\ [X + \xi(X), E_3] &= -x_4 Z_1 + x_1 Z_2 + x_2 Z_3 - \xi_1 E_4 - \xi_2 E_1 + \xi_3 E_2,\\ [X + \xi(X), E_4] &= +x_3 Z_1 - x_2 Z_2 + x_1 Z_3 + \xi_1 E_3 - \xi_2 E_2 - \xi_3 E_1. \end{aligned}$$

Now we use these expressions again in formula (5) from Corollary 7, where we put $V = v_1 Z_1 + v_2 Z_2 + v_3 Z_3$ and we substitute, step by step, $U = E_1, \dots, E_4$,

$Z_1, \dots, Z_3$. We obtain the system of equations

$$\begin{aligned}
-\xi_1 x_2 - \xi_2 x_3 - \xi_3 x_4 &= -x_2(x_5 + v_1\|x\|) - x_3(x_6 + v_2\|x\|) - x_4(x_7 + v_3\|x\|),\\
\xi_1 x_1 - \xi_2 x_4 + \xi_3 x_3 &= x_1(x_5 + v_1\|x\|) + x_4(x_6 + v_2\|x\|) - x_3(x_7 + v_3\|x\|),\\
\xi_1 x_4 + \xi_2 x_1 - \xi_3 x_2 &= -x_4(x_5 + v_1\|x\|) + x_1(x_6 + v_2\|x\|) + x_2(x_7 + v_3\|x\|),\\
-\xi_1 x_3 + \xi_2 x_2 + \xi_3 x_1 &= x_3(x_5 + v_1\|x\|) - x_2(x_6 + v_2\|x\|) + x_1(x_7 + v_3\|x\|).
\end{aligned}$$

Here $\|x\|$ stands for $\sqrt{x_1^2 + \dots + x_7^2}$. The last three equations for $U = Z_i$ are satisfied identically. The rank of this system is equal to 3 and using Cramer's rule, we obtain the unique solution

$$\begin{aligned}
\xi_1 = \Big(&(x_5 + v_1\|x\|)(x_1^2 + x_2^2 - x_3^2 - x_4^2) + 2(x_6 + v_2\|x\|)(x_2x_3 + x_1x_4)\\
&+ 2(x_7 + v_3\|x\|)(x_2x_4 - x_1x_3)\Big)\frac{1}{x_1^2 + \dots + x_4^2},\\
\xi_2 = \Big(&2(x_5 + v_1\|x\|)(x_2x_3 - x_1x_4) + (x_6 + v_2\|x\|)(x_1^2 - x_2^2 + x_3^2 - x_4^2)\\
&+ 2(x_7 + v_3\|x\|)(x_1x_2 + x_3x_4)\Big)\frac{1}{x_1^2 + \dots + x_4^2},\\
\xi_3 = \Big(&2(x_5 + v_1\|x\|)(x_1x_3 + x_2x_4) + 2(x_6 + v_2\|x\|)(x_3x_4 - x_1x_2)\\
&+ (x_7 + v_3\|x\|)(x_1^2 - x_2^2 - x_3^2 + x_4^2)\Big)\frac{1}{x_1^2 + \dots + x_4^2}.
\end{aligned}$$

These formulas determine the geodesic graph ξ. □

Again, the construction can be also generalized to the arbitrary modified H-type algebra $\mathfrak{n}$ with $\dim(\mathfrak{z}) = 3$ and isotypic $\mathfrak{z}$-modules. The components of geodesic graph to each corresponding triplet of the operators D_1, D_2, D_3 have the same expressions as the components ξ_i above, see [5] for similar construction with Riemannian metrics. We obtain the following conclusion.

Corollary 16. *Each modified H-type group with* $\dim(\mathfrak{z}) = 3$, $\dim(\mathfrak{v}) = 4k$ *and isotypic* $\mathfrak{z}$*-modules admits a* 3*-parameter family of invariant Randers metrics with respect to the isometry group* $\overline{G} = N \rtimes \mathrm{SU}(2)^k$ *whose isotropy representation on each* $\mathfrak{z}$*-module* $\mathfrak{v}_i$ *is given by the operators* D_1, D_2, D_3*. Corresponding homogeneous spaces* $\overline{G}/\overline{H}$ *are Randers g.o. spaces of degree* 2.

We remark here once more the interesting point, that Riemannian g.o. metrics on these groups are g.o. manifolds of degree 0. The crucial difference here is that the isotropy group for Riemannian metrics is bigger than the maximal isotropy group for the Randers g.o. metric.

3.4 $\dim(\mathfrak{z}) = 5$. For the modified H-type groups with $\dim(\mathfrak{z}) = 5$, we obtain the following result.

Proposition 17. *Modified H-type groups with* $\dim(\mathfrak{z}) = 5$ *and* $\dim(\mathfrak{v}) = 8$ *do not admit invariant non-Riemannian Randers g.o. metrics.*

Proof: According to [13], a modified H-type group of the given type is Riemannian g.o. space if and only if $S = k \cdot \mathrm{Id}_{\mathfrak{z}}$. Any such 13-dimensional group admits two presentations as a Riemannian g.o. space. First, $N = G/H$, where $H = \mathrm{SO}(5)$ and second, $N = G'/H'$, where $H' = \mathrm{SO}(5) \times \mathrm{SO}(2)$. The isotropy representation of $H = \mathrm{SO}(5)$ on $\mathfrak{n}$ is transitive on $\mathfrak{z}$ and also on $\mathfrak{v}$, see [8] for details about the isotropy representation and geodesic graphs for Riemannian metrics. Hence there are no invariant vectors $V \in \mathfrak{n}$ and consequently no invariant Randers g.o. metrics on N with respect to any of these groups G or G'. On the other hand, the group H is the smallest isotropy group for which we obtain the g.o. property of a Riemannian metric, hence there are no invariant Randers g.o. metrics with respect to any smaller isotropy group. $\square$

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