Further properties of Stepanov–Orlicz almost periodic functions

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Abstract. We revisit the concept of Stepanov–Orlicz almost periodic functions introduced by Hillmann in terms of Bochner transform. Some structural properties of these functions are investigated. A particular attention is paid to the Nemytskii operator between spaces of Stepanov–Orlicz almost periodic functions. Finally, we establish an existence and uniqueness result of Bohr almost periodic mild solution to a class of semilinear evolution equations with Stepanov–Orlicz almost periodic forcing term.

Keywords: Bohr almost periodic; Bochner transform; Stepanov–Orlicz almost periodic function; semilinear evolution equations; Nemytskii operator

Classification: 34C27, 35B15, 46E30

1. Introduction

The concept of Stepanov almost periodic functions introduced by Stepanov in [30] in 1926 is a natural and important generalization of Bohr almost periodicity. In recent decades, this concept has been well developed in connection with the theory of differential equations, because of their significance and applications in different areas such as physics, mathematical biology, control theory, and other related fields, see [7], [17], [20]. Important progresses have been made on this subject, we can cite several works which give a beautiful presentation of the methods and results: S. Zaidman [32], A. S. Rao [29], S. Stoiński [31], J. Andres and D. Pennequin [5], [4], T. Diagana [13], H. S. Ding et al. [15], Y. Hu and A. B. Mingarelli [18], F. Bedouhene et al. [6], D. Bugajewski and A. Nawrocki [7], P. Kasprzak et al. [20].

Although there is an abundant literature dedicated to the different extension of the Stepanov almost periodicity for Lebesgue p-integrable functions, such as Stepanov pseudo almost periodicity in both deterministic and stochastic cases,

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see [13], [6], there are very little works devoted to Stepanov almost periodicity in Orlicz and Musielak–Orlicz spaces. To our knowledge, the only reference that introduces and deals with such functions in Orlicz spaces is the paper [16] by T. R. Hillmann, where some structural and topological properties have been investigated. Moreover, their applications to the qualitative theory of differential equations have not been examined so far. This constitues a first motivation of the present study.

On the other hand, T. Diagana and M. Zitane in [14] introduced a new class of Stepanov–Musielak–Orlicz-pseudo almost periodic functions, but with the restriction that only the ergodicity property holds in Lebesgue space with variable exponents $\mathrm{L}^{p(\cdot)}$. Indeed, as speculated in [14, Remark 5.9], there are some difficulties to define Stepanov almost periodicity in $\mathrm{L}^{p(\cdot)}$ spaces. This is due mainly to the fact that theses spaces are not translation invariant. The question of considering Stepanov almost periodicity in Orlicz spaces where the translation invariant property occurs becomes more naturally. This gives rise to a second motivation for our study.

Another source of inspiration for our study comes from J. Andres and D. Pennequin in [5], in which the authors studied the nonexistence of purely Stepanov almost periodic solutions of ordinary differential equations in Banach spaces.

The main goal of this article is to continue and further explore the study [16] of T. R. Hillmann. First, we show that unlike Stepanov almost periodicity, Stepanov–Orlicz almost periodicity cannot be characterized in terms of its Bochner transform, when the generating Orlicz function, φ, fails the Δ_2-condition (Theorem 3.3). Second, based on a superposition result for Stepanov–Olicz almost periodic functions (Theorem 3.13), we address the issue of existence and uniqueness of Bohr almost periodic solution to the abstract differential equation

$$u'(t) = Au(t) + f(t, u(t)), \qquad t \in \mathbb{R} \tag{1.1}$$

with $A\colon D(A) \subset \mathbb{X} \to \mathbb{X}$ being a linear operator (unbounded) which generates an exponentially stable C_0-semigroup on $\mathbb{X}$ and $f\colon \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ being Stepanov–Orlicz almost periodic. We show that even with a Stepanov–Orlicz almost periodic coefficient, the unique bounded mild solution to (1.1) is Bohr almost periodic.

The organization of this paper is as follows. In Section 2, we recall some basic definitions and facts concerning Orlicz spaces and almost periodic functions. Special attention is paid to Stepanov–Orlicz almost periodic functions. Section 3 is devoted to our main result. First, we characterize the class of Stepanov–Orlicz almost periodic functions via the Bochner transform. Then, we give a link between the Stepanov–Orlicz almost periodic functions and measurably almost periodic functions, in order to establish a superposition theorem. Finally, an application of the above-mentioned results to the qualitative theory of almost periodic functions is indicated. We study the problem of existence and uniqueness of Bohr almost periodic mild solution to (1.1).

2. Preliminaries

At the beginning of this section we give some basic notations which are used throughout this paper.

For a Banach space $(\mathbb{X}, \|\cdot\|)$, let $\mathrm{M}(\mathbb{R}, \mathbb{X})$ be the space of all measurable functions from $\mathbb{R}$ into $\mathbb{X}$. We denote by $\mathrm{C}(\mathbb{R}, \mathbb{X})$ the space of continuous functions from $\mathbb{R}$ into $\mathbb{X}$. The notation $\mathrm{BC}(\mathbb{R}, \mathbb{X})$ stands for the space of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{X}$. For $f \in \mathrm{BC}(\mathbb{R}, \mathbb{X})$, let $\|f\|_\infty = \sup\{\|f(t)\| : t \in \mathbb{R}\}$ denote the norm of uniform convergence (sup-norm) on $\mathbb{R}$, under this norm $\mathrm{BC}(\mathbb{R}, \mathbb{X})$ is a Banach space. Space $\mathrm{L}^\infty(\mathbb{R}, \mathbb{X})$ is the Banach space of $\mathbb{X}$-valued essentially bounded functions on $\mathbb{R}$. We denote by $\Sigma := \Sigma(\mathbb{R})$ the σ-algebra of all Lebesgue-measurable subsets of $\mathbb{R}$, and by meas the Lebesgue measure on Σ. Finally, for a function $f \in \mathrm{M}(\mathbb{R}, \mathbb{X})$, we denote by $f_\tau : \mathbb{R} \to \mathbb{X}$, $\tau \in \mathbb{R}$, its *translation mapping* defined by $f_\tau(\cdot) = f(\cdot + \tau)$.

Orlicz spaces. In all the sequel, the notation φ is used for a *Young function*, i.e. a symmetric convex function $\varphi : \mathbb{R} \to \mathbb{R}^+$, satisfying $\varphi(u) = 0$ if and only if $u = 0$, moreover $\lim_{|u|\to\infty} \varphi(u) = \infty$.

This function φ is said to satisfy the Δ_2-*condition* for large values (we write $\varphi \in \Delta_2$), when there exist constants $k > 0$ and $u_0 > 0$ such that,

$$\varphi(2u) \le k\varphi(u), \qquad \forall\, |u| \ge u_0.$$

The function $\psi(y) = \sup\{x|y| - \varphi(x) : x \ge 0\}$ is called conjugate to φ. It is a Young function when φ is.

Let $\mathrm{E}^\varphi_{\mathrm{loc}}(\mathbb{R}, \mathbb{X})$ ($\mathrm{L}^\varphi_{\mathrm{loc}}(\mathbb{R}, \mathbb{X})$, respectively) be the subspace of all Lebesgue-measurable functions f defined on $\mathbb{R}$ such that for each bounded interval U and all $\alpha > 0$, (there exists $\alpha := \alpha_{U,f} > 0$, respectively),

$$\varrho_{\mathrm{L}^\varphi(U)}(\alpha f) := \int_U \varphi(\alpha\|f(t)\|)\,\mathrm{d}t < \infty.$$

When $U = [0, 1]$, we get the classical Morse–Transue space $\mathrm{E}^\varphi([0, 1], \mathbb{X})$ (the Orlicz space $\mathrm{L}^\varphi([0, 1], \mathbb{X})$, respectively). Both spaces are modular spaces under the modular $\varrho_{\mathrm{L}^\varphi([0,1])}$. Moreover, they are Banach spaces under the Luxemburg norm,

$$\|f\|_{\mathrm{L}^\varphi([0,1])} = \inf\Big\{k > 0 : \varrho_{\mathrm{L}^\varphi([0,1])}\Big(\frac{f}{k}\Big) \le 1\Big\}.$$

Clearly, $\mathrm{E}^{\varphi}([0,1],\mathbb{X}) \subset \mathrm{L}^{\varphi}([0,1],\mathbb{X})$. Equality holds if and only if φ satisfies the Δ_2-condition.

Two important properties hold in $\mathrm{E}^{\varphi}([0,1],\mathbb{X})$. If $f \in \mathrm{E}^{\varphi}([0,1],\mathbb{X})$, then we have [23]:

(i) Function f is *φ-mean continuous*, namely for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|f_h - f\|_{\mathrm{L}^{\varphi}([0,1])} < \varepsilon$ for $h \in \mathbb{R}$ with $|h| < \delta$, where

$$f_h(t) = \begin{cases} f(t+h), & \text{if } \ t \in [0,1] \ \text{ and } \ t+h \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Function f is *absolutely continuous in* L^{φ}*-norm*, that is for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $A \in \Sigma$ with $\operatorname{meas}(A) < \delta$, we have $\|f \mathbb{1}_A\|_{\mathrm{L}^{\varphi}[0,1])} < \varepsilon$, where $\mathbb{1}_A$ denotes the characteristic function of the set A.

For more details about Orlicz and modular spaces, we refer the reader to [23], [8], [22], [26].

Now, let us recall the definition of Stepanov–Orlicz space. The functional

$$\tag{2.1} \begin{aligned} \varrho_{\mathbb{S}^{\varphi}} \colon \ \mathrm{L}^{\varphi}_{\mathrm{loc}}(\mathbb{R},\mathbb{X}) \ &\to \ \overline{\mathbb{R}}^{+} \\ f \ &\mapsto \ \sup_{t \in \mathbb{R}} \int_t^{t+1} \varphi(\|f(s)\|) \, \mathrm{d}s \end{aligned}$$

is the Stepanov–Orlicz modular. The corresponding modular space, namely the Stepanov–Orlicz space, is given by:

$$\mathrm{B}\mathbb{S}^{\varphi}(\mathbb{R},\mathbb{X}) = \{f \in \mathrm{L}^{\varphi}_{\mathrm{loc}}(\mathbb{R},\mathbb{X})\colon \text{ s.t. } \varrho_{\mathbb{S}^{\varphi}}(\alpha f) < \infty \ \text{ for some } \ \alpha > 0\}.$$

We endow this space with its natural Luxemburg norm defined by

$$\tag{2.2} \|f\|_{\mathbb{S}^{\varphi}} := \inf \Big\{ k > 0 \colon \varrho_{\mathbb{S}^{\varphi}}\Big(\frac{f}{k}\Big) \le 1 \Big\}.$$

T. R. Hillmann in [16] has shown that the Luxemburg norm (2.2) can be reformulated as follows

$$\tag{2.3} \|f\|_{\mathbb{S}^{\varphi}} = \sup_{t \in \mathbb{R}} \inf \Big\{ k > 0 \colon \int_t^{t+1} \varphi\Big(\frac{\|f(s)\|}{k}\Big) \, \mathrm{d}s \le 1 \Big\}.$$

Using similar arguments as those of Orlicz space theory in [26], see also [16, Lemma 1.2], the following property holds: for any $f \in \mathrm{B}\mathbb{S}^{\varphi}(\mathbb{R},\mathbb{X})$, there exists $\beta_{\varphi,f} > 0$ such that

$$\tag{2.4} \|f\|_{\mathbb{S}^1} \le \beta_{\varphi,f} \|f\|_{\mathbb{S}^{\varphi}},$$

where the notation $\|\cdot\|_{\mathbb{S}^1}$ stands for the norm (2.2) when $\varphi = |\cdot|$.

Bohr and Stepanov–Orlicz almost periodicity. Let us start with the definition of relatively dense set. A set $\mathrm{E} \subset \mathbb{R}$ is called *relatively dense* if there exists a number $l > 0$ such that $(a, a+l) \cap \mathrm{E} \neq \emptyset$ for all $a \in \mathbb{R}$.

Definition 2.1 ([2], [10])**.** A continuous function $f\colon \mathbb{R} \to \mathbb{X}$ is said to be *Bohr almost periodic* if for all $\varepsilon > 0$ the set

$$\mathrm{T}(f, \varepsilon) := \{\tau \in \mathbb{R}\colon \|f_\tau - f\|_\infty < \varepsilon\},$$

is relatively dense in $\mathbb{R}$. The number τ above is called ε-translation number of f. We denote by $\mathrm{AP}(\mathbb{R}, \mathbb{X})$ the space of all Bohr almost periodic functions.

Bohr almost periodic functions enjoy important properties, in particular they can be defined as uniform limits of sequences belonging to $\mathrm{Trig}(\mathbb{R}; \mathbb{X})$, the set of generalized trigonometric polynomials $P_n(t) = \sum_{k=1}^n a_k \exp(i\lambda_k t)$, $t \in \mathbb{R}$, where $a_k \in \mathbb{X}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$, are mutually different real numbers. Moreover, they are bounded and uniformly continuous. Note that $(\mathrm{AP}(\mathbb{R}, \mathbb{X}), \|\cdot\|_\infty)$ is a Banach space.

Now, we pass to the definition of Stepanov–Orlicz almost periodicity introduced by T. R. Hillmann in [16]. Following J. Albrycht work [1], where the theory of Marcinkiewicz–Orlicz spaces has been introduced, T. R. Hillmann in [16] has proposed various natural generalizations of almost periodicity for complex-valued functions by considering his study in $\mathrm{L}^\varphi_{\mathrm{loc}}(\mathbb{R}, \mathbb{X})$.

Definition 2.2 ([16])**.** A function $f \in \mathrm{L}^\varphi_{\mathrm{loc}}(\mathbb{R}, \mathbb{X})$ is said to be *Stepanov–Orlicz almost periodic* $\mathbb{S}^\varphi_{\mathrm{a.p.}}(\mathbb{R}, \mathbb{X})$ *function* if f belongs to the closure of the linear trigonometric polynomials set $\mathrm{Trig}(\mathbb{R}; \mathbb{X})$ in $\mathrm{L}^\varphi_{\mathrm{loc}}(\mathbb{R}, \mathbb{X})$, with respect to the norm $\|\cdot\|_{\mathbb{S}^\varphi}$. More exactly:

$$\mathbb{S}^\varphi_{\mathrm{a.p.}}(\mathbb{R}, \mathbb{X}) = \overline{\mathrm{Trig}(\mathbb{R}; \mathbb{X})}^{\|\cdot\|_{\mathbb{S}^\varphi}}. \tag{2.5}$$

Another definition in Bohr sense is given by the following:

Definition 2.3 ([16])**.** A function f in $\mathrm{L}^\varphi_{\mathrm{loc}}(\mathbb{R}, \mathbb{X})$ is said to satisfy the $\mathbb{S}^\varphi$-*translation property* and we write $f \in \mathbb{S}^\varphi_{\mathrm{t.p.}}(\mathbb{R}, \mathbb{X})$, if for every $\varepsilon > 0$, the set

$$\mathbb{S}^\varphi\mathrm{T}(f, \varepsilon) := \{\tau \in \mathbb{R}\colon \|f_\tau - f\|_{\mathbb{S}^\varphi} < \varepsilon\}, \tag{2.6}$$

is relatively dense in $\mathbb{R}$. Elements of $\mathbb{S}^\varphi\mathrm{T}(f, \varepsilon)$ are called ε-$\mathbb{S}^\varphi$-almost periods of f.

The space $(\mathbb{S}^\varphi_{\mathrm{t.p.}}(\mathbb{R}, \mathbb{X}), \|\cdot\|_{\mathbb{S}^\varphi})$ is a Banach space, see [16].

3. Main results

3.1 Characterization of $\mathbb{S}^\varphi\mathrm{AP}$-functions via the Bochner transform. In very important theorem [16, Theorem 1.1], T. R. Hillmann has shown the following inclusion

$$\mathbb{S}^\varphi_{\text{a.p.}}(\mathbb{R},\mathbb{X}) \subset \mathbb{S}^\varphi_{\text{t.p.}}(\mathbb{R},\mathbb{X}).$$

Equality holds if and only if φ satisfies the Δ_2-condition. This is the case in particular when $\varphi = |\cdot|^p$, $p \geq 1$. In this case, we obtain the usual space of Stepanov almost periodic functions, that we denote by $\mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{X})$. This space is well investigated. The literature is abundant, see for instance [20], [18], [3], [27].

Let $f \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$. It is well-know, see e.g. [2], [10], [3], [27], that Stepanov almost periodicity of f is equivalent to the $\mathrm{L}^p([0,1],\mathbb{X})$-almost periodicity of its Bochner transform f^b defined as

$$f^b\colon \mathbb{R} \to \mathbb{X}^{[0,1]}, \qquad t \mapsto f^b(t) = f(t+\cdot).$$

A question arises then naturally:

(**Q**) Can the previous definitions be characterized via the Bochner transform?

Prior giving an answer, let us make the following remark:

Remark 3.1.

1. Thanks to (2.3), the Luxemburg norm of $f \in \mathrm{B}\mathbb{S}^\varphi(\mathbb{R},\mathbb{X})$ can be seen as the sup-norm of its Bochner transform $f^b\colon \mathbb{R} \to \mathrm{L}^\varphi([0,1],\mathbb{X})$. That is,

$$\|f\|_{\mathbb{S}^\varphi} = \sup_{t\in\mathbb{R}} \|f(t+\cdot)\|_{\mathrm{L}^\varphi([0,1])} = \|f^b\|_\infty. \tag{3.1}$$

   Due to this identification, we conclude that boundedness of f^b is equivalent to boundedness of f with respect to the Luxemburg norm $\|\cdot\|_{\mathbb{S}^\varphi}$.
2. Using again (2.3), we have $\mathbb{S}^\varphi\mathrm{T}(f,\varepsilon) = \mathrm{T}(f^b,\varepsilon)$.
3. A sequence (f_n) converges to f in $\mathrm{B}\mathbb{S}^\varphi(\mathbb{R},\mathbb{X})$ if and only if (f_n^b) converges to f^b in $\mathrm{L}^\infty(\mathbb{R},\mathrm{L}^\varphi([0,1],\mathbb{X}))$.

A partial answer to (**Q**) is provided by the following proposition:

Proposition 3.2. *The question is affirmative for:*

1. $\mathbb{S}^\varphi_{\text{t.p.}}(\mathbb{R},\mathbb{X})$, *if* φ *satisfies the* Δ_2*-condition;*
2. $\mathbb{S}^\varphi_{\text{a.p.}}(\mathbb{R},\mathbb{X})$ *for every Young function* φ.

Proof: 1. Using Remark 3.1, both f and f^b have the same ε-translation numbers. To get the almost periodicity of f^b it reminds only to check its continuity. Let $t_0 \in \mathbb{R}$. Since $\varphi \in \Delta_2$ then for every $\varepsilon > 0$, there exists $\delta = \delta(t_0,\varepsilon) > 0$ such that for every $h \in \mathbb{R}$ with $|h| \le \delta$ we have

$$\|f_h - f\|_{\mathrm{L}^\varphi([t_0,t_0+1])} \le \varepsilon.$$

The continuity of f^b follows then from the equality

$$\|f^b(t_0+h) - f^b(t_0)\|_{\mathrm{L}^\varphi([0,1])} = \|f_h - f\|_{\mathrm{L}^\varphi([t_0,t_0+1])} \le \varepsilon$$

combined with the φ-mean continuity of f in $\mathrm{L}^\varphi([t_0, t_0+1])$.

2. Let $f \in \mathbb{S}^\varphi_{\text{a.p.}}(\mathbb{R}, \mathbb{X})$. Let (f_n) be an approximating sequence of generalized trigonometric polynomials. That is

$$(f_n) \subset \mathrm{Trig}_f(\mathbb{R}, \mathbb{X}) = \{P_n \in \mathrm{Trig}(\mathbb{R}; \mathbb{X})\colon \text{ such that } \lim_{n\to\infty} \|f - P_n\|_{\mathbb{S}^\varphi} = 0\}.$$

As for every n, f_n^b belongs to $\mathrm{Trig}_{f^b}(\mathbb{R}, \mathbb{X}) \subset \mathrm{Trig}(\mathbb{R}, \mathrm{L}^\varphi_{\text{loc}}(\mathbb{R}, \mathbb{X}))$, see [27, Proposition 4.3], and

$$\lim_{n\to\infty} \|f^b - f_n^b\|_\infty = \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \|f^b(t) - f_n^b(t)\|_{\mathrm{L}^\varphi([0,1])} = \lim_{n\to\infty} \|f - f_n\|_{\mathbb{S}^\varphi} = 0,$$

we deduce that $f^b \in \mathrm{AP}(\mathbb{R}, \mathrm{L}^\varphi([0,1], \mathbb{X}))$, and the claim follows. □

Hereafter, we see that when φ fails the Δ_2-condition, the answer is negative when dealing with $\mathbb{S}^\varphi_{\text{t.p.}}(\mathbb{R}, \mathbb{X})$. It seems to us that this is due, mainly, to the fact that functions in $\mathrm{L}^\varphi([0,1], \mathbb{X})$ are not absolutely continuous in L^φ-norm.

Theorem 3.3. *Without the Δ_2 condition, there exists $\widetilde{g} \in \mathbb{S}^\varphi_{\text{t.p.}}(\mathbb{R}, \mathbb{X})$ such that $\widetilde{g}^b \notin \mathrm{AP}(\mathbb{R}, \mathrm{L}^\varphi([0,1], \mathbb{X}))$.*

Proof: Since functions in $\mathrm{L}^\varphi([0,1], \mathbb{X}) \setminus \mathrm{E}^\varphi([0,1], \mathbb{X})$ are not absolutely continuous in L^φ-norm, see e.g. A. Kufner et al. [23, Theorem 3.15.6], then we can find a function $g \in \mathrm{L}^\varphi([0,1], \mathbb{X}) \setminus \mathrm{E}^\varphi([0,1], \mathbb{X})$ and $\varepsilon_0 > 0$ such that for every $\delta > 0$, a measurable set, A_δ, exists with $\mathrm{meas}(A_\delta) \le \delta$ and $\|g \mathbb{1}_{A_\delta}\|_{\mathrm{L}^\varphi([0,1])} > \varepsilon_0$.

We can ask that A_δ be included in $[1-\delta, 1[$ and $g \equiv 0$ on $[0, 1-\delta[$.

Let $\widetilde{g}$ be the 1-periodic extension of g to the whole $\mathbb{R}$. Then $\widetilde{g}$ satisfies the $\mathbb{S}^\varphi$-translation property, that is, $\widetilde{g} \in S^\varphi_{\text{t.p.}}(\mathbb{R}, \mathbb{X})$, see also [16]. Now, if we assume that $\widetilde{g}^b\colon \mathbb{R} \to \mathrm{L}^\varphi([0,1], \mathbb{X})$ is continuous, then we can choose δ' (small enough) such that $\varepsilon_0 \ge \|\widetilde{g}^b(h) - \widetilde{g}^b(0)\|_{\mathrm{L}^\varphi([0,1])}$ for every $h \in]0, \delta']$. Let $\delta^* = \min(\delta, \delta')$. Since $\widetilde{g}(\delta^* + \cdot) \equiv 0$ on $[1-\delta^*, 1[$, we get for $h = \delta^*$,

$$\begin{aligned}\|\widetilde{g}^b(\delta^*) - \widetilde{g}^b(0)\|_{\mathrm{L}^\varphi([0,1])} &\ge \|(\widetilde{g}(\delta^* + \cdot) - \widetilde{g}(\cdot))\mathbb{1}_{[1-\delta^*,1]}\|_{\mathrm{L}^\varphi([0,1])} \\ &= \|g(\cdot)\mathbb{1}_{[1-\delta^*,1]}\|_{\mathrm{L}^\varphi([0,1])} \\ &= \|g(\cdot)\mathbb{1}_{A_{\delta^*}}\|_{\mathrm{L}^\varphi([0,1])} > \varepsilon_0,\end{aligned}$$

which contradicts the fact that $\widetilde{g}^b\colon \mathbb{R} \to \mathrm{L}^\varphi([0,1], \mathbb{X})$ is continuous. □

Our aim here is to characterize the Stepanov–Orlicz almost periodicity via the Bochner transform without imposing any restriction on the Young function. We propose an alternative idea, that is, restrict our study to space $\mathrm{E}^{\varphi}_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$. This allows us to benefit from the richness of the Morse–Transue space. In particular, we take the advantage of the φ-mean continuity and the absolutely continuity in L^{φ}-norm properties of $\mathrm{E}^{\varphi}_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$ space.

Define the space

$$\mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R},\mathbb{X}) := \mathbb{S}^{\varphi}_{\mathrm{t.p.}}(\mathbb{R},\mathbb{X}) \cap \mathrm{E}^{\varphi}_{\mathrm{loc}}(\mathbb{R},\mathbb{X}). \tag{3.2}$$

Elements from $\mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R},\mathbb{X})$ will be called $\mathbb{S}^{\varphi}$-almost periodic functions. Clearly, $\mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R},\mathbb{X})$ is a closed subspace of $\mathbb{S}^{\varphi}_{\mathrm{t.p.}}(\mathbb{R},\mathbb{X})$. Thus, $(\mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R},\mathbb{X}), \|\cdot\|_{\mathbb{S}^{\varphi}})$ is a Banach space.

Remark 3.4. Let us mention that, the space $\mathbb{S}^{\varphi}_{\mathrm{a.p.}}(\mathbb{R},\mathbb{X})$ is invariant by intersection with $\mathrm{E}^{\varphi}_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$, even if the closure in (2.5) is taken in $\mathrm{L}^{\varphi}_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$. More precisely,

$$\mathbb{S}^{\varphi}_{\mathrm{a.p.}}(\mathbb{R},\mathbb{X}) = \mathbb{S}^{\varphi}_{\mathrm{a.p.}}(\mathbb{R},\mathbb{X}) \cap \mathrm{E}^{\varphi}_{\mathrm{loc}}(\mathbb{R},\mathbb{X}).$$

To see how the property $\mathbb{S}^{\varphi}_{\mathrm{a.p.}}(\mathbb{R},\mathbb{X}) \subset \mathrm{E}^{\varphi}_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$ occurs naturally, it is enough to show that for every $\alpha > 0$, $\varrho_{\mathbb{S}^{\varphi}}(\alpha f) < \infty$. For, let $f \in \mathbb{S}^{\varphi}_{\mathrm{a.p.}}(\mathbb{R},\mathbb{X})$. Fix $\varepsilon > 0$ and choose $P_\varepsilon \in \mathrm{Trig}(\mathbb{R};\mathbb{X})$ such that $\|f - P_\varepsilon\|_{\mathbb{S}^{\varphi}} \le \varepsilon$. Then taking into account the boundedness of P_ε, we obtain for every $\alpha > 0$,

$$\varrho_{\mathbb{S}^{\varphi}}(\alpha f) \le \frac{1}{2}\varrho_{\mathbb{S}^{\varphi}}(2\alpha(f - P_\varepsilon)) + \frac{1}{2}\varrho_{\mathbb{S}^{\varphi}}(2\alpha P_\varepsilon) < \infty,$$

which achieves the claimed property.

As an immediate consequence, we obtain

Corollary 3.5. *The following properties hold true for every Young function φ:*

1. *if $f \in \mathrm{E}^{\varphi}_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$, then $f^b \in \mathrm{C}(\mathbb{R},\mathrm{E}^{\varphi}([0,1];\mathbb{X}))$;*
2. *$\mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R},\mathbb{X}) = \mathbb{S}^{\varphi}_{\mathrm{a.p.}}(\mathbb{R},\mathbb{X})$;*
3. *$f \in \mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R},\mathbb{X})$ if and only if $f^b \in \mathrm{AP}(\mathbb{R},\mathrm{E}^{\varphi}([0,1];\mathbb{X}))$.*

Remark 3.6. Item 3. of Corollary 3.5 allows us to deduce some properties of a function $f \in \mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R},\mathbb{X})$ via the almost periodicity of $f^b \colon \mathbb{R} \to \mathrm{E}^{\varphi}([0,1],\mathbb{X})$. In particular, Bohr Bohl Amerio's result [2, Theorem II, page 82] remains valid in $\mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R},\mathbb{X})$. Indeed, we have:

Proposition 3.7. *Let $\mathbb{X}$ be a uniformly convex Banach space. Let φ be a uniformly convex function satisfying the Δ_2-condition, and let $f \in \mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R},\mathbb{X})$. Let $F(t) = \int_0^t f(s)\,\mathrm{d}s$. If $F \in \mathrm{B}\mathbb{S}^{\varphi}(\mathbb{R},\mathbb{X})$, then F is Bohr almost periodic.*

The proof that F is Bohr almost periodic is provided by the fact that the space $\mathrm{L}^{\varphi}([0,1];\mathbb{X})$ is uniformly convex under the uniform convexity and the Δ_2-condition on φ and the uniform convexity of the space $\mathbb{X}$, see [19].

It should be noted that the case $\varphi = |\cdot|$ is considered in [5], [11].

Almost periodicity in Lebesgue measure (meas-AP$(\mathbb{R},\mathbb{X})$**).** W. Stepanov in [30] has introduced the notion of almost periodicity for measurable function and called them *measurably almost periodic functions* (meas-AP$(\mathbb{R},\mathbb{X})$). Since then, there has been a significant attention devoted to these functions. One can mention the works by S. Stoiński [31], L. I. Danilov [12], D. Bugajewski and A. Nawrocki [7] and P. Kasprzak et al. [20].

Let $\overline{\mu}_{\mathbb{S}}$ be the set function defined on Σ by:

$$\overline{\mu}_{\mathbb{S}}(A) = \sup_{\xi\in\mathbb{R}} \operatorname{meas}(A \cap [\xi, \xi+1]). \tag{3.3}$$

For a set $A \in \Sigma$, the notation A^c means the complementary of A.

Definition 3.8 ([30], [7], [20], [31])**.** We say that a function $f \in \mathrm{M}(\mathbb{R},\mathbb{X})$ is *measurably almost periodic* and we write $f \in$ meas-AP$(\mathbb{R},\mathbb{X})$, if for any ε, $\delta > 0$, there exists $l(\varepsilon, \delta) > 0$ such that any interval of length $l(\varepsilon, \delta)$ contains at least a real number τ for which

$$\sup_{\xi\in\mathbb{R}} \operatorname{meas}(\{t \in [\xi, \xi+1] \colon \|f(t+\tau) - f(t)\| \geq \varepsilon\}) < \delta.$$

As pointed out by [7], [31], this notion coincides with the classical Stepanov almost periodicity when replacing the norm $\|\cdot\|$ by the truncated one $\|\cdot\|' = \min(\|\cdot\|, 1)$. In other words, we have the following characterization

$$\text{meas-AP}(\mathbb{R},\mathbb{X}) = \mathbb{S}^1\mathrm{AP}(\mathbb{R},(\mathbb{X},\|\cdot\|')). \tag{3.4}$$

Using (2.4) and (3.4), we deduce that

$$\mathrm{AP}(\mathbb{R},\mathbb{X}) \subset \mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R},\mathbb{X}) \subset \mathbb{S}^1\mathrm{AP}(\mathbb{R},\mathbb{X}) \subset \text{meas-AP}(\mathbb{R},\mathbb{X}). \tag{3.5}$$

Notice that the class meas-AP$(\mathbb{R},\mathbb{X})$ enjoys an important compactness property, see [12], that we denote by (**D**):

(**D**) If $f \in$ meas-AP$(\mathbb{R},\mathbb{X})$, then for all $\varepsilon > 0$ there exist a measurable subset T_{ε} of $\mathbb{R}$ and a compact subset K_{ε} of $\mathbb{X}$ such that $\overline{\mu}_{\mathbb{S}}(\mathrm{T}_{\varepsilon}^{c}) < \varepsilon$ and $f(t) \in \mathrm{K}_{\varepsilon}$ for all $t \in \mathrm{T}_{\varepsilon}$.

This compactness property is very useful when establishing our superposition result.

3.2 Link between the spaces $\mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R},\mathbb{X})$ and meas-AP$(\mathbb{R},\mathbb{X})$**.** Below, we are concerned with establishing a necessary and sufficient condition for a measurably almost periodic function to be in $\mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R},\mathbb{X})$. It should be noticed that a similar result is obtained by W. Stepanov in [30] in the case when $\varphi = |\cdot|$, see also [7] for the proof. L. I. Danilov in [12] has considered such result for metric-valued functions. An extension to the case of Besicovitch–Musielak–Orlicz spaces of almost periodic functions is considered in [21].

First, we need to introduce the following definition, inspired from [21].

Definition 3.9. A function $f \in \mathrm{E}^{\varphi}_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$ is said to be absolutely φ-integrable in $\overline{\mu}_{\mathbb{S}}$ sense, if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that for every measurable subset $A \in \Sigma$ with $\overline{\mu}_{\mathbb{S}}(A) < \delta$ we have

$$\|f\mathbb{1}_A\|_{\mathbb{S}^{\varphi}} < \varepsilon.$$

The class of these functions will be denoted by $\mathbb{M}^{\varphi}(\mathbb{R},\mathbb{X})$.

For $f \in \mathrm{E}^{\varphi}_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$, let us define the following quantity:

$$\kappa(f) := \lim_{\delta\to 0^+} \sup_{\substack{A\subset\mathbb{R}\\ \overline{\mu}_{\mathbb{S}}(A)\le\delta}} \|f\mathbb{1}_A\|_{\mathbb{S}^{\varphi}}. \tag{3.6}$$

Then, it is easy to see that f is absolutely φ-integrable in $\overline{\mu}_{\mathbb{S}}$ sense if and only if $\kappa(f) = 0$. So, we can write

$$\mathbb{M}^{\varphi}(\mathbb{R},\mathbb{X}) = \{f \in \mathrm{E}^{\varphi}_{\mathrm{loc}}(\mathbb{R},\mathbb{X})\colon \text{ s.t. } \kappa(f) = 0\}.$$

The following lemma shows that the functions in $\mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R},\mathbb{X})$ enjoy the absolutely φ-integrable in the $\overline{\mu}_{\mathbb{S}}$ sense.

Lemma 3.10. *The inclusion*

$$\mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R},\mathbb{X}) \subset \mathbb{M}^{\varphi}(\mathbb{R},\mathbb{X}) \tag{3.7}$$

holds true.

Proof: First, let us show that bounded functions are absolutely φ-integrable in $\overline{\mu}_{\mathbb{S}^{\varphi}}$ sense, that is $\mathrm{L}^{\infty}(\mathbb{R},\mathbb{X}) \subset \mathbb{M}^{\varphi}(\mathbb{R},\mathbb{X})$.

Let $\varepsilon > 0$ and $A \in \Sigma$. Let $f\colon \mathbb{R} \to \mathbb{X}$ be a bounded function. Put $\mathcal{C} = \sup_{t\in\mathbb{R}} \|f(t)\|$. Here, we exclude for simplicity the trivial case, when $\overline{\mu}_{\mathbb{S}}(A) = 0$. Clearly, we have

$$\|\mathbb{1}_A\|_{\mathbb{S}^{\varphi}} = \frac{1}{\varphi^{-1}(1/\overline{\mu}_{\mathbb{S}}(A))} \qquad \text{and} \qquad \|f\mathbb{1}_A\|_{\mathbb{S}^{\varphi}} \le \mathcal{C}\|\mathbb{1}_A\|_{\mathbb{S}^{\varphi}}.$$

Since the function $t \to (\varphi^{-1}(1/t))^{-1}$ is continuous and increasing on $]0,\infty[$, we deduce by choosing $\delta := (\varphi(\mathcal{C}/\varepsilon))^{-1}$ that $\|f\mathbb{1}_A\|_{\mathbb{S}^{\varphi}} \le \varepsilon$, whenever $\overline{\mu}_{\mathbb{S}}(A) < \delta$.

Now, let us assume that $f \in \mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R}, \mathbb{X})$. Using Corollary 3.5, there exists a trigonometric polynomial $P_{\varepsilon} \in \mathrm{Trig}(\mathbb{R}; \mathbb{X})$ such that

$$\|f - P_{\varepsilon}\|_{\mathbb{S}^{\varphi}} \leq \frac{\varepsilon}{2}. \tag{3.8}$$

Since $P_{\varepsilon} \in \mathrm{L}^{\infty}(\mathbb{R}, \mathbb{X})$, there exists $\delta > 0$ such that $\|P_{\varepsilon}\mathbb{1}_A\|_{\mathbb{S}^{\varphi}} \leq \varepsilon/2$ whenever $\overline{\mu}_{\mathbb{S}}(A) < \delta$. For such δ, we have

$$\|f\mathbb{1}_A\|_{\mathbb{S}^{\varphi}} \leq \|(f - P_{\varepsilon})\mathbb{1}_A\|_{\mathbb{S}^{\varphi}} + \|P_{\varepsilon}\mathbb{1}_A\|_{\mathbb{S}^{\varphi}} \leq \|f - P_{\varepsilon}\|_{\mathbb{S}^{\varphi}} + \|P_{\varepsilon}\mathbb{1}_A\|_{\mathbb{S}^{\varphi}} \leq \varepsilon.$$

This completes the proof of the lemma. □

As a consequence, we have the following theorem:

Theorem 3.11. *The following characterization holds*

$$\mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R}, \mathbb{X}) = \mathbb{M}^{\varphi}(\mathbb{R}, \mathbb{X}) \cap \mathrm{meas}\text{-}\mathrm{AP}(\mathbb{R}, \mathbb{X}).$$

Proof: Firstly, using (3.5) and Lemma 3.10, we obtain the following inclusion

$$\mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R}, \mathbb{X}) \subset \mathrm{meas}\text{-}\mathrm{AP}(\mathbb{R}, \mathbb{X}) \cap \mathbb{M}^{\varphi}(\mathbb{R}, \mathbb{X}).$$

Conversely, let us assume that $f \in \mathrm{meas}\text{-}\mathrm{AP}(\mathbb{R}, \mathbb{X}) \cap \mathbb{M}^{\varphi}(\mathbb{R}, \mathbb{X})$. Let $\varepsilon > 0$. We can find a trigonometric polynomial $P_{\varepsilon} \in \mathrm{Trig}(\mathbb{R}; \mathbb{X})$ such that $\overline{\mu}_{\mathbb{S}}(A_{\varepsilon}) < \varepsilon$, where

$$A_{\varepsilon} = \{t \in \mathbb{R} \colon \text{ s.t. } \|f(t) - P_{\varepsilon}(t)\| > \varepsilon\varphi^{-1}(1)\}.$$

Now, let $\eta > 0$. From the absolute φ-integrability in $\overline{\mu}_{\mathbb{S}}$ sense of both f and P_{ε}, we can choose $\varepsilon > 0$ small enough such that $\max(\|f\mathbb{1}_{A_{\varepsilon}}\|_{\mathbb{S}^{\varphi}}, \|P_{\varepsilon}\mathbb{1}_{A_{\varepsilon}}\|_{\mathbb{S}^{\varphi}}) < \eta/2$.

Hence, using triangular inequality, we get

$$\begin{aligned}\|f - P_{\varepsilon}\|_{\mathbb{S}^{\varphi}} &\leq \|(f - P_{\varepsilon})\mathbb{1}_{A_{\varepsilon}}\|_{\mathbb{S}^{\varphi}} + \|(f - P_{\varepsilon})\mathbb{1}_{A_{\varepsilon}^c}\|_{\mathbb{S}^{\varphi}} \\ &\leq \|f\mathbb{1}_{A_{\varepsilon}}\|_{\mathbb{S}^{\varphi}} + \|P_{\varepsilon}\mathbb{1}_{A_{\varepsilon}}\|_{\mathbb{S}^{\varphi}} + \varepsilon \\ &\leq \varepsilon + \eta.\end{aligned}$$

This means that $f \in \mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R}, \mathbb{X})$, since η is arbitrary, which proves our claim. □

3.3 The Nemytskii operator in $\mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R}, \mathbb{X})$. Let $F \colon \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ be a measurable function. Recall that the usual Nemytskii operator (or superposition operator), $\mathcal{N}_F$, associated to F is the mapping defined by

$$\mathcal{N}_F(x) := [t \mapsto F(t, x(t))].$$

This operator is well studied in the literature, in spaces of Stepanov almost periodic functions [5], [15] as well as in Orlicz spaces, see e.g. [9]. This section is devoted to the study of this operator in the space of Stepanov–Orlicz almost periodic functions. Inspired from [6], we show that under some conditions on F, $\mathcal{N}_F$ maps $\mathbb{S}^\varphi\mathrm{AP}(\mathbb{R},\mathbb{X})$ into itself.

We begin by introducing the following definition:

Definition 3.12. Let $F\colon \mathbb{R}\times\mathbb{X}\to\mathbb{X}$ be a mapping.

a) We say that a function F is almost periodic in Stepanov–Orlicz sense if the mapping $t\to F(t,x)$ belongs to $\mathbb{S}^\varphi\mathrm{AP}(\mathbb{R},\mathbb{X})$ uniformly on any compact subset K of $\mathbb{X}$, that is for each $x\in\mathbb{X}$, $F(\cdot,x)\in \mathrm{E}^{\varphi}_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$, and for every $\varepsilon>0$ and any compact subset K of $\mathbb{X}$, the set

$$\Big\{\tau\in\mathbb{R}\colon \sup_{x\in \mathrm{K}}\|F(\cdot+\tau,x)-F(.,x)\|_{\mathbb{S}^\varphi}\le\varepsilon\Big\}$$

is relatively dense in $\mathbb{R}$. The collection of such functions is denoted by $\mathbb{S}^\varphi\mathrm{AP}_{\mathrm{K}}(\mathbb{R}\times\mathbb{X},\mathbb{X})$.

b) Fuction F is said to be $\mathbb{S}^\varphi$-bounded if $|\mathcal{N}_F(0)|_{\mathbb{S}^\varphi}\le C$ for some $C>0$.

Note that item b) from Definition 3.12 coincides with the property that $\mathcal{N}_F(0)\in \mathrm{B}\mathbb{S}^\varphi(\mathbb{R},\mathbb{X})$, see [25, Definition 1, page 43].

From now on we make the following basic assumption on the mapping $F\colon \mathbb{R}\times\mathbb{X}\to\mathbb{X}$.

(**Lip**) There exists $L>0$ such that for all $u,v\in\mathbb{X}$

$$\|F(\cdot,u)-F(\cdot,v)\|_\infty\le L\|u-v\|.$$

We are now in position of establishing a superposition theorem for $\mathbb{S}^\varphi$-almost periodic functions.

Theorem 3.13. *Let $F\in\mathbb{S}^\varphi\mathrm{AP}_{\mathrm{K}}(\mathbb{R}\times\mathbb{X},\mathbb{X})$. Assume that F satisfies condition* (**Lip**). *Then, for every $x(\cdot)\in\mathbb{S}^\varphi\mathrm{AP}(\mathbb{R},\mathbb{X})$, we have $\mathcal{N}_F(x(\cdot))\in\mathbb{S}^\varphi\mathrm{AP}(\mathbb{R},\mathbb{X})$.*

Proof: We first examine that $F(\cdot,x(\cdot))\in\mathrm{E}^\varphi_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$. We have $x(\cdot)\in\mathrm{E}^\varphi_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$ and $F(\cdot,0)\in\mathrm{E}^\varphi_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$, then for every bounded set $U\subset\mathbb{R}$, and any $\alpha>0$, we get

$$\begin{aligned}
\varrho_\varphi\Big(\frac{\alpha}{2}\big(F(\cdot,x(\cdot))\big)\Big) &= \int_U \varphi\Big(\frac{\alpha}{2}\|F(t,x(t))\|\Big)\,\mathrm{d}t\\
&\le \frac12\int_U\varphi(\alpha\|F(t,x(t))-F(t,0)\|)\,\mathrm{d}t+\frac12\int_U\varphi(\alpha\|F(t,0)\|)\,\mathrm{d}t\\
&\le \frac12\int_U\varphi(\alpha L\|x(t)\|)\,\mathrm{d}t+\frac12\int_U\varphi(\alpha\|F(t,0)\|)\,\mathrm{d}t\\
&\le \frac12\varrho_\varphi(\alpha Lx(\cdot))+\frac12\varrho_\varphi(\alpha F(\cdot,0))<\infty.
\end{aligned}$$

Now, let K be a compact subset of $\mathbb{X}$. Let $x(\cdot) \in \mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R}, \mathbb{X})$. Fix $\varepsilon > 0$. In view of Theorem 3.11 and Danilov's property (**D**), we can find $\eta := \eta(\varepsilon) > 0$ and a compact subset $\mathbb{K}_{\eta(\varepsilon)} \subset \mathbb{X}$ such that

$$\text{(3.9)} \qquad \overline{\mu}_{\mathbb{S}}\{t \in \mathbb{R} \colon x(t) \notin \mathbb{K}_{\eta(\varepsilon)}\} < \eta \qquad \text{and} \qquad \big\| x(\cdot) \mathbb{1}_{\mathrm{T}^c_{\eta(\varepsilon)}} \big\|_{\mathbb{S}^{\varphi}} \leq \frac{\varepsilon}{12L},$$

where $\mathrm{T}_{\eta(\varepsilon)} = \{t \in \mathbb{R} \colon x(t) \in \mathbb{K}_{\eta(\varepsilon)}\}$.

The compactness of $\mathbb{K}_{\eta(\varepsilon)}$ ensures the existence of a finite sequence $(x_i)_{1 \leq i \leq m}$ in $\mathbb{K}_{\eta(\varepsilon)}$ such that

$$\text{(3.10)} \qquad \mathbb{K}_{\eta(\varepsilon)} \subset \bigcup_{i=1}^{m} B\Big(x_i, \frac{\varepsilon}{12L}\Big).$$

Now, since for every $i = 1, \ldots, m$, $F(\cdot, x_i) \in \mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R}, \mathbb{X})$ and $x(\cdot) \in \mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R}, \mathbb{X})$, we can choose a common ε-$\mathbb{S}^{\varphi}$-almost period associated to $x(\cdot)$ and $(F(\cdot, x_i))_{1 \leq i \leq m}$, see, for instance, [24, page 6, Property 7]. Namely, let τ be an ε-$\mathbb{S}^{\varphi}$-almost period satisfying

$$\text{(3.11)} \qquad \tau \in \bigcap_{i=1}^{m} \mathbb{S}^{\varphi}\mathrm{T}\Big(F(\cdot, x_i), \frac{\varepsilon}{4}\Big) \cap \mathbb{S}^{\varphi}\mathrm{T}\Big(x(\cdot), \frac{\varepsilon}{4L}\Big).$$

Using the Lipschitz condition (**Lip**) and the increasing of φ, we have for every $k > 0$

$$\varrho_{\mathbb{S}^{\varphi}}\Big(\frac{1}{k}\|F(\cdot + \tau, x(\cdot + \tau)) - F(\cdot + \tau, x(\cdot))\|\Big) \leq \varrho_{\mathbb{S}^{\varphi}}\Big(\frac{L}{k}\|x(\cdot + \tau) - x(\cdot)\|\Big).$$

Hence

$$\Big\{k > 0 \colon \varrho_{\mathbb{S}^{\varphi}}\Big(\frac{L}{k}\|x(\cdot + \tau) - x(\cdot)\|\Big) \leq 1\Big\} \subset \Big\{k > 0 \colon \varrho_{\mathbb{S}^{\varphi}}\Big(\frac{1}{k}\|F(\cdot + \tau, x(\cdot + \tau)) - F(\cdot + \tau, x(\cdot))\|\Big) \leq 1\Big\},$$

passing to the infimum, we deduce that

$$\text{(3.12)} \qquad \|F(\cdot + \tau, x_{\tau}(\cdot)) - F(\cdot + \tau, x(\cdot))\|_{\mathbb{S}^{\varphi}} \leq L\|x_{\tau}(\cdot) - x(\cdot)\|_{\mathbb{S}^{\varphi}}.$$

Using again the condition (**Lip**), inequality (3.12) and the triangle inequality, we obtain

$$\begin{aligned}
\|F_\tau(\cdot,x_\tau(\cdot))-F(\cdot,x(\cdot))\|_{\mathbb{S}^\varphi} &\le \|F(\cdot+\tau,x(\cdot+\tau))-F(\cdot+\tau,x(\cdot))\|_{\mathbb{S}^\varphi}\\
&\quad+\|F(\cdot+\tau,x(\cdot))-F(\cdot,x(\cdot))\|_{\mathbb{S}^\varphi}\\
&\le L\|x(\cdot+\tau)-x(\cdot)\|_{\mathbb{S}^\varphi}+\max_{1\le i\le m}\|F(\cdot+\tau,x(\cdot))-F(\cdot+\tau,x_i)\|_{\mathbb{S}^\varphi}\\
&\quad+\max_{1\le i\le m}\|F(\cdot+\tau,x_i)-F(\cdot,x_i)\|_{\mathbb{S}^\varphi}\\
&\quad+\max_{1\le i\le m}\|F(\cdot,x_i)-F(\cdot,x(\cdot))\|_{\mathbb{S}^\varphi}\\
&\le L\|x_\tau(\cdot)-x(\cdot)\|_{\mathbb{S}^\varphi}+2L\max_{1\le i\le m}\|x(\cdot)-x_i\|_{\mathbb{S}^\varphi}\\
&\quad+\max_{1\le i\le m}\|\{F(\cdot+\tau,x_i)-F(\cdot,x_i)\}\|_{\mathbb{S}^\varphi}.
\end{aligned}\tag{3.13}$$

Let us estimate the term $\max_{1\le i\le m}\|x(\cdot)-x_i\|_{\mathbb{S}^\varphi}$. Using (3.9) and the fact that the constant function $t\to x_i$ (for each i) is in $\mathbb{M}^\varphi(\mathbb{R},\mathbb{X})$, we can choose $\eta(\varepsilon)$ small enough such that

$$\max_{1\le i\le m}\{\|x_i\mathbb{1}_{T_{\eta(\varepsilon)}}\|_{\mathbb{S}^\varphi}+\|x(\cdot)\mathbb{1}_{T_{\eta(\varepsilon)}}\|_{\mathbb{S}^\varphi}\}\le\frac{\varepsilon}{6L}.$$

Hence, from the above inequality and (3.10), we deduce that for every $i=1,\dots,m$

$$\begin{aligned}
\|x(\cdot)-x_i\|_{\mathbb{S}^\varphi} &\le \|\{x(\cdot)-x_i\}\mathbb{1}_{T_{\eta(\varepsilon)}}\|_{\mathbb{S}^\varphi}+\|\{x(\cdot)-x_i\}\mathbb{1}_{T^c_{\eta(\varepsilon)}}\|_{\mathbb{S}^\varphi}\\
&\le\frac{\varepsilon}{12L}+\|x(\cdot)\mathbb{1}_{T^c_{\eta(\varepsilon)}}\|_{\mathbb{S}^\varphi}+\|x_i\mathbb{1}_{T^c_{\eta(\varepsilon)}}\|_{\mathbb{S}^\varphi}\le\frac{\varepsilon}{4L}.
\end{aligned}\tag{3.14}$$

Combining (3.11), (3.13), and (3.14) we obtain

$$\|F_\tau(\cdot,x_\tau(\cdot))-F(\cdot,x(\cdot))\|_{\mathbb{S}^\varphi}\le L\frac{\varepsilon}{4L}+2L\frac{\varepsilon}{4L}+\frac{\varepsilon}{4}=\varepsilon.$$

Finally, the $\mathbb{S}^\varphi$-almost periodicity of $\mathcal{N}_F(x(\cdot))$ is obtained using the inclusion

$$\bigcap_{i=1}^{m}\mathbb{S}^\varphi\mathrm{T}\Big(F(\cdot,x_i),\frac{\varepsilon}{4}\Big)\cap\mathbb{S}^\varphi\mathrm{T}\Big(x(\cdot),\frac{\varepsilon}{4L}\Big)\subset\mathbb{S}^\varphi\mathrm{T}(\mathcal{N}_F(x(\cdot)),\varepsilon).$$

□

3.4 Bohr almost periodic mild solutions to abstract evolution equations with $\mathbb{S}^\varphi$AP-coefficients. This section is devoted to the existence and uniqueness of Bohr almost periodic mild solution to the following abstract linear and semilinear differential equations

$$u'(t)=Au(t)+f(t),\tag{3.15}$$

and

$$u'(t) = Au(t) + F(t, u(t)), \tag{3.16}$$

respectively, where $A\colon \mathrm{Dom}(A) \subset \mathbb{X} \to \mathbb{X}$ is a densely defined closed (possibly unbounded) linear operator, and $f\colon \mathbb{R} \to \mathbb{X}$, $F\colon \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ are measurable functions (their continuity is not required here). We first list the following basic assumptions:

(H1) The operator A generates an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$, that is, there exist constants $M > 0$, $\omega > 0$ such that

$$\|T(t)\| \leq M \exp(-\omega t), \qquad \forall\, t \geq 0.$$

(H2) $f \in \mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R}, \mathbb{X})$.

(H3) $F \in \mathbb{S}^{\varphi}\mathrm{AP}_{\mathrm{K}}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$.

Such problem is well investigated in the literature, see [32], [15], [28] when the forcing term f is continuous and Stepanov almost periodic. Our objective here is to extend these results to the Stepanov–Orlicz context, without imposing the continuity assumption on f.

We need to recall the definition of mild solution.

Definition 3.14. A continuous function $u\colon \mathbb{R} \mapsto \mathbb{X}$ given by

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\sigma) f(\sigma, u(\sigma))\,\mathrm{d}\sigma, \qquad t \geq s,\ s \in \mathbb{R}. \tag{3.17}$$

is a mild solution to the semilinear differential equation (3.16).

The first result about existence and uniqueness of almost periodic solution to (3.15) is reported in the following theorem:

Theorem 3.15. *Under* (H1) *and* (H2), *equation* (3.15) *has a unique Bohr almost periodic mild solution given by*

$$u(t) = \int_{-\infty}^t T(t-s) f(s)\,\mathrm{d}s, \qquad t \in \mathbb{R}. \tag{3.18}$$

Proof: Consider for each $n \geq 1$ the function $u_n\colon \mathbb{R} \to \mathbb{X}$ defined by the integral:

$$u_n(t) = \int_{t-n}^{t-n+1} T(t-s) f(s)\,\mathrm{d}s = \int_{n-1}^{n} T(s) f(t-s)\,\mathrm{d}s, \qquad t \in \mathbb{R}.$$

Let us show that for each $n \geq 1$, u_n belongs to $\mathrm{C}(\mathbb{R}, \mathbb{X})$. For, fix $t_0 \in \mathbb{R}$. Let $(t_m)_m$ be a sequence converging to t_0. Using (H1), Hölder inequality in Orlicz-space, see e.g. [23], and the fact that $\|\mathbb{1}_{\mathbb{R}}\|_{\mathrm{L}^{\psi}[t_0-n, t_0-n+1]} = (\psi^{-1}(1))^{-1}$, we get for each $n \geq 1$ and every $m \geq 1$

$$
\begin{aligned}
\|u_n(t_m) - u_n(t_0)\| &\leq \int_{n-1}^{n} \|T(s)\| \, \|f(t_m - s) - f(t_0 - s)\| \, \mathrm{d}s \\
&\leq M \int_{n-1}^{n} \exp(-\omega s) \|f(t_m - s) - f(t_0 - s)\| \, \mathrm{d}s \\
&\leq M \|1\|_{\mathrm{L}^{\psi}[t_0 - n, t_0 - n + 1]} \|f_{(t_m - t_0)} - f\|_{\mathrm{L}^{\varphi}[t_0 - n, t_0 - n + 1]} \\
&\leq \frac{M}{\psi^{-1}(1)} \|f_{(t_m - t_0)} - f\|_{\mathrm{L}^{\varphi}[t_0 - n, t_0 - n + 1]}.
\end{aligned}
\tag{3.19}
$$

In view of (3.2), functions in $\mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R}, \mathbb{X})$ are φ-mean continuous. Combining this fact with (3.19) we obtain that

$$
\lim_{m \to \infty} \|u_n(t_m) - u_n(t_0)\| = 0.
$$

Since $t_0 \in \mathbb{R}$ is arbitrary, we deduce that $u_n \in \mathrm{C}(\mathbb{R}, \mathbb{X})$.

Next, let us verify that for each n, $u_n(\cdot) \in \mathrm{AP}(\mathbb{R}, \mathbb{X})$. Similarly as in (3.19), we have for every $\tau \in \mathbb{R}$

$$
\|u_n(\cdot + \tau) - u_n(\cdot)\|_{\infty} \leq \frac{M}{\psi^{-1}(1)} \|f_{\tau} - f\|_{\mathbb{S}^{\varphi}},
$$

which means that for every $\varepsilon > 0$,

$$
\mathbb{S}^{\varphi}\mathrm{T}\Big(f, \frac{\varepsilon \psi^{-1}(1)}{M}\Big) \subset \mathrm{T}(u_n, \varepsilon).
$$

The almost periodicity of u_n is then followed.

Finally, let us show that the series $\sum_{n \geq 1} u_n(t)$ is uniformly convergent on $\mathbb{R}$. Using (H1) and (2.4), it follows that for any $t \in \mathbb{R}$,

$$
\|u_n(t)\| \leq M \beta_{\varphi} \mathrm{e}^{-\omega(n-1)} \|f\|_{\mathbb{S}^{\varphi}}. \tag{3.20}
$$

According to the Weierstrass test and the previous estimation, we get that the series $\sum_{n \geq 1} u_n(t)$ is uniformly convergent on $\mathbb{R}$. For each $t \in \mathbb{R}$ let $u(t)$ be its sum. Then, u is almost periodic. Moreover

$$
\|u\|_{\infty} \leq \frac{M \beta_{\varphi}}{1 - \mathrm{e}^{-\omega}} \|f\|_{\mathbb{S}^{\varphi}}.
$$

The uniqueness is obtained using similar arguments as in Stepanov almost periodic case, see [15]. □

Next, let us investigate the solution to the semilinear differential equation (3.16). Using Theorem 3.15 one easily proves the following theorem

Theorem 3.16. *Under* (H1), (**Lip**), *and* $\mathbb{S}^\varphi$*-boundedness of* F, (3.16) *has a unique bounded mild solution given by*

$$u(t) = \int_{-\infty}^{t} T(t-s)F(s,u(s))\,\mathrm{d}s \qquad \text{for each } t \in \mathbb{R},$$

provided that $LM/(1-\mathrm{e}^{-\omega}) < 1$. *If in addition* (H3) *is satisfied, then this solution is almost periodic.*

Proof: For each $t \in \mathbb{R}$ and $n \geq 1$, we consider the nonlinear operators defined by

$$\Gamma(u)(t) = \int_{-\infty}^{t} T(t-s)F(s,u(s))\,\mathrm{d}s,$$
$$\Gamma_n(u)(t) = \int_{t-n}^{t-n+1} T(t-s)F(s,u(s))\,\mathrm{d}s = \int_{n}^{n-1} T(s)F(t-s,u(t-s))\,\mathrm{d}s.$$

Let us show that for each $n \geq 1$, the operator Γ_n maps $\mathrm{E}^{\varphi}_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$ into $\mathrm{C}(\mathbb{R},\mathbb{X})$. Let $u \in \mathrm{E}^{\varphi}_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$. To simplify notations, let $g := \mathcal{N}_F(u)$.

In view of Theorem 3.13, $g \in \mathrm{E}^{\varphi}_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$. Repeating the same reasoning as in the proof of Theorem 3.15, we get easily that for each $n \geq 1$, $\Gamma_n(u) \in \mathrm{C}(\mathbb{R},\mathbb{X})$.

Let us show that the operator Γ maps $\mathrm{E}^{\varphi}_{\mathrm{loc}}(\mathbb{R},\mathbb{X}) \cap \mathrm{B}\mathbb{S}^{\varphi}(\mathbb{R},\mathbb{X})$ into $\mathrm{BC}(\mathbb{R},\mathbb{X})$. We begin by showing that for each $n \geq 1$,

$$\Gamma_n(\mathrm{B}\mathbb{S}^{\varphi}(\mathbb{R},\mathbb{X})) \subset \mathrm{L}^{\infty}(\mathbb{R},\mathbb{X}).$$

To this end, let $u \in \mathrm{B}\mathbb{S}^{\varphi}(\mathbb{R},\mathbb{X})$. Using assumptions (H1), (H3), (**Lip**), and inequality (2.4), we get for each $n \geq 1$ and any $t \in \mathbb{R}$

$$\begin{aligned}
\|\Gamma_n u(t)\| &\leq M\mathrm{e}^{-\omega(n-1)} \int_{t-n}^{t-n+1} (\|F(s,u(s)) - F(s,0)\| + \|F(s,0)\|)\,\mathrm{d}s \\
&\leq M\mathrm{e}^{-\omega(n-1)} \int_{t-n}^{t-n+1} (L\|u(s)\| + \|F(s,0)\|)\,\mathrm{d}s \\
&\leq M\mathrm{e}^{-\omega(n-1)} (L\|u\|_{\mathbb{S}^1} + \|F(\cdot,0)\|_{\mathbb{S}^1}) \\
&\leq M\mathrm{e}^{-\omega(n-1)} (L\alpha_{u,\varphi}\|u\|_{\mathbb{S}^\varphi} + \beta_{F,\varphi}\|F(\cdot,0)\|_{\mathbb{S}^\varphi}),
\end{aligned} \tag{3.21}$$

where the constants $\alpha_{u,\varphi} > 0$ and $\beta_{F,\varphi} > 0$ come from (2.4). According to the $\mathbb{S}^\varphi$-boundedness of both F and (3.21), it follows that the series $\sum_{n\geq 1} \Gamma_n u$ is uniformly convergent on $\mathbb{R}$. Clearly, $\Gamma(u) = \sum_{n\geq 1} \Gamma_n(u) \in \mathrm{C}(\mathbb{R},\mathbb{X})$ for every $u \in \mathrm{E}^{\varphi}_{\mathrm{loc}}(\mathbb{R},\mathbb{X})$. Moreover,

$$\|\Gamma(u)\|_\infty \leq \frac{M}{1-\mathrm{e}^{-\omega}} (L\alpha_{u,\varphi}\|u\|_{\mathbb{S}^\varphi} + \beta_{F,\varphi}\|F(\cdot,0)\|_{\mathbb{S}^\varphi}).$$

Therefore,

$$\Gamma(\mathrm{E}^{\varphi}_{\mathrm{loc}}(\mathbb{R},\mathbb{X}) \cap \mathrm{B}\mathbb{S}^{\varphi}(\mathbb{R},\mathbb{X})) \subset \mathrm{BC}(\mathbb{R},\mathbb{X}).$$

Now, let us prove that if in addition $F \in \mathbb{S}^{\varphi}\mathrm{AP}_{\mathrm{K}}(\mathbb{R}\times\mathbb{X},\mathbb{X})$, the operator Γ maps $\mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R},\mathbb{X})$ into $\mathrm{AP}(\mathbb{R},\mathbb{X})$. For, let $u \in \mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R},\mathbb{X})$. By Theorem 3.13, we infer that $g \in \mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R},\mathbb{X})$. Hence, using the proof of Theorem 3.15, we have that $\Gamma u \in \mathrm{AP}(\mathbb{R},\mathbb{X})$. Thus

$$\Gamma(\mathbb{S}^{\varphi}\mathrm{AP}(\mathbb{R},\mathbb{X})) \subset \mathrm{AP}(\mathbb{R},\mathbb{X}).$$

To complete the proof, we need to prove that Γ is a contraction mapping on $\mathrm{AP}(\mathbb{R},\mathbb{X})$. It suffices to apply the Banach fixed-point theorem to the nonlinear operator Γ. The calculations are similar to the existing ones in the literature, see for instance [15], [32]. We repeat them for convenience of the reader. Under assumptions (H1) and (**Lip**), we have

$$\|\Gamma u - \Gamma v\|_{\infty} \leq \frac{LM}{1-\mathrm{e}^{-\omega}}\|u-v\|,$$

which implies that Γ is a contraction mapping on $\mathrm{AP}(\mathbb{R},\mathbb{X})$ provided that $LM/(1-\mathrm{e}^{-\omega}) < 1$. So by the Banach contraction principle, there exists a unique fixed point $u \in \mathrm{AP}(\mathbb{R},\mathbb{X})$, such that $\Gamma u = u$. Moreover, using the reasoning as in [15], we can see that u is a mild solution to equation (3.16). □

References

[1] Albrycht J., *The theory of Marcinkiewic–Orlicz spaces*, Rozprawy Mat. **27** (1962), 56 pages.

[2] Amerio L., Prouse G., *Almost-Periodic Functions and Functional Equations*, Van Nostrand Reinhold, New York, Ont.-Melbourne, 1971.

[3] Andres J., Bersani A. M., Grande R. F., *Hierarchy of almost-periodic function spaces*, Rend. Mat. Appl. (7) **26** (2006), no. 2, 121–188.

[4] Andres J., Pennequin D., *On Stepanov almost-periodic oscillations and their discretizations*, J. Difference Equ. Appl. **18** (2012), no. 10, 1665–1682.

[5] Andres J., Pennequin D., *On the nonexistence of purely Stepanov almost-periodic solutions of ordinary differential equations*, Proc. Amer. Math. Soc. **140** (2012), no. 8, 2825–2834.

[6] Bedouhene F., Challali N., Mellah O., Raynaud de Fitte P., Smaali M., *Almost periodic solution in distribution for stochastic differential equations with Stepanov almost periodic coefficients*, available at arXiv: 1703.00282v3 [math.PR] (2017), 42 pages.

[7] Bugajewski D., Nawrocki A., *Some remarks on almost periodic functions in view of the Lebesgue measure with applications to linear differential equations*, Ann. Acad. Sci. Fenn., Math. **42** (2017), no. 2, 809–836.

[8] Chen S., *Geometry of Orlicz Spaces*, Dissertationes Math. (Rozprawy Mat.), 356, 1996.

[9] Cichoń M., Metwali M. M. A., *On quadratic integral equations in Orlicz spaces*, J. Math. Anal. Appl. **387** (2012), no. 1, 419–432.

[10] Corduneanu C., *Almost Periodic Functions*, Interscience Tracts in Pure and Applied Mathematics, 22, Interscience Publishers, John Wiley, New York, 1968.

[11] Dads A. E. H., Es-Sebbar B., Ezzinbi K., Ziat M., *Behavior of bounded solutions for some almost periodic neutral partial functional differential equations*, Math. Methods Appl. Sci. **40** (2017), no. 7, 2377–2397.

[12] Danilov L. I., *On the uniform approximation of a function that is almost periodic in the sense of Stepanov*, Izv. Vyssh. Uchebn. Zaved. Mat (1998), no. 5, 10–18.

[13] Diagana T., *Stepanov-like pseudo-almost periodicity and its applications to some nonautonomous differential equations*, Nonlinear Anal. **69** (2008), no. 12, 4277–4285.

[14] Diagana T., Zitane M., *Stepanov-like pseudo-almost automorphic functions in Lebesgue spaces with variable exponents $L^{p(x)}$*, Electron. J. Differential Equations **2013** (2013), No. 188, 20 pages.

[15] Ding H.-S., Long W., N'Guérékata G. M., *Almost periodic solutions to abstract semilinear evolution equations with Stepanov almost periodic coefficients*, J. Comput. Anal. Appl. **13** (2011), no. 2, 231–242.

[16] Hillmann T. R., *Besicovitch–Orlicz spaces of almost periodic functions*, Real and stochastic analysis, Wiley Ser. Probab. Math. Statist. Probab. Math. Statist., Wiley, 1986, 119–167.

[17] Hu Z., *Boundedness and Stepanov's almost periodicity of solutions*, Electron. J. Differential. Equations **2005** (2005), no. 35, 7 pages.

[18] Hu Z., Mingarelli A. B., *Bochner's theorem and Stepanov almost periodic functions*, Ann. Mat. Pura Appl. (4) **187** (2008), no. 4, 719–736.

[19] Hudzik H., *Uniform convexity of Musielak–Orlicz spaces with Luxemburg's norm*, Comment. Math. Prace Mat. **23** (1983), no. 1, 21–32.

[20] Kasprzak P., Nawrocki A., Signerska-Rynkowska J., *Integrate-and-fire models with an almost periodic input function*, J. Differential Equations **264** (2018), no. 4, 2495–2537.

[21] Kourat H., *Caractérisation de quelques propriétés géométriques locales dans les espaces de type Musielak–Orlicz*, PhD. Thesis, Mouloud Mammeri University of Tizi–Ouzou, Tizi–Ouzou, 2016 (French).

[22] Kozlowski W. M., *Modular Function Spaces*, Monographs and Textbooks in Pure and Applied Mathematics, 122, Marcel Dekker, New York, 1988.

[23] Kufner A., John O., Fučík S., *Function Spaces*, Monographs and Textsbooks on Mechanics of Solids and Fluids, Mechanics: Analysis, Noordhoff International Publishing, Leyden, Publishing House of the Czechoslovak Academy of Sciences, Prague, 1977.

[24] Levitan B. M., Zhikov V. V., *Almost Periodic Functions and Differential Equations*, Cambridge University Press, Cambridge, 1982.

[25] Luxemburg W. A. J., *Banach Function Spaces*, PhD. Dissertation, Delft University of Technology, Delft, 1955.

[26] Musielak J., *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, 1034, Springer, Berlin, 1983.

[27] Pankov A. A., *Bounded and Almost Periodic Solutions of Nonlinear Operator Differential Equations*, Mathematics and Its Applications (Soviet Series), 55, Kluwer Academic Publishers Group, Dordrecht, 1990.

[28] Radová L., *Theorems of Bohr–Neugebauer-type for almost-periodic differential equations*, Math. Slovaca **54** (2004), no. 2, 191–207.

[29] Rao A. S., *On the Stepanov-almost periodic solution of a second-order operator differential equation*, Proc. Edinburgh Math. Soc. (2) **19** (1974/75), 261–263.

[30] Stepanoff W., *Über einige Verallgemeinerungen der fast periodischen Funktionen*, Math. Ann. **95** (1926), no. 1, 473–498 (German).

[31] Stoiński S., *Almost periodic functions in the Lebesgue measure*, Comment. Math. (Prace Mat.) **34** (1994), 189–198.

[32] Zaidman S., *An existence result for Stepanoff almost-periodic differential equations*, Canad. Math. Bull. **14** (1971), 551–554.

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