Comment.Math.Univ.Carolin. 61,3 (2020) 277–312

Approximating solutions of split equality of some nonlinear optimization problems using an inertial algorithm

Lateef O. Jolaoso, Oluwatosin T. Mewomo

Abstract. This paper presents an inertial iterative algorithm for approximating a common solution of split equalities of generalized mixed equilibrium problem, monotone variational inclusion problem, variational inequality problem and common fixed point problem in real Hilbert spaces. The algorithm is designed in such a way that it does not require prior knowledge of the norms of the bounded linear operators. We prove a strong convergence theorem under some mild conditions of the control sequences and also give a numerical example to show the efficiency and accuracy of our algorithm. We see that the inertial algorithm performs better in terms of number of iteration and CPU-time than the non-inertial algorithm. This result improves and generalizes many recent results in the literature.

Keywords: split equality; generalized equilibrium problem; variational inclusion problem; variational inequality; quasi-nonexpansive mapping; fixed point problem

Classification: 47H06, 47H09, 47J05, 47J25

1. Introduction

Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively, and let $A\colon H_1 \to H_3$ and $B\colon H_2 \to H_3$ be bounded linear operators. The split equality problem (SEP) according to A. Moudafi, see [40], is defined as:

$$\text{Find } \ x \in C \ \text{ and } \ y \in Q \ \text{ such that } \ Ax = By. \tag{1.1}$$

The SEP allows asymmetric and partial relation between the variables x and y. In fact, many problems arising in mathematics and other sciences can be regarded as SEP. This include decomposition in PDE's, optimal control in decision making and image modulated radiation therapy (IMRT), see [1], [2], [12], [13], [45], [46]. One of the common methods used for solving the SEP (1.1) is the following

DOI 10.14712/1213-7243.2020.029

projection method by C. L. Bryne and A. Moudafi in [11]:

$$x_{n+1} = P_C(x_n - \gamma_n A^*(Ax_n - By_n)),$$
$$y_{n+1} = P_Q(y_n + \gamma_n B^*(Ax_n - By_n)), \tag{1.2}$$

where $\gamma_n \in (\varepsilon, 2/(\lambda_A + \lambda_B) - \varepsilon)$, λ_A and λ_B are the operator (matrix) norms $\|A\|$ and $\|B\|$ (or the largest eigenvalues of A^*A and B^*B), respectively. To determine stepsize γ_n, one will need to first calculate (or at least) estimate the operator norms $\|A\|$ and $\|B\|$. In general it is difficult or even impossible to determine the norms $\|A\|$ and $\|B\|$.

When C and Q are the sets of fixed points (which are closed and convex) of some nonlinear operators, then (1.1) becomes the split equality common fixed point problem (SECFP) which is defined as:

$$\text{find } \ x \in F(S) \ \text{ and } \ y \in F(T) \ \text{ such that } \ Ax = By, \tag{1.3}$$

where $S\colon H_1 \to H_1$ and $T\colon H_2 \to H_2$ are nonlinear mappings and $F(S)$ and $F(T)$ are the sets of fixed points of S and T, respectively, (i.e., $F(S) := \{x \in C\colon Sx = x\}$ and $F(T) := \{y \in Q\colon Ty = y\}$). The SECFP was first studied by A. Moudafi in [40] and he introduced the following algorithm for finding its solutions:

$$x_{n+1} = S(x_n - \gamma_n A^*(Ax_n - By_n)),$$
$$y_{n+1} = T(y_n + \gamma_n B^*(Ax_n - By_n)), \tag{1.4}$$

where S and T are firmly quasi-nonexpansive mappings and $\gamma_n \in (\varepsilon, 2/(\lambda_A + \lambda_B) - \varepsilon)$. A lot of researchers have further studied the SECFP and several modifications of (1.4) were introduced, see for instance [21], [22], [37], [41], [54], [55].

G. López et al. in [31] and J. Zhao and Q. Yang in [56] presented helpful methods for estimating the step-sizes which does not require a prior knowledge of the operator norms for solving the SEP. In this direction, J. Zhao in [53] studied the SEP and presented the following step-size selection which guarantees convergence of the iterative scheme without a prior information about the operator norms of A and B,

$$\gamma_n \in \Big(0, \frac{2\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2}\Big).$$

Definition 1.1. A nonlinear mapping $T\colon H \to H$ is said to be

(a) L-Lipschitz continuous mapping if there is a constant $L > 0$ such that

$$\|Tx - Ty\| \le L\|x - y\|, \qquad \forall\, x, y \in H. \tag{1.5}$$

If $L \in [0,1)$, then T is called a contraction and if $L = 1$, T is called a nonexpansive mapping;

(b) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|, \qquad \forall\, x \in H \ \text{ and } \ p \in F(T);$$

(c) firmly quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - p\|^2 \leq \|x - p\|^2 - \|x - Tx\|^2, \qquad \forall\, x \in H \ \text{ and } \ p \in F(T);$$

(d) monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \qquad \forall\, x, y \in H;$$

(e) α-inverse strongly monotone (shortly, α-ism), if there is a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2, \qquad \forall\, x, y \in H.$$

Clearly every nonexpansive mapping is quasi-nonexpansive if $F(T) \neq \emptyset$. Also, the class of α-ism mapping is $\frac{1}{\alpha}$-Lipschitz continuous.

Definition 1.2. An operator $T\colon H \to H$ is said to be

(i) convex if there is $\alpha \in (0,1)$ such that

$$T(\alpha x + (1-\alpha)y) \leq \alpha T(x) + (1-\alpha)T(y), \qquad \forall\, x, y \in H;$$

(ii) proper if the effective domain of T, $\operatorname{dom} T := \{x \in H\colon T(x) \in H\}$ is nonempty;

(iii) weakly lower semicontinuous if for every subsequence $\{x_{n_k}\} \subset H$ and every point x in H, we have $x_{n_k} \rightharpoonup x$ implies $T(x) \leq \liminf_{k\to\infty} T(x_{n_k})$.

A set-valued mapping $T\colon H \to 2^H$ is called monotone if for all $x, y \in H$, with $u \in T(x)$ and $v \in T(y)$ then

$$\langle x - y, u - v\rangle \geq 0,$$

and T is maximal monotone if the graph of T denoted by $G(T)$ which is define by $G(T) := \{(x,y)\colon y \in T(x)\}$ is not properly contain in the graph of any other monotone mapping. It is also known that T is maximal if and only if for $(x,u) \in H \times H$, $\langle x - y, u - v\rangle \geq 0$ for all $(y,v) \in G(T)$ implies $u \in T(x)$.

1.1 Generalized mixed equilibrium problem. Let C be a nonempty, closed and convex subset of H, $F\colon C \times C \to \mathbb{R}$ be a nonlinear bifunction, $\phi\colon C \to H$ be a nonlinear mapping and $U\colon C \to \mathbb{R} \cup \{\infty\}$ be a proper convex lower semicontinuous function. The generalized mixed equilibrium problem (GMEP) is defined as finding a point $x \in C$ such that

$$F(x,y) + \langle \phi x, y - x \rangle + U(y) - U(x) \geq 0, \qquad \forall\, y \in C. \tag{1.6}$$

The set of solutions of (1.6) is denoted y GMEP(F, U, ϕ) which is a closed and convex set, see [29].

If $\phi = 0$, Problem (1.6) reduces to the mixed equilibrium problem (MEP) which is to find a point $x \in C$ such that

$$F(x,y) + U(y) - U(x) \geq 0, \qquad \forall\, y \in C.$$

In particular, if $U = 0$ in (1.1), the MEP reduces to the classical equilibrium problem which was introduced by E. Blum and W. Oettli in [6] and defined s finding a point $x \in C$ such that

$$F(x,y) \geq 0, \qquad \forall\, y \in C.$$

The GMEP is very general in the sense that it includes as special cases, optimization problem, variational inequality problem, fixed point problem, Nash equilibrium problem in noncooperative games and many others.

When C and Q in (1.1) are the solution sets of GMEP, then (1.1) becomes the split equality generalized mixed equilibrium problem (SEGMEP). Recently some authors have studying the SEGMEP and introduced some iterative schemes for approximating solutions of SEGMEP in real Hilbert spaces. See for instance [14], [20], [23], [24], [25], [43], [48], [49].

1.2 Variational inequality problems. Let C be a nonempty, closed and convex subset of a Hilbert space H and $T\colon C \to H$ be a nonlinear mapping. The variational inequality problem (VIP) is define as finding a point $x \in C$ such that

$$\langle Tx, y - x \rangle, \qquad \forall\, y \in C. \tag{1.7}$$

We denote the set of solutions of VIP by VP(T, C). It is well known that a point x is a solution of the VIP (1.7) if and only if it is a solution of the fixed point equation

$$P_C(x - \lambda Tx) = x,$$

where $\lambda > 0$ and P_C is the metric projection onto C. Several iterative methods have been introduced in the literature for finding the solutions of VIP (1.7). If the sets C and Q in (1.1) are the solution sets of VIP, then (1.1) becomes a split equality variational inequality problem (SEVIP). Recently, some authors have also studying the SEVIP and several iterative methods were proposed for approximating its solutions. See for instance [27].

1.3 Monotone variational inclusion problems. Let $f\colon H \to H$ be α-inverse strongly monotone operator and $M\colon H \to 2^H$ be a maximal monotone operator. The monotone variational inclusion problem is defined as finding a point $x \in H$ such that

$$0 \in (f + M)x. \tag{1.8}$$

The set of solutions of the MVIP (1.8) is denoted by $\mathrm{VI}(f, M)$. The MVIP was first introduced by R. T. Rockafellar in [44] and has been studied extensively in recent days, see for instance [30]. The MVIP has successfully been applied to study some concrete problems in machine learning, image processing and linear inverse problem. The resolvent operator J_λ^M associated with M and λ is the mapping $J_\lambda^M\colon H \to H$ defined by

$$J_\lambda^M(x) = (I + \lambda M)^{-1}(x), \qquad x \in H,\ \lambda > 0. \tag{1.9}$$

It is well-known that the resolvent operator J_λ^M is single-valued, nonexpansive and 1-inverse strongly monotone and that a solution of problem (1.8) is a fixed point of $J_\lambda^M(I - \lambda f)$, i.e.,

$$0 \in f(x) + M(x) \iff x = J_\lambda^M(I - \lambda f)(x).$$

If f is α-ism with $0 < \lambda < 2\alpha$, then $J_\lambda^M(I - \lambda f)$ is nonexpansive and $\mathrm{VI}(f, M)$ is closed and convex.

Taking C and Q in (1.1) to be the solution sets of MVIP, the SEP becomes a split equality monotone variational inclusion problem (SEMVIP). The SEMVIP was first studied by A. Moudafi in [39] and C. L. Byrne in [11] in 2011 and have been studied recently by many other authors, see for example [19], [26] and references there in.

In 2018, R. Shukla and R. Pant in [47] proposed the following algorithm for finding a common solution of SEGMEP and SECFP:

$$\begin{aligned}
&F(u_n, w) + \langle \phi(w_n), w - u_n\rangle + U_1(w) - U_1(w_n)\\
&\qquad + \frac{1}{r_n}\langle w - u_n, u_n - w_n\rangle \geq 0, \qquad \forall\, w \in C,\\
&G(v_n, z) + \langle \varphi(z_n), z - v_n\rangle + U_2(z) - U_2(z_n)\\
&\qquad + \frac{1}{r_n}\langle z - v_n, v_n - z_n\rangle \geq 0, \qquad \forall\, z \in Q,\\
&z_n = u_n - \gamma_n A^*(Au_n - Bv_n),\\
&w_n = v_n + \gamma_n B^*(Au_n - Bv_n),
\end{aligned} \tag{1.10}$$

$$x_{n+1} = \alpha_n f_1(x_n) + (1-\alpha_n)S_\beta(z_n),$$
$$y_{n+1} = \alpha_n f_2(y_n) + (1-\alpha_n)T_\beta(w_n),$$

where $S_\beta = \beta I + (1-\beta)S$ and $T_\beta = \beta I + (1-\beta)T$, $S\colon H_1 \to H_1$ and $T\colon H_2 \to H_2$ are quasi-nonexpansive mappings and $f_i\colon H_i \to H_i$ for $i = 1, 2$, is a contraction mapping. Under mild conditions, the authors proved a strong convergence of algorithm 1.10 to a common solution $x \in \Omega$, where $\Omega := \{(x, y) \in H_1 \times H_2\colon x \in \mathrm{GMEP}(F, U_1\phi) \cap F(S),\ y \in \mathrm{GMEP}(G, U_2, \varphi) \cap F(T)\ \text{ and }\ Ax = By\}$.

On the other hand, let us mention the inertial type algorithm which is based upon a discrete version of a second-order dissipative dynamical system, see [3], [4], and can be regarded as a procedure for accelerating the convergence properties of an algorithm. A lot of researchers have constructed fast iterative algorithms by using inertial extrapolation techniques, including inertial forward-backward splitting methods, see [5], [42], inertial Douglas–Rachford splitting method, see [9], inertial ADMM, see [15], inertial forward-backward-forward algorithm, see [10], [8], inertial proximal-extragradient method, see [7], inertial split equilibrium method, see [24] and inertial Mann method, see [50].

W. Cholomjiak et al. in [16] also introduced an inertial iterative algorithm for solving the fixed point problem of a quasi-nonexpansive mapping and MVIP (1.8). They proved the following weak convergence theorem.

Theorem 1.3. *Let H be a real Hilbert space and $T\colon H \to CB(H)$ be a quasi-nonexpansive mapping. Let $f\colon H \to H$ be an α-inverse strongly monotone operator and $M\colon H \to 2^H$ a maximal monotone operator. Assume that $\mathrm{VI}(f, M) \cap F(T) \neq \emptyset$ and $I - T$ is demiclosed at 0. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by $x_0,\ x_1 \in H$ and*

$$z_n = x_n + \theta_n(x_n - x_{n-1}),$$
$$z_n = \alpha_n y_n + (1-\alpha_n)Ty_n,$$
$$x_{n+1} = \beta_n z_n + (1-\beta_n)J^M_{r_n}(I - r_n f)z_n, \qquad n \geq 1,$$

where $J^M_{r_n} = (I + r_n f)^{-1}$, $\{r_n\} \subset (0, 2\alpha)$, $\{\theta_n\} \subset [0, \theta]$ for some $\theta \in [0, 1)$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$. Assume that the following conditions hold:

(i) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$;
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(iii) $\limsup_{n\to\infty} \beta_n < 1$;
(iv) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\alpha$.

Then, the sequence $\{x_n\}$ converges weakly to $q \in S$.

Motivated by the works of R. Shukla and R. Pant, see [47], and W. Cholamjiak et al., see [16], in this paper, we introduce an inertial algorithm for approximating a common solution of SEGMEP, SEMVIP, SEVIP and SECFP in real Hilbert spaces. The algorithm is designed in such a way that the stepsize is chosen without prior knowledge of the operator norms $\|A\|$ and $\|B\|$. We also prove a strong convergence result and further provide a numerical test to show the relevance of our algorithm. This result improve and generalize many recent results in literature.

Our contribution in this paper is in two folds:

- We consider a more general problem than the problems considered by R. Shukla and R. Pant in [47], Z. Ma et al. [32], M. Rahman et al. [43], A. Moudafi [39], L. O. Jolaoso et al. [24] and K. R. Kazmi and S. H. Rizvi [26].
- The inertial algorithm technique used in this paper is new and it guarantees strong convergence which is more desirable than the weak convergence obtained by W. Cholamjiak et al. in [16], C.-S. Chuang [17], Q.-L. Dong et al. [18] and D. V. Thong and D. V. Hieu [50], [51].

2. Preliminaries

In this section, we recall some preliminary results which will be needed in the sequel. We denote the strong convergence and weak convergence of a sequence $\{x_n\}$ to a point $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Lemma 2.1 ([35], [52]). *Let H be a real Hilbert space. Then the following result holds $\forall\, x, y \in H$:*

(i) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$;
(ii) $\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2,\ \ t \in (0,1)$;
(iii) $\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$.

Lemma 2.2 (Demiclosedness principle in [38]). *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $S\colon C \to C$ be a nonlinear mapping with $F(S) \neq \emptyset$, then $I - S$ is said to be demiclosed at 0 if $x_n \rightharpoonup x^* \in C$ and $x_n - Sx_n \to 0$, then $x^* = Sx^*$.*

For solving the equilibrium problem, we assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfy the following:

(L1) $F(x,x) = 0, \quad \forall\, x \in C$;
(L2) bifunction F is monotone, i.e. $F(x,y) + F(y,x) \le 0, \quad \forall\, x, y \in C$;
(L3) for each $x, y \in C$, $\lim_{t\to 0} F(ty + (1-t)x, y) \le F(x,y)$;
(L4) for each $x \in C$, $y \longmapsto F(x,y)$ is convex and lower semi-continuous.

Lemma 2.3 ([29]). *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $F\colon C \times C \to \mathbb{R}$ be a bifunction which satisfies assumption*

(L1)–(L4), $\phi\colon C \to H_1$ *be a nonlinear mapping and let* $U\colon C \to \mathbb{R} \cup \{\infty\}$ *be a proper lower semicontinuous and convex function. For* $r > 0$ *and* $x \in H_1$*, define a resolvent function*

$$T_r^F(x) = \Big\{z \in C\colon F(z,y)+\langle\phi(z), y-z\rangle+U(y)-U(z)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0,\ \forall\, y \in C\Big\}$$

for all $x \in H$*. Then the following conclusions hold:*

(i) *for each* $x \in H$, $T_r^F(x) \neq \emptyset$;
(ii) *function* T_r^F *is single-valued;*
(iii) *function* T_r^F *is firmly nonexpansive, i.e. for any* $x, y \in H$,

$$\|T_r^F x - T_r^F y\|^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle,$$

(iv) $F(T_r^F) = \mathrm{GMEP}(F, U, \phi)$,
(v) $\mathrm{GMEP}(F, U, \phi)$ *is closed and convex.*

Lemma 2.4 ([28]). *Let* $M\colon H \to 2^H$ *be a maximal monotone mapping and* $f\colon H \to H$ *be a Lipschitz continuous mapping. Then the mapping* $G := f + M\colon H \to 2^H$ *is a maximal monotone mapping.*

Lemma 2.5 ([33]). *Let* $\{\alpha_n\}$ *and* $\{\gamma_n\}$ *be sequences of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \Theta_n)\alpha_n + \beta_n + \gamma_n, \qquad n \geq 1,$$

where $\{\Theta_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\beta_n\}$ *is a real sequence. Assume that* $\sum_{n=0}^{\infty} \beta_n < \infty$*. Then, the following results hold:*

(i) *If* $\beta_n \leq \Theta_n M$ *for some* $M \geq 0$*, then* $\{\alpha_n\}$ *is a bounded sequence.*
(ii) *If* $\sum_{n=0}^{\infty} \Theta_n = \infty$ *and* $\limsup_{n\to\infty} \beta_n/\Theta_n \leq 0$*, then* $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.6 ([36]). *Assume that* $\{s_n\}$ *is a sequence of nonnegative real numbers such that*

$$\begin{aligned} s_{n+1} &\leq (1 - \theta_n)s_n + \theta_n\delta_n, \qquad n \geq 0,\\ s_{n+1} &\leq s_n - \tau_n + \sigma_n, \qquad n \geq 0, \end{aligned} \tag{2.1}$$

where $\{\theta_n\}$ *is a sequence in* $(0, 1)$, $\{\tau_n\}$ *is a sequence of nonnegative real numbers and* $\{\delta_n\}$ *and* $\{\sigma_n\}$ *are two sequences in* $\mathbb{R}$ *such that*

(i) $\sum_{n=0}^{\infty} \theta_n = \infty$;
(ii) $\lim_{n\to\infty} \sigma_n = 0$;
(iii) $\lim_{k\to\infty} \tau_{n_k} = 0$ *implies that* $\limsup_{k\to\infty} \delta_{n_k} \leq 0$ *for any subsequence* $\{n_k\} \subset \{n\}$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.7 ([34]). *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ with $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Consider the integer $\{m_k\}$ defined by*

$$m_k = \max\{j \leq k \colon a_j < a_{j+1}\}.$$

Then $\{m_k\}$ is a nondecreasing sequence verifying $\lim_{n\to\infty} m_n = \infty$, and for all $k \in \mathbb{N}$, the following estimates hold:

$$a_{m_k} \leq a_{m_k+1} \qquad \text{and} \qquad a_k \leq a_{m_k+1}.$$

3. Main results

In this section, we give a precise statement of our propose algorithm and discuss its convergence analysis. We first state the assumptions that we assume to hold through the rest of the paper.

(a) Let H_1, H_2 and H_3 are real Hilbert spaces, C and Q are nonempty, closed and convex subsets of H_1 and H_2, respectively, and $A\colon H_1 \to H_3$ and $B\colon H_2 \to H_3$ are bounded linear operators.

(b) Let $F\colon C\times C \to \mathbb{R}$ and $G\colon Q\times Q \to \mathbb{R}$ are bifunctions satisfying conditions (L1)–(L4), $\phi\colon C \to H_1$ and $\varphi\colon Q \to H_2$ are inverse strongly monotone operators with coefficients η_1 and η_2, respectively, $U_1\colon H_1 \to (-\infty,\infty]$ and $U_2\colon H_2 \to (-\infty,\infty]$ are proper, convex and lower semicontinuous functions.

(c) For $l = 1,2$, $f_l\colon H_l \to H_l$ is μ_l-inverse strongly monotone, $M_l\colon H_l \to 2^{H_l}$ is multivalued maximal monotone and $h_l\colon H_l \to H_l$ contraction mapping with coefficient $k_l \in (0,1)$.

(d) Let $\{S_i\}_{i=1}^{\infty}$ and $\{T_j\}_{j=1}^{\infty}$ are infinite families of quasi-nonexpansive mappings on H_1 and H_2, respectively, satisfying demiclosedness principle.

(e) The solution set $\Gamma := \big\{(x^*,y^*) \in H_1 \times H_2\colon x^* \in EP(F,U_1,\phi) \cap \mathrm{VI}(M_1,f_1) \cap \bigcap_{i=1}^{\infty} F(S_i),\ y^* \in \mathrm{EP}(G,U_2,\varphi) \cap \mathrm{VI}(M_2,f_2) \cap \bigcap_{j=1}^{\infty} F(T_j)$ and $Ax^* = By^*\big\}$ is nonempty.

We next give a precise statement of our iterative algorithm.

Algorithm 3.1. *Step 1*: *Take $x_0, x_1 \in H_1$, $y_0, y_1 \in H_2$ arbitrarily. Choose $\xi_n \in (0,1)$ and $\{\beta_{n,i}\}_{i=0}^{\infty}, \{\delta_{n,j}\}_{j=0}^{\infty} \subset [0,1)$ such that $\beta_{n,i} \neq 0$ for all $i \leq n$ and $\delta_{n,j} \neq 0$ for all $j \leq n$. Set $n = 1$.*

Step 2: *Given the nth and the $(n-1)$th iterates, calculate the (x_{n+1}, y_{n+1}) by the following process:*

$$
(3.1)\quad
\begin{aligned}
(\bar{x}_n, \bar{y}_n) &= (x_n, y_n) + \alpha_n(x_n - x_{n-1}, y_n - y_{n-1}),\\
w_n &= J_\lambda^{M_1}(I - \lambda f_1)(\bar{x}_n - \gamma_n A^*(A\bar{x}_n - B\bar{y}_n)),\\
z_n &= J_\lambda^{M_2}(I - \lambda f_2)(\bar{y}_n + \gamma_n B^*(A\bar{x}_n - B\bar{y}_n)),\\
F(u_n, w) &+ \langle \phi(w_n), w - u_n\rangle + U_1(w) - U_1(w_n)\\
&+ \frac{1}{r_n}\langle w - u_n, u_n - w_n\rangle \geq 0, \qquad \forall\, w \in C,\\
G(v_n, z) &+ \langle \varphi(z_n), z - v_n\rangle + U_2(z) - U_2(z_n)\\
&+ \frac{1}{r_n}\langle z - v_n, v_n - z_n\rangle \geq 0, \qquad \forall\, z \in Q,\\
x_{n+1} &= \xi_n h_1(x_n) + (1 - \xi_n)\Big(\beta_{n,0}u_n + \sum_{i=1}^{\infty}\beta_{n,i}S_i(u_n)\Big),\\
y_{n+1} &= \xi_n h_2(y_n) + (1 - \xi_n)\Big(\delta_{n,0}v_n + \sum_{j=1}^{\infty}\delta_{n,j}T_j(v_n)\Big),
\end{aligned}
$$

where α_n is chosen such that

$$
(3.2)\qquad \alpha_n = \begin{cases} \frac{\omega_n}{\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|}, & \text{if } (x_n, y_n) \neq (x_{n-1}, y_{n-1}),\\ \theta, & \text{if } (x_n, y_n) = (x_{n-1}, y_{n-1}) \end{cases}
$$

for $\theta > 0$ and $w_n \in [0, 1)$. The stepsize γ_n is chosen such that

$$
(3.3)\qquad \gamma_n \in \Big(0, \frac{2\|A\bar{x}_n - B\bar{y}_n\|^2}{\|A^*(A\bar{x}_n - B\bar{y}_n)\|^2 + \|B^*(A\bar{x}_n - B\bar{y}_n)\|^2}\Big), \qquad n \in \Omega,
$$

otherwise, $\gamma_n = \gamma$ (γ being any nonnegative value), where the set of indexes $\Omega = \{n\colon A\bar{x}_n - B\bar{y}_n \neq 0\}$, A^* and B^* are adjoints of A and B, respectively, $\lambda \in (0, 2\mu)$ where $\mu = \max\{\mu_1, \mu_2\}$ and $\{r_n\} \subset (0, \infty)$.

Since the convergence of our algorithm depends on the choice of the sequences $\{r_n\}, \{\alpha_n\}, \{\beta_{n,i}\}$ and $\{\delta_{n,j}\}$, we make the following assumption regarding these parameters.

Assumption 3.2. *Suppose that $\{r_n\}, \{\alpha_n\}, \{\beta_{n,i}\}$ and $\{\delta_{n,j}\}$ satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \xi_n = 0$ *and* $\sum_{n=1}^{\infty} \xi_n = \infty$;

(C2) $\omega_n = o(\xi_n)$, *i.e.*, $\lim_{n\to\infty} \omega_n/\xi_n = 0$;

(C3) $\sum_{i=0}^{n} \beta_{n,i} = 1$ *and* $\liminf_{n\to\infty} \beta_{n,i} > 0$, $\sum_{j=0}^{n} \delta_{n,j} = 1$ *and* $\liminf_{n\to\infty} \delta_{n,j} > 0$;

(C4) $0 < \liminf_{n\to\infty} r_n < 2\eta$, *where* $\eta = \max\{\eta_1, \eta_2\}$.

Remark 3.3. We remark that the first line in Step 2 can easily be implemented in numerical computation since the values of $\|x_n - x_{n-1}\|$ and $\|y_n - y_{n-1}\|$ are known before choosing α_n. Also observe that condition (C2) implies that

$$\lim_{n\to\infty} \alpha_n(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|) = 0,$$
$$\lim_{n\to\infty} \frac{\alpha_n}{\xi_n}(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|) = 0.$$

Now, we show the boundedness of the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1.

Lemma 3.4. *The sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1 is bounded.*

Proof: Let $(x^*, y^*) \in \Gamma$. This implies that $x^* \in \mathrm{EP}(F, U_1, \phi) \cap \mathrm{VI}(f_1, M_1) \cap \bigcap_{i=1}^{\infty} F(S_i)$ and $y^* \in \mathrm{EP}(G, U_2, \varphi) \cap \mathrm{VI}(f_2, M_2) \cap \bigcap_{j=1}^{\infty} F(T_j)$. Then, from Step 2 in Algorithm 3.1 and Lemma 2.1 (iii), we have

$$\begin{aligned}
\|\bar{x}_n - x^*\|^2 &= \|(x_n - x^*) + \alpha_n(x_n - x_{n-1})\|^2 \\
&= \|x_n - x^*\|^2 + 2\alpha_n\langle x_n - x^*, x_n - x_{n-1}\rangle + \alpha_n^2\|x_n - x_{n-1}\|^2 \\
&= \|x_n - x^*\|^2 + \alpha_n(\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2 \\
&\quad + \|x_n - x_{n-1}\|^2) + \alpha_n^2\|x_n - x_{n-1}\|^2 \\
&\le \|x_n - x^*\|^2 + \alpha_n(\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2) \\
&\quad + 2\alpha_n\|x_n - x_{n-1}\|^2 \\
&\le \|x_n - x^*\|^2 + \alpha_n(\|x_n - x^*\| + \|x_{n-1} - x^*\|)\|x_n - x_{n-1}\| \\
&\quad + 2\alpha_n\|x_n - x_{n-1}\|^2 \\
&= \|x_n - x^*\|^2 + \alpha_n(\|x_n - x^*\| + \|x_{n-1} - x^*\| \\
&\quad + 2\alpha_n\|x_n - x_{n-1}\|)\|x_n - x_{n-1}\| \\
&\le \|x_n - x^*\|^2 + \alpha_n c_1\|x_n - x_{n-1}\|,
\end{aligned} \tag{3.4}$$

where

$$c_1 = \sup_{n\ge1}\{\|x_n - x^*\| + \|x_{n-1} - x^*\| + 2\alpha_n\|x_n - x_{n-1}\|\}.$$

Similarly, we get

$$\|\bar{y}_n - y^*\|^2 \le \|y_n - y^*\|^2 + \alpha_n c_2\|y_n - y_{n-1}\|, \tag{3.5}$$

where

$$c_2 = \sup_{n\ge1}\{\|y_n - y^*\| + \|y_{n-1} - y^*\| + 2\alpha_n\|y_n - y_{n-1}\|\}.$$

Hence from (3.4) and (3.5), we obtain

$$\begin{aligned}\|\bar{x}_n - x^*\|^2 + \|\bar{y}_n - y^*\|^2 \le \|x_n - x^*\|^2 + \|y_n - y^*\|^2 \\ + \alpha_n c^*(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|),\end{aligned} \tag{3.6}$$

where $c^* = \max\{c_1, c_2\}$.

Since $(x^*, y^*) \in \mathrm{VI}(f_1, M_1) \times \mathrm{VI}(f_2, M_2)$, then $x^* = J_\lambda^{M_1}(I - \lambda f_1)x^*$ and $y^* = J_\lambda^{M_2}(I - \lambda f_2)y^*$. Then by the nonexpansivity of $J_\lambda^{M_l}(I - \lambda f_l)$ for $l = 1, 2$, we have

$$\begin{aligned}\|w_n - x^*\|^2 &= \|J_\lambda^{M_1}(I - \lambda f_1)(\bar{x}_n - \gamma_n A^*(A\bar{x}_n - B\bar{y}_n) - J_\lambda^{M_1}(I - f_1)x^*\|^2 \\ &\le \|\bar{x}_n - \gamma_n A^*(A\bar{x}_n - B\bar{y}_n) - x^*\|^2 \\ &= \|\bar{x}_n - x^*\|^2 + \gamma_n^2\|A^*(A\bar{x}_n - B\bar{y}_n)\|^2 \\ &\quad - 2\gamma_n\langle A\bar{x}_n - Ax^*, A\bar{x}_n - B\bar{y}_n\rangle.\end{aligned} \tag{3.7}$$

But

$$2\langle A\bar{x}_n - Ax^*, A\bar{x}_n - B\bar{y}_n\rangle = \|A\bar{x}_n - Ax^*\|^2 + \|A\bar{x}_n - B\bar{y}_n\|^2 - \|B\bar{y}_n - Ax^*\|^2. \tag{3.8}$$

Substituting (3.8) into (3.7), we have

$$\begin{aligned}\|w_n - x^*\|^2 \le \|\bar{x}_n - x^*\|^2 + \gamma_n^2\|A^*(A\bar{x}_n - B\bar{y}_n)\|^2 - \gamma_n\|A\bar{x}_n - Ax^*\|^2 \\ - \gamma_n\|A\bar{x}_n - B\bar{y}_n\|^2 + \gamma_n\|B\bar{y}_n - Ax^*\|^2.\end{aligned} \tag{3.9}$$

Similarly,

$$\begin{aligned}\|z_n - y^*\|^2 \le \|\bar{y}_n - y^*\|^2 + \gamma_n^2\|B^*(A\bar{x}_n - B\bar{y}_n)\|^2 + \gamma_n\|A\bar{x}_n - By^*\|^2 \\ - \gamma_n\|B\bar{y}_n - By^*\|^2 - \gamma_n\|A\bar{x}_n - B\bar{y}_n\|^2.\end{aligned} \tag{3.10}$$

Adding (3.9) and (3.10) and noting that $Ax^* = By^*$, we have

$$\begin{aligned}\|w_n - x^*\|^2 + \|z_n - y^*\|^2 \le \|\bar{x}_n - x^*\|^2 + \|\bar{y}_n - y^*\|^2 \\ - \gamma_n\big(2\|A\bar{x}_n - \gamma_n\big(2\|A\bar{x}_n - B\bar{y}_n\|^2 \\ - \gamma_n(\|A^*(A\bar{x}_n - B\bar{y}_n)\|^2 + \|B^*(A\bar{x}_n - B\bar{y}_n)\|^2)\big),\end{aligned} \tag{3.11}$$

and from (3.3), we get

$$\|w_n - x^*\|^2 + \|z_n - y^*\|^2 \le \|\bar{x}_n - x^*\|^2 + \|\bar{y}_n - y^*\|^2. \tag{3.12}$$

Observe that from Algorithm 3.1, $u_n = T_{r_n}^F(w_n - r_n\phi w_n)$ and $v_n = T_{r_n}^G(z_n - r_n\varphi z_n)$.

Since $(x^*, y^*) \in \mathrm{EP}(F, U_1, \phi) \times \mathrm{EP}(G, U_2, \varphi)$, then

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|T^F_{r_n}(w_n - r_n\phi w_n) - x^*\|^2 \\
&= \|T^F_{r_n}(w_n - r_n\phi w_n) - T^F_{r_n}(x^* - r_n\phi x^*)\|^2 \\
&\leq \|(I - r_n\phi)w_n - (I - r_n\phi)x^*\|^2 \\
&\leq \|(w_n - x^*) - r_n(\phi w_n - \phi x^*)\|^2 \\
&= \|w_n - x^*\|^2 - 2r_n\langle w_n - x^*, \phi w_n - \phi x^*\rangle + r_n^2\|\phi w_n - \phi x^*\|^2 \\
&\leq \|w_n - x^*\|^2 - 2r_n\eta_1\|\phi w_n - \phi x^*\|^2 + r_n^2\|\phi w_n - \phi x^*\|^2 \\
&= \|w_n - x^*\|^2 - r_n(2\eta_1 - r_n)\|\phi w_n - \phi x^*\|^2.
\end{aligned} \tag{3.13}$$

Similarly, we have

$$\|v_n - y^*\|^2 \leq \|z_n - y^*\|^2 - r_n(2\eta_2 - r_n)\|\varphi z_n - \varphi y^*\|^2. \tag{3.14}$$

Thus, from (3.12), (3.13) and (3.14), we have

$$\begin{aligned}
\|u_n - x^*\|^2 + \|v_n - y^*\|^2 &\leq \|w_n - x^*\|^2 + \|z_n - y^*\|^2 - r_n(2\eta - r_n) \\
&\quad \times (\|\phi w_n - \phi x^*\|^2 + \|\varphi z_n - \varphi y^*\|^2) \\
&\leq \|\bar{x}_n - x^*\|^2 + \|\bar{y}_n - y^*\|^2,
\end{aligned} \tag{3.15}$$

where $\eta = \max\{\eta_1, \eta_2\}$.

Furthermore, since $(x^*, y^*) \in \bigcap_{i=1}^{\infty} F(S_i) \times \bigcap_{j=1}^{\infty} F(T_j)$, then

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|\xi_n h_1(x_n) + (1 - \xi_n)\left(\beta_{n,0}u_n + \sum_{i=1}^{\infty}\beta_{n,i}S_iu_n\right) - x^*\right\|^2 \\
&= \left\|\xi_n(h_1(x_n) - x^*) + (1 - \xi_n)\left(\beta_{n,0}u_+ \sum_{i=1}^{\infty}\beta_{n,i}S_iu_n - x^*\right)\right\|^2 \\
&\leq \xi_n\|h_1(x_n) - x^*\|^2 + (1 - \xi_n)\left\|\beta_{n,0}u_n + \sum_{i=1}^{\infty}\beta_{n,i}S_iu_n - x^*\right\|^2 \\
&= \xi_n\|h_1(x_n) - h_1(x^*) + h_1(x^*) - x^*\|^2 + (1 - \xi_n)\Bigg(\beta_{n,0}\|u_n - x^*\|^2 \\
&\quad + \sum_{i=1}^{\infty}\beta_{n,i}\|S_iu_n - x^*\|^2 - \beta_{n,0}\sum_{i=1}^{\infty}\beta_{n,i}\|S_iu_n - u_n\|^2\Bigg) \\
&\leq \xi_n(\|h_1(x_n) - h_1(x^*)\| + \|h_1(x^*) - x^*\|)^2 + (1 - \xi_n)\|u_n - x^*\|^2 \\
&\quad - (1 - \xi_n)\beta_{n,0}\sum_{i=1}^{\infty}\beta_{n,i}\|S_iu_n - u_n\|^2 \\
&\leq \xi_n k_1^2\|x_n - x^*\|^2 + \xi_n\|h_1(x^*) - x^*\|^2 + (1 - \xi_n)\|u_n - x^*\|^2.
\end{aligned} \tag{3.16}$$

Similarly, we have

$$\|y_{n+1}-y^*\|^2 \le \xi_n k_2^2\|y_n-y^*\|^2+\xi_n\|h_2(y^*)-y^*\|^2+(1-\xi_n) \times \|v_n-y^*\|^2-(1-\xi_n)\delta_{n,0}\sum_{j=1}^{\infty}\delta_{n,j}\|T_jv_n-v_n\|^2. \tag{3.17}$$

Hence from (3.15), (3.16) and (3.17), we get

$$\begin{aligned}\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2 &\le \xi_n(k_1^2\|x_n-x^*\|^2+k_2^2\|y_n-y^*\|^2)\\ &\quad+\xi_n(\|h_1(x^*)-x^*\|^2+\|h_2(y^*)-y^*\|^2)\\ &\quad+(1-\xi_n)(\|u_n-x^*\|^2+\|v_n-y^*\|^2)\\ &\le \xi_n k^2(\|x_n-x^*\|^2+\|y_n-y^*\|^2)\\ &\quad+\xi_n(\|h_1(x^*)-x^*\|^2+\|h_2(y^*)-y^*\|^2)\\ &\quad+(1-\xi_n)(\|\bar{x}_n-x^*\|^2+\|\bar{y}_n-y^*\|^2),\end{aligned} \tag{3.18}$$

where $k=\max\{k_1,k_2\}$. Therefore from (3.6) and (3.18), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2 &\le \xi_n k^2(\|x_n-x^*\|^2+\|y_n-y^*\|^2)\\ &\quad+\xi_n(\|h_1(x^*)-x^*\|^2+\|h_2(y^*)-y^*\|)\\ &\quad+(1-\xi_n)(\|x_n-x^*\|^2+\|y_n-y^*\|^2)\\ &\quad+\alpha_n c^*(1-\xi_n)(\|x_n-x_{n-1}\|+\|y_n-y_{n-1}\|)\\ &=(1-\xi_n(1-k^2))(\|x_n-x^*\|^2+\|y_n-y^*\|^2)\\ &\quad+\xi_n(\|h_1(x^*)-x^*\|^2+\|h_2(y^*)-y^*\|^2)\\ &\quad+\alpha_n c^*(\|x_n-x_{n-1}\|+\|y_n-y_{n-1}\|)\\ &\quad-\alpha_n c^*\xi_n(\|x_n-x_{n-1}\|+\|y_n-y_{n-1}\|)\\ &\le(1-\xi_n(1-k^2))(\|x_n-x^*\|^2+\|y_n-y^*\|^2)\\ &\quad+\xi_n(\|h_1(x^*)-x^*\|^2+\|h_2(y^*)-y^*\|^2)\\ &\quad+\alpha_n c^*(\|x_n-x_{n-1}\|+\|y_n-y_{n-1}\|).\end{aligned} \tag{3.19}$$

Putting $\varrho_n(x^*,y^*)=\|x_n-x^*\|^2+\|y_n-y^*\|^2$, then we have

$$\begin{aligned}\varrho_{n+1}(x^*,y^*) &\le (1-\xi_n(1-k^2))\varrho_n(x^*,y^*)\\ &\quad+\xi_n(1-k^2)\Big[\frac{\|h_1(x^*)-x^*\|^2+\|h_2(y^*)-y^*\|^2}{1-k^2}\\ &\quad+\frac{\alpha_n c^*(\|x_n-x_{n-1}\|+\|y_n-y_{n-1}\|)}{\xi_n(1-k^2)}\Big].\end{aligned} \tag{3.20}$$

Note that $\sup_{n\geq 1}\{\alpha_n c^*(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|)/(\xi_n(1-k^2))\}$ exists by Remark 3.3. Let

$$M := \max\Big\{\frac{\|h_1(x^*) - x^*\|^2 + \|h_2(y^*) - y^*\|^2}{1-k^2},\\ \sup_{n\geq 1}\Big\{\frac{\alpha_n c^*(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|)}{\xi_n(1-k^2)}\Big\}\Big\},$$

then from (3.20), we have

$$\varrho_{n+1}(x^*, y^*) \leq (1 - \xi_n(1-k^2))\varrho_n(x^*, y^*) + \xi_n(1-k^2)M. \tag{3.21}$$

Then using Lemma 2.5 and (3.21), we obtain that $\{\varrho_n(x^*, y^*)\}$ is bounded. This implies that $\{(x_n, y_n)\}$ is bounded. Consequently, $\{x_n\}$ and $\{y_n\}$ are bounded. □

Next, we show an important result which is crucial in establishing the convergence of Algorithm 3.1.

Lemma 3.5. *The sequence $\{(x_n, y_n)\}$ generated by Algorithm* 3.1 *satisfied the following estimate:*

$$\varrho_{n+1}(x^*, y^*) \leq (1 - \Theta_n)\varrho_n(x^*, y^*) + \Theta_n \sigma_n, \tag{3.22}$$

where

$$\varrho_n(x^*, y^*) = \|x_n - x^*\|^2 + \|y_n - y^*\|^2, \qquad \Theta_n = \frac{2\xi_n(1-k)}{1-\xi_n k}, \tag{3.23}$$

and

$$\sigma_n = \frac{\langle h_1(x^*) - x^*, x_{n+1} - x^*\rangle + \langle h_2(y^*) - y^*, y_{n+1} - y^*\rangle}{1-k} + \frac{\xi_n}{2(1-k)}\varrho_n(x^*, y^*)\\ + \frac{\alpha_n c^*}{2\xi_n(1-k)}(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|),$$

for some $c^* > 0$, *and* $(x^*, y^*) \in \Gamma$.

Proof: Set $\bar{u}_n = \beta_{n,0}u_n + \sum_{i=1}^{\infty} \beta_{n,i}S_i u_n$ and $\bar{v}_n = \delta_{n,0}v_n + \sum_{j=1}^{\infty} \delta_{n,j}T_j v_n$. Let $(x^*, y^*) \in \Gamma$, then

$$\begin{aligned}
\|\bar{u}_n - x^*\|^2 &= \left\|\beta_{n,0}u_n + \sum_{i=1}^{\infty} \beta_{n,i}S_i u_n - x^*\right\|^2 \\
&= \beta_{n,0}\|u_n - x^*\|^2 + \sum_{i=1}^{\infty} \beta_{n,i}\|S_i u_n - S_i x^*\|^2 \\
&\quad - \beta_{n,0}\sum_{i=1}^{\infty} \beta_{n,i}\|S_i u_n - u_n\|^2 \\
&\leq \|u_n - x^*\|^2 - \beta_{n,0}\sum_{i=1}^{\infty} \beta_{n,i}\|S_i u_n - u_n\|^2.
\end{aligned}$$

Similarly,

$$\|\bar{v}_n - y^*\|^2 \leq \|v_n - y^*\|^2 - \delta_{n,0}\sum_{j=1}^{\infty} \delta_{n,j}\|T_j v_n - v_n\|^2.$$

Hence

$$\|\bar{u}_n - x^*\|^2 + \|\bar{v}_n - y^*\|^2 \leq \|u_n - x^*\|^2 + \|u_n - x^*\|^2.$$

Also

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq (1-\xi_n)^2\|\bar{u}_n - x^*\|^2 + 2\xi_n\langle h_1(x_n) - x^*, x_{n+1} - x^*\rangle \\
&= (1-\xi_n)^2\|\bar{u}_n - x^*\|^2 + 2\xi_n\langle h_1(x_n) - h_1(x^*) \\
&\quad + h_1(x^*) + x^*, x_{n+1} - x^*\rangle \\
&= (1-\xi_n)^2\|\bar{u}_n - x^*\|^2 + 2\xi_n k_1\|x_n - x^*\|\,\|x_{n+1} - x^*\| \\
&\quad + 2\xi_n\langle h_1(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\leq (1-\xi_n)^2\|\bar{u}_n - x^*\|^2 + \xi_n k(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
&\quad + 2\xi_n\langle h_1(x^*) - x^*, x_{n+1} - x^*\rangle.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|y_{n+1} - y^*\|^2 \leq (1-\xi_n)^2\|\bar{v}_n - y^*\|^2 &+ \xi_n k(\|y_n - y^*\|^2 \\
&+ \|y_{n+1} - y^*\|^2) + 2\xi_n\langle h_2(y^*) - y^*, y_{n+1} - y^*\rangle.
\end{aligned}$$

Then

$$\begin{aligned}
\varrho_{n+1}(x^*, y^*) \leq (1-\xi_n)^2(\|\bar{u}_n - x^*\|^2 &+ \|\bar{v}_n - y^*\|^2) + \xi_n k(\varrho_n(x^*, y^*) \\
&+ \varrho_{n+1}(x^*, y^*)) + 2\xi_n(\langle h_1(x^*) - x^*, x_{n+1} - x^*\rangle \\
&+ \langle h_2(y^*) - y^*, y_{n+1} - y^*\rangle)
\end{aligned} \tag{3.24}$$

$$\leq (1-\xi_n)^2(\|\bar{x}_n - x^*\|^2 + \|\bar{y}_n - y^*\|^2) + \xi_n k(\varrho_n(x^*,y^*) + \varrho_{n+1}(x^*,y^*)) + 2\xi_n(\langle h_1(x^*) - x^*, x_{n+1} - x^*\rangle + \langle h_2(y^*) - y^*, y_{n+1} - y^*\rangle).$$

Using (3.6) in (3.24), we have

$$\begin{aligned}\varrho_{n+1}(x^*,y^*) \leq{}& (1-2\xi_n+\xi_n^2)\varrho_n(x^*,y^*) + \xi_n k(\varrho_n(x^*,y^*) \\ &+ \varrho_{n+1}(x^*,y^*)) + 2\xi_n(\langle h_1(x^*) - x^*, x_{n+1} - x^*\rangle \\ &+ \langle h_2(y^*) - y^*, y_{n+1} - y^*\rangle) + \alpha_n c^*(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|).\end{aligned}$$

Therefore we get

$$\begin{aligned}\varrho_{n+1}(x^*,y^*) \leq{}& \frac{1-2\xi_n+\xi_n k}{1-\xi_n k}\varrho_n(x^*,y^*) + \frac{\xi_n^2}{1-\xi_n k}\varrho_n(x^*,y^*) \\ &+ \frac{2\xi_n}{1-\xi_n k}(\langle h_1(x^*) - x^*, x_{n+1} - x^*\rangle \\ &+ \langle h_2(y^*) - y^*, y_{n+1} - y^*\rangle) \\ &+ \frac{\alpha_n c^*}{1-\xi_n k}(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|) \\ ={}& \Big(1 - \frac{2\xi_n(1-k)}{1-\xi_n k}\Big)\varrho_n(x^*,y^*) + \frac{2\xi_n(1-k)}{1-\xi_n k} \\ &\times \Big(\frac{\langle h_1(x^*) - x^*, x_{n+1} - x^*\rangle + \langle h_2(y^*) - y^*, y_{n+1} - y^*\rangle}{1-k} \\ &+ \frac{\xi_n}{2(1-k)}\varrho_n(x^*,y^*) + \frac{\alpha_n c^*}{2\xi_n(1-k)}(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|)\Big) \\ ={}& (1-\Theta_n)\varrho_n(x^*,y^*) + \Theta_n\sigma_n.\end{aligned}$$

This establishes (3.22). □

In the next lemma, we show that every weak cluster point of $\{(x_n, y_n)\}$ belongs to the solution set Γ.

Lemma 3.6. *Suppose that $\varrho_n(x^*,y^*) - \varrho_{n+1}(x^*,y^*) \to 0$ as $n \to \infty$. Let $(p,q) \in H_1 \times H_2$ denotes the weak subsequential limit of $\{(x_n, y_n)\}$. Then $(p,q) \in \Gamma$.*

Proof: We divide the proof of this result into three steps:

Step 1: Let $\Upsilon_n = \|A^*(A\bar{x}_n - B\bar{y}_n)\|^2 + \|B^*(A\bar{x}_n - B\bar{y}_n)\|^2$. Then we show that

$$\lim_{n\to\infty} \Upsilon_n = 0, \qquad \text{and} \qquad \lim_{n\to\infty} \|A\bar{x}_n - B\bar{y}_n\| = 0. \tag{3.25}$$

From (3.16) and (3.17), we have

$$\begin{aligned}\varrho_{n+1}(x^*,y^*) \leq \xi_n k^2 \varrho_n(x^*,y^*) + \xi_n(\|h_1(x^*)-x^*\|^2 + \|h_2(y^*)-y^*\|^2) \\ + (1-\xi_n)(\|u_n - x^*\|^2 + \|v_n - y^*\|^2),\end{aligned} \tag{3.26}$$

and from (3.15) and the last inequality, we get

$$\begin{aligned}\varrho_{n+1}(x^*,y^*) \leq \xi_n k^2 \varrho_n(x^*,y^*) + \xi_n(\|h_1(x^*)-x^*\|^2 + \|h_2(y^*)-y^*\|^2) \\ + (1-\xi_n)(\|w_n - x^*\|^2 + \|z_n - y^*\|^2).\end{aligned} \tag{3.27}$$

Also from (3.11) and the above inequality, we have

$$\begin{aligned}\varrho_{n+1}(x^*,y^*) \leq \xi_n k^2 \varrho_n(x^*,y^*) + \xi_n(\|h_1(x^*)-x^*\|^2 \\ + \|h_2(y^*)-y^*\|^2) + (1-\xi_n)(\|\bar{x}_n - x^*\|^2 + \|\bar{y}_n - y^*\|^2 \\ - \gamma_n(2\|A\bar{x}_n - B\bar{y}_n\|^2 - \gamma_n\Upsilon_n)).\end{aligned} \tag{3.28}$$

Hence from (3.6) and (3.28), we have

$$\begin{aligned}\varrho_{n+1}(x^*,y^*) \leq (1-\xi_n(1-k^2))\varrho_n(x^*,y^*) + \xi_n(\|h_1(x^*)-x^*\|^2 + \|h_2(y^*)-y^*\|^2) \\ + (1-\xi_n)(\alpha_n c^*(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|) \\ - \gamma_n(2\|A\bar{x}_n - B\bar{y}_n\|^2 - \gamma_n\Upsilon_n)).\end{aligned}$$

This implies that

$$\begin{aligned}(1-\xi_n)\gamma_n(2\|A\bar{x}_n - B\bar{y}_n\|^2 - \gamma_n\Upsilon_n) \leq (1-\xi_n(1-k^2))\varrho_n(x^*,y^*) \\ + \xi_n(\|h_1(x^*)-x^*\|^2 + \|h_2(y^*)-y^*\|^2) \\ + \alpha_n c^*(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|) \\ - \varrho_{n+1}(x^*,y^*).\end{aligned}$$

Since $\xi_n \to 0$ and the fact that $\alpha_n(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|) \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \gamma_n(2\|A\bar{x}_n - B\bar{y}_n\|^2 - \gamma_n\Upsilon_n) = 0. \tag{3.29}$$

By the choice of γ_n in (3.3) for a very small $\varepsilon > 0$, we have

$$\gamma_n < \frac{2\|A\bar{x}_n - B\bar{y}_n\|^2}{\Upsilon_n} - \varepsilon$$

which implies that

$$\gamma_n\Upsilon_n < 2\|A\bar{x}_n - B\bar{y}_n\|^2 - \varepsilon\Upsilon_n,$$

and then from (3.29), we have

$$\frac{\varepsilon\Upsilon_n}{2} < \|A\bar{x}_n - B\bar{y}_n\|^2 - \frac{\gamma_n}{2}\Upsilon_n \to 0. \tag{3.30}$$

Hence

$$\lim_{n\to\infty} \Upsilon_n = 0. \tag{3.31}$$

Consequently

$$\lim_{n\to\infty} \|A^*(A\bar{x}_n - B\bar{y}_n)\| = \lim_{n\to\infty} \|B^*(A\bar{x}_n - B\bar{y}_n)\| = 0. \tag{3.32}$$

Also

$$\lim_{n\to\infty} \|A\bar{x}_n - B\bar{y}_n\| = 0. \tag{3.33}$$

Step 2: We show that the following limits hold:

$$\begin{aligned}
&\lim_{n\to\infty} (\|x_n - w_n\| + \|y_n - z_n\|) = 0,\\
&\lim_{n\to\infty} (\|x_n - u_n\| + \|y_n - v_n\|) = 0,\\
&\lim_{n\to\infty} (\|S_i u_n - u_n\| + \|T_j v_n - v_n\|) = 0.
\end{aligned} \tag{3.34}$$

Observe that

$$\lim_{n\to\infty} (\|\bar{x}_n - x_n\| + \|\bar{y}_n - y_n\|) = \lim_{n\to\infty} \alpha_n(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|) = 0. \tag{3.35}$$

Also

$$\begin{aligned}
\|w_n - x^*\|^2 &\le \|J_\lambda^{M_1}(I - \lambda f_1)(\bar{x}_n - \gamma_n A^*(A\bar{x}_n - B\bar{y}_n)) - x^*\|^2\\
&\le \langle w_n - x^*, \bar{x}_n - \gamma_n A^*(A\bar{x}_n - B\bar{y}_n) - x^*\rangle\\
&= \frac{1}{2}\big[\|w_n - x^*\|^2 + \|\bar{x}_n - \gamma_n A^*(A\bar{x}_n - B\bar{y}_n) - x^*\|^2\\
&\quad - \|w_n - \bar{x}_n - \gamma_n A^*(A\bar{x}_n - B\bar{y}_n)\|^2\big].
\end{aligned}$$

Hence

$$\begin{aligned}
\|w_n - x^*\|^2 \le\ & \|\bar{x}_n - x^*\|^2 + \gamma_n^2\|A^*(A\bar{x}_n - B\bar{y}_n)\|^2\\
& - 2\gamma_n\langle A\bar{x}_n - Ax^*, A\bar{x}_n - B\bar{y}_n\rangle\\
& - \|w_n - \bar{x}_n\|^2 - \gamma_n^2\|A^*(A\bar{x}_n - B\bar{y}_n)\|^2\\
& + 2\gamma_n\langle Aw_n - A\bar{x}_n, A\bar{x}_n - B\bar{y}_n\rangle\\
\le\ & \|\bar{x}_n - x^*\|^2 - \|w_n - \bar{x}_n\|^2 + 2\gamma_n\|Aw_n - Ax^*\|\,\|A\bar{x}_n - B\bar{y}_n\|.
\end{aligned}$$

Similarly, we have

$$\|z_n - y^*\|^2 \leq \|\bar{y}_n - y^*\|^2 - \|z_n - \bar{y}_n\|^2 + 2\gamma_n\|Bz_n - By^*\|\,\|A\bar{x}_n - B\bar{y}_n\|.$$

Therefore

$$\begin{aligned}\|w_n - x^*\|^2 + \|z_n - y^*\|^2 \leq {} & \|\bar{x}_n - x^*\|^2 + \|\bar{y}_n - y^*\|^2 \\ & - (\|w_n - \bar{x}_n\|^2 + \|z_n - \bar{y}_n\|^2) \\ & + 2\gamma_n(\|Aw_n - Ax^*\| + \|Bz_n - By^*\|) \\ & \times \|A\bar{x}_n - B\bar{y}_n\|.\end{aligned} \tag{3.36}$$

Therefore from (3.6), (3.27) and (3.36), we get

$$\begin{aligned}\varrho_{n+1}(x^*, y^*) \leq {} & (1 - \xi_n(1 - k^2))\varrho_n(x^*, y^*) + \xi_n(\|h_1(x^*) - x^*\|^2 \\ & + \|h_2(y^*) - y^*\|^2) - (\|w_n - \bar{x}_n\|^2 + \|z_n - \bar{y}_n\|^2) \\ & + 2\gamma_n(\|Aw_n - Ax^*\| + \|Bz_n - By^*\|)\|A\bar{x}_n - B\bar{y}_n\| \\ & + \alpha_n c^*(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|).\end{aligned}$$

This implies that

$$\begin{aligned}\|w_n - \bar{x}_n\|^2 + \|z_n - \bar{y}_n\|^2 \leq {} & (1 - \xi_n(1 - k^2))\varrho_n(x^*, y^*) - \varrho_{n+1}(x^*, y^*) \\ & + \xi_n(\|h_1(x^*) - x^*\|^2 + \|h_2(y^*) - y^*\|^2) \\ & + 2\gamma_n(\|Aw_n - Ax^*\| + \|Bz_n - By^*|)\|A\bar{x}_n - B\bar{y}_n\| \\ & + \alpha_n c^*(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|).\end{aligned}$$

Since $\xi_n \to 0$ as $n \to \infty$ and using (3.33), we have

$$\lim_{n\to\infty}(\|w_n - \bar{x}_n\|^2 + \|z_n - \bar{y}_n\|^2) = 0,$$

which implies that

$$\lim_{n\to\infty}\|w_n - \bar{x}_n\| = \lim_{n\to\infty}\|z_n - \bar{y}_n\| = 0. \tag{3.37}$$

Therefore from (3.35) and (3.37), we get

$$\lim_{n\to\infty}\|w_n - x_n\| = \lim_{n\to\infty}\|z_n - y_n\| = 0. \tag{3.38}$$

Furthermore, from (3.6), (3.15) and (3.26), we have

$$\begin{aligned}\varrho_{n+1}(x^*,y^*) \le\ & (1-\xi_n(1-k^2))\varrho_n(x^*,y^*)\\ & + \xi_n(\|h_1(x^*)-x^*\|^2 + \|h_2(y^*)-y^*\|^2)\\ & + \alpha_n c^*(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|)\\ & - r_n(2\eta - r_n)(\|\phi w_n - \phi x^*\|^2 + \|\varphi z_n - \varphi y^*\|^2).\end{aligned}$$

Hence

$$\begin{aligned} r_n(2\eta - r_n)(\|\phi w_n - \phi x^*\|^2 + \|\varphi z_n - \varphi y^*\|^2) \le\ & (1-\xi_n(1-k^2))\varrho_n(x^*,y^*)\\ & - \varrho_{n+1}(x^*,y^*) + \xi_n(\|h_1(x^*)-x^*\|^2 + \|h_2(y^*)-y^*\|^2)\\ & + \alpha_n c^*(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|).\end{aligned}$$

Therefore

(3.39) $$\lim_{n\to\infty} r_n(2\eta - r_n)(\|\phi w_n - \phi x^*\|^2 + \|\varphi z_n - \varphi y^*\|^2) = 0.$$

Since $0 < \liminf_{n\to\infty} r_n < 2\eta$, then

$$\lim_{n\to\infty}(\|\phi w_n - \phi x^*\|^2 + \|\varphi z_n - \varphi y^*\|^2) = 0,$$

which implies that

(3.40) $$\lim_{n\to\infty}\|\phi w_n - \phi x^*\| = \lim_{n\to\infty}\|\varphi z_n - \varphi y^*\| = 0.$$

Again from (3.1), we have

$$\begin{aligned}\|u_n - x^*\|^2 &= \|T^F_{r_n}(w_n - r_n\phi w_n) - x^*\|^2\\ &\le \|T^F_{r_n}(w_n - r_n\phi w_n) - T^F_{r_n}(x^* - r_n\phi x^*)\|^2\\ &\le \langle u_n - x^*, (w_n - r_n\phi w_n) - (x^* - r_n\phi x^*)\rangle\\ &= \frac{1}{2}\big[\|u_n - x^*\|^2 + \|(w_n - r_n\phi w_n) - (x^* - r_n\phi x^*)\|^2\\ &\quad - \|w_n - r_n\phi w_n - (x^* - r_n\phi x^*) - (u_n - x^*)\|^2\big].\end{aligned}$$

Hence

$$\begin{aligned}\|u_n - x^*\|^2 &= \|w_n - x^*\|^2 - \|(w_n - r_n\phi w_n)\\ &\quad - (x^* - r_n\phi x^*) - (u_n - x^*)\|^2\\ &= \|w_n - x^*\|^2 - \|w_n - u_n\|^2\\ &\quad + 2r_n\langle w_n - u_n, \phi w_n - \phi x^*\rangle - r_n^2\|\phi w_n - \phi x^*\|^2\\ &\le \|w_n - x^*\|^2 - \|w_n - u_n\|^2 + 2r_n\|w_n - u_n\|\,\|\phi w_n - \phi x^*\|.\end{aligned}$$

Similarly, we have

$$\|v_n - y^*\|^2 \leq \|z_n - y^*\|^2 - \|z_n - v_n\|^2 + 2r_n\|z_n - v_n\|\,\|\varphi z_n - \varphi y^*\|.$$

Then we obtain

$$\begin{aligned}\|u_n - x^*\|^2 + \|v_n - y^*\|^2 \leq{}& \|w_n - x^*\|^2\|z_n - y^*\|^2 \\ &- (\|w_n - u_n\|^2 + \|z_n - v_n\|^2) \\ &+ 2r_n\|w_n - u_n\|\,\|\phi w_n - \phi x^*\| \\ &+ 2r_n\|z_n - v_n\|\,\|\varphi z_n - \varphi y^*\|.\end{aligned} \tag{3.41}$$

Therefore from (3.6), (3.26) and (3.41), we get

$$\begin{aligned}\varrho_{n+1}(x^*, y^*) \leq{}& (1 - \xi_n(1 - k^2))\varrho_n(x^*, y^*) + \xi_n(\|h_1(x^*) - x^*\|^2 \\ &+ \|h_2(y^*) - y^*\|^2) - (\|w_n - u_n\|^2 + \|z_n - v_n\|^2) \\ &+ 2r_n\|w_n - u_n\|\,\|\phi w_n - \phi x^*\| \\ &+ 2r_n\|z_n - v_n\|\,\|\varphi z_n - \varphi y^*\| \\ &+ \alpha_n c^*(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|).\end{aligned}$$

Thus

$$\begin{aligned}\|w_n - u_n\|^2 + \|z_n - v_n\|^2 \leq{}& (1 - \xi_n(1 - k^2))\varrho_n(x^*, y^*) - \varrho_{n+1}(x^*, y^*) \\ &+ \xi_n(\|h_1(x^*) - x^*\|^2 + \|h_2(y^*) - y^*\|^2) \\ &+ 2r_n\|w_n - u_n\|\,\|\phi w_n - \phi x^*\| \\ &+ 2r_n\|z_n - v_n\|\,\|\varphi z_n - \varphi y^*\| \\ &+ \alpha_n c^*(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|).\end{aligned}$$

Also from (3.40) and the fact that $\xi_n \to 0$, we get

$$\lim_{n\to\infty} (\|w_n - u_n\|^2 + \|z_n - v_n\|^2) = 0.$$

This implies that

$$\lim_{n\to\infty} \|w_n - u_n\| = \lim_{n\to\infty} \|z_n - v_n\| = 0. \tag{3.42}$$

Hence from (3.38) and (3.42), we have

$$\lim_{n\to\infty} \|x_n - u_n\| = \lim_{n\to\infty} \|y_n - v_n\| = 0. \tag{3.43}$$

Again from (3.16) and (3.17), we have

$$
\begin{aligned}
\varrho_{n+1}(x^*, y^*) \leq{} & \xi_n k^2(\|x_n - x^*\|^2 + \|y_n - y^*\|^2) \\
& + \xi_n(\|h_1(x^*) - x^*\|^2 + \|h_2(y^*) - y^*\|) \\
& + (\|u_n - x^*||^2 + ||v_n - y^*\|^2) \\
& - (1 - \xi_n)\Big(\beta_{n,0}\sum_{i=1}^{\infty}\beta_{n,i}\|S_i u_n - u_n\|^2 \\
& + \delta_{n,0}\sum_{j=1}^{\infty}\delta_{n,j}\|T_j v_n - v_n\|^2\Big).
\end{aligned}
\tag{3.44}
$$

Hence from (3.6), (3.15) and (3.44), we have

$$
\begin{aligned}
(1 - \xi_n)\Big(\beta_{n,0}\sum_{i=1}^{\infty}&\beta_{n,i}\|S_i u_n - u_n\|^2 + \delta_{n,0}\sum_{j=1}^{\infty}\delta_{n,j}\|T_j v_n - v_n\|^2\Big) \\
&\leq (1 - \xi_n(1 - k^2))\varrho_n(x^*, y^*) + \xi_n(\|h_1(x^*) - x^*\|^2 \\
&\quad + \|h_2(y^*) - y^*\|^2) + \alpha_n c^*(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|).
\end{aligned}
$$

Using the fact that $\xi_n \to 0$, we have

$$
\lim_{n\to\infty}\Big(\beta_{n,0}\sum_{i=1}^{\infty}\beta_{n,i}\|S_i u_n - u_n\|^2 + \delta_{n,0}\sum_{j=1}^{\infty}\delta_{n,j}\|T_j v_n - v_n\|^2\Big) = 0.
$$

Also, by using condition (C3), we have

$$
\lim_{n\to\infty}\|S_i u_n - u_n\| = \lim_{n\to\infty}\|T_j v_n - v_n\| = 0. \tag{3.45}
$$

Step 3: Let $\{(x_{n_k}, y_{n_k})\}$ be a subsequence of $\{(x_n, y_n)\}$ such that $(x_{n_k}, y_{n_k}) \rightharpoonup (p, q)$. Note that the existence of (p, q) is guaranteed since $\{(x_n, y_n)\}$ is bounded, see Lemma 3.4. We now show that $(p, q) \in \Gamma$.

First, we show that $(p, q) \in \mathrm{EP}(F, U_1, \phi) \times \mathrm{EP}(G, U_2, \varphi)$. Since $u_n = T^F_{r_n}(w_n - r_n\phi w_n)$, $n \geq 1$, we have for any $x \in C$

$$
F(u_n, x) + \langle\phi(u_n), x - u_n\rangle + U_1(x) - U_1(u_n) + \frac{1}{r_n}\langle x - u_n, u_n - w_n\rangle \geq 0.
$$

It follows from the monotonicity of F that

$$
\langle\phi(u_n), x - u_n\rangle + U_1(x) - U_1(u_n) + \frac{1}{r_n}\langle x - u_n, u_n - w_n\rangle \geq F(x, u_n).
$$

Replacing n by n_k, we get

$$\langle \phi(u_{n_k}), x-u_{n_k}\rangle + U_1(x) - U_1(u_{n_k}) + \frac{1}{r_{n_k}}\langle x-u_{n_k}, u_{n_k}-w_{n_k}\rangle \geq F(x,u_{n_k}). \tag{3.46}$$

For any $t\in(0,1]$ and $x\in C$, let $x_t = tx+(1-t)p$. Since $p\in C$ and $x\in C$, then $x_t\in C$. So from (3.46), we have

$$\begin{aligned}\langle x_t-u_{n_k},\phi(x_t)\rangle &\geq \langle x_t-u_{n_k},\phi(x_t)\rangle - \langle x_t-u_{n_k},\phi(u_{n_k})\rangle\\ &\quad - \left\langle x_t-u_{n_k}, \frac{u_{n_k}-w_{n_k}}{r_{n_k}}\right\rangle\\ &\quad + F(x_t,u_{n_k}) + U_1(u_{n_k}) - U_1(x_t)\\ &= \langle x_t-u_{n_k},\phi(x_t)-\phi(u_{n_k})\rangle - \left\langle x_t-u_{n_k}, \frac{u_{n_k}-w_{n_k}}{r_{n_k}}\right\rangle\\ &\quad + F(x_t,u_{n_k}) + U_1(u_{n_k}) - U_1(x_t).\end{aligned} \tag{3.47}$$

Since ϕ is monotone, then $\langle x_t-u_{n_k},\phi(x_t)-\phi(u_{n_k})\rangle \geq 0$. Therefore by (L4) and the weak lower semicontinuity of U_1, taking the limit of (3.47) (noting that $\|u_{n_k}-w_{n_k}\|\to 0$ and $u_{n_k}\rightharpoonup p$) as $k\to\infty$, we get

$$\langle x_t-p,\phi(x_t)\rangle \geq F(x_t,p) + U_1(p) - U_1(x_t). \tag{3.48}$$

Hence from (L1) and (3.48), we get

$$\begin{aligned}0 &= F(x_t,x_t) + U_1(x_t) - U_1(x_t)\\ &\leq tF(x_t,x) + (1-t)F(x_t,p) + tU_1(x) + (1-t)U_1(p) - U_1(x_t)\\ &= t(F(x_t,x)+U_1(x)-U_1(x_t)) + (1-t)(F(x_t,p)+U_1(p)-U_1(x_t))\\ &\leq t(F(x_t,x)+U_1(x)-U_1(x_t)) + (1-t)\langle x_t-p,\phi(x_t)\rangle\\ &= t(F(x_t,x)+U_1(x)-U_1(x_t)) + (1-t)t\langle x-p,\phi(x_t)\rangle,\end{aligned}$$

which implies that

$$F(x_t,x) + (1-t)\langle x-p,\phi(x_t)\rangle + U_1(x) - U_1(x_t) \geq 0.$$

Letting $t\to 0$, we have

$$F(p,x) + \langle x-p,\phi(p)\rangle + U_1(x) - U_1(p) \geq 0, \qquad \forall\, x\in C.$$

This means that $p\in \mathrm{EP}(F,U_1,\phi)$. Similarly, we can show that $q\in\mathrm{EP}(G,U_2,\varphi)$.

Next, we show that $(p,q)\in \mathrm{VI}(f_1,M_1)\times\mathrm{VI}(f_2,M_2)$. Since M_1 is $\frac{1}{\mu_1}$- Lipschitz monotone and the domain of M_1 is H_1, we obtain from Lemma 2.5 that f_1+M_1 is maximal monotone. Let $(u,w)\in G(f_1,M_1)$, i.e., $w-f_1u\in M_1(u)$ and put $\overline{w}_{n_k} = \bar{x}_{n_k} - \gamma_{n_k}A^*(A\overline{X}_{n_k} - B\bar{y}_{n_k})$. Then $w_{n_k} = J_\lambda^{M_1}(I-f_1)\overline{w}_{n_k}$, which implies that $(I-\lambda f_1)\overline{w}_{n_k} \in (I+\lambda M_1)w_{n_k}$. Using the maximal monotonicity of (f_1+M_1), we get

$$\Big\langle u - w_{n_k}, w - f_1 u - \frac{1}{\lambda}(\overline{w}_{n_k} - \lambda f_1 \overline{w}_{n_k} - w_{n_k})\Big\rangle \geq 0$$

and so

$$\begin{aligned}\langle u - w_{n_k}, w\rangle &\geq \Big\langle u - w_{n_k}, f_1 u + \frac{1}{\lambda}(\overline{w}_{n_k} - \lambda f_1 \overline{w}_{n_k} - w_{n_k})\Big\rangle \\ &= \Big\langle u - w_{n_k}, f_1 u - f_1 w_{n_k} + f_1 w_{n_k} \\ &\quad - f_1 \overline{w}_{n_k} + \frac{1}{\lambda}(\overline{w}_{n_k} - w_{n_k})\Big\rangle \\ &\geq \langle u - w_{n_k}, f_1 w_{n_k} - f_1 \overline{w}_{n_k}\rangle + \Big\langle u - w_{n_k}, \frac{1}{\lambda}(\overline{w}_{n_k} - w_{n_k})\Big\rangle.\end{aligned} \tag{3.49}$$

Note that

$$\|\overline{w}_{n_k} - \bar{x}_{n_k}\| = \gamma_{n_k}\|A^*(A\bar{x}_{n_k} - B\bar{y}_{n_k})\| \to 0, \qquad \text{as } \ k \to \infty,$$

and

$$\|w_{n_k} - \overline{w}_{n_k}\| \leq \|w_{n_k} - \bar{x}_{n_k}\| + \|\bar{x}_{n_k} - \overline{w}_{n_k}\| \to 0, \qquad \text{as } \ k \to \infty.$$

Therefore, taking limit of (3.49), we have

$$\lim_{k\to\infty}\langle u - w_{n_k}, w\rangle = \langle u - p, w\rangle \geq 0.$$

Since $f_1 + M_1$ is maximal monotone, hence $0 \in (f_1 + M_1)p$. Therefore $p \in \mathrm{VI}(f_1, M_1)$. Similarly, we can show that $q \in \mathrm{VI}(f_2, M_2)$.

Finally, we show that $(p, q) \in \big(\bigcap_{i=1}^{\infty} F(S_i)\big) \times \big(\bigcap_{j=1}^{\infty} F(T_j)\big)$ and $Ap = Bq$. Since $\|S_i u_n - u_n\| \to 0$ and $\|u_n - x_n\| \to 0$ as $n \to \infty$, using the demiclosedness principle, $p \in F(S_i)$ for each $i \in \mathbb{N}$. Hence $p \in \bigcap_{i=1}^{\infty} F(S_i)$. Similarly, $q \in \bigcap_{j=1}^{\infty} F(T_j)$. Also, A and B are bounded linear operators, then $A\bar{x}_{n_k} \rightharpoonup Ap$ and $B\bar{y}_{n_k} \rightharpoonup Bq$. By the weakly lower semicontinuity of the squared norm and (3.33), we have

$$\|Ap - Bq\|^2 \leq \liminf_{n\to\infty}\|A\bar{x}_{n_k} - B\bar{y}_{n_k}\|^2 = 0.$$

Therefore, $Ap = Bq$. Thus, we have shown that $(p, q) \in \Gamma$. □

We now show the main strong convergence result.

Theorem 3.7. *The sequence $\{(x_n, y_n)\}$ generated by Algorithm* 3.1 *strongly converges to a solution $(p^*, q^*) \in \Gamma$ which solves the variational inequality*

$$\begin{aligned}&\langle (I - h_1)p^*, p - p^*\rangle \geq 0, \\ &\langle (I - h_2)q^*, q - q^*\rangle \geq 0,\end{aligned} \tag{3.50}$$

for all $(p, q) \in \Gamma$.

Proof: Let $(p^*, q^*) \in \Gamma$ be the solution of the variational inequality (3.50) , from Lemma 3.5, we see that

$$\varrho_{n+1}(p^*, q^*) \leq (1-\Theta_n)\varrho_n(p^*, q^*) + \Theta_n\sigma_n, \tag{3.51}$$

and

$$\begin{aligned}
\varrho_n(p^*, q^*) &= \|x_n - p^*\|^2 + \|y_n - q^*\|^2, \\
\Theta_n &= \frac{2\xi_n(1-k)}{1-\xi_n k}, \\
\sigma_n &= \frac{\langle h_1(p^*) - p^*, x_{n+1} - p^*\rangle + \langle h_2(q^*) - q^*, y_{n+1} - q^*\rangle}{1-k} \\
&\quad + \frac{\xi_n}{2(1-k)}\varrho_n(p^*, q^*) + \frac{\alpha_n c^*}{2\xi_n(1-k)}(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|)
\end{aligned}$$

for some $c^* > 0$.

Secondly, from (3.1) and (3.6), we have

$$\begin{aligned}
&\varrho_{n+1}(p^*, q^*) \\
&\quad \leq (1-\xi_n)(\|\bar{u}_n - p^*\|^2 + \|\bar{v}_n - q^*\|^2) \\
&\qquad + \xi_n(\|h_1(x_n) - p^*\|^2 + \|h_2(y_n) - q^*\|^2) \\
&\quad \leq (1-\xi_n)(\|u_n - p^*\|^2 + \|v_n - q^*\|^2) \\
&\qquad - (1-\xi_n)\Big(\beta_{n,0}\sum_{i=1}^{\infty}\beta_{n,i}\|S_i u_n - u_n\|^2 + \delta_{n,0}\sum_{j=1}^{\infty}\delta_{n,j}\|T_j v_n - v_n\|^2\Big) \\
&\qquad + \xi_n(\|h_1(x_n) - p^*\|^2 + \|h_2(y_n) - q^*\|^2) \\
&\quad \leq \|u_n - p^*\|^2 + \|v_n - q^*\|^2 \qquad (3.52)\\
&\qquad - (1-\xi_n)\Big(\beta_{n,0}\sum_{i=1}^{\infty}\beta_{n,i}\|S_i u_n - u_n\|^2 + \delta_{n,0}\sum_{j=1}^{\infty}\delta_{n,j}\|T_j v_n - v_n\|^2\Big) \\
&\qquad + \xi_n(\|h_1(x_n) - p^*\|^2 + \|h_2(y_n) - q^*\|^2) \\
&\quad \leq \varrho_n(p^*, q^*) + \alpha_n c^*(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|) \\
&\qquad - (1-\xi_n)\Big(\beta_{n,0}\sum_{i=1}^{\infty}\beta_{n,i}\|S_i u_n - u_n\|^2 + \delta_{n,0}\sum_{j=1}^{\infty}\delta_{n,j}\|T_j v_n - v_n\|^2\Big) \\
&\qquad + \xi_n(\|h_1(x_n) - p^*\|^2 + \|h_2(y_n) - q^*\|^2).
\end{aligned}$$

By setting

$$(3.53)\quad \begin{aligned} \lambda_n &= \xi_n(\|h_1(x_n) - p^*\|^2 + \|h_2(y_n) - q^*\|^2) \\ &\qquad + \alpha_n c^*(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|), \\ \tau_n &= (1-\xi_n)\Bigg(\beta_{n,0}\sum_{i=1}^{\infty}\beta_{n,i}\|S_i u_n - u_n\|^2 + \delta_{n,0}\sum_{j=1}^{\infty}\delta_{n,j}\|T_j v_n - v_n\|^2\Bigg), \end{aligned}$$

(3.53) can be rewritten in the form

$$(3.54)\quad \varrho_{n+1}(p^*, q^*) \le \varrho_n(p^*, q^*) - \tau_n + \lambda_n.$$

Using condition (C1) of Assumption 3.2 in (3.51) and (3.54), it is easy to see that $\{\Theta_n\} \subset (0,1)$, $\Theta_n \to 0$, $\sum_{n=0}^{\infty}\Theta_n = \infty$ and $\lambda_n \to 0$. In order to use Lemma 2.6, it suffices to verify that for each subsequence $\{n_j\} \subset \{n\}$, $\lim_{j\to\infty}\tau_{n_j} = 0$ implies that $\limsup_{j\to\infty}\sigma_{n_j} \le 0$. Note that from Remark 3.3 and the fact that $\xi_{n_j} \to 0$, we have

$$(3.55)\quad \frac{\xi_{n_j}}{2(1-k)}\varrho_{n_j}(p^*, q^*) + \frac{\alpha_{n_j}c^*}{2\xi_{n_j}(1-k)}(\|x_{n_j} - x_{n_j-1}\| + \|y_{n_j} - y_{n_j-1}\|) \to 0,$$

as $j \to \infty$. Now we show that

$$\limsup_{j\to\infty}(\langle h_1(p^*) - p^*, x_{n_j+1} - p^*\rangle + \langle h_2(q^*) - q^*, y_{n_j+1} - q^*\rangle) \le 0.$$

Indeed

$$(3.56)\quad \begin{aligned} &\limsup_{j\to\infty}(\langle h_1(p^*) - p^*, x_{n_j+1} - p^*\rangle + \langle h_2(q^*) - q^*, y_{n_j+1} - q^*\rangle) \\ &\qquad = -\liminf_{j\to\infty}(\langle (I-h_1)p^*, x_{n_j+1} - p^*\rangle + \langle (I-h_2)q^*, y_{n_j+1} - q^*\rangle). \end{aligned}$$

We can take a subsequence of $\{(x_{n_j}, y_{n_j})\}$ still denoted by $\{(x_{n_j}, y_{n_j})\}$ such that $(x_{n_j}, y_{n_j}) \rightharpoonup (p, q)$ as $j \to \infty$. Then

$$(3.57)\quad \begin{aligned} & -\liminf_{j\to\infty}(\langle (I-h_1)p^*, x_{n_j+1} - p^*\rangle + \langle (I-h_2)q^*, y_{n_j+1} - q^*\rangle) \\ &= -\lim_{j\to\infty}(\langle (I-h_1)p^*, x_{n_j+1} - p^*\rangle + \langle (I-h_2)q^*, y_{n_j+1} - q^*\rangle) \\ &= -(\langle (I-h_1)p^*, p - p^*\rangle + \langle (I-h_2)q^*, q - q^*\rangle). \end{aligned}$$

Since (p^*, q^*) is the solution of the variational inequality (3.50), then (3.56) and (3.57), we have

$$(3.58)\quad \limsup_{j\to\infty}(\langle h_1(p^*) - p^*, x_{n_j+1} - p^*\rangle + \langle h_2(q^*) - q^*, y_{n_j+1} - q^*\rangle) \le 0.$$

Hence from (3.55) and (3.58), we have $\limsup_{j\to\infty} \sigma_{n_j} \le 0$. Therefore from Lemma 2.6, it follows that

$$\lim_{n\to\infty} (\|x_n - p^*\|^2 + \|y_n - q^*\|^2) = 0,$$

which implies that $(x_n, y_n) \to (p^*, q^*)$. Hence, the sequence $\{(x_n, y_n)\}$ converges strongly to the solution (p^*, q^*). This completes the proof. □

4. Numerical example

In this section, we consider a numerical example to demonstrate the efficiency and accuracy of our Algorithm 3.1. We compare the inertial Algorithm 3.1 with non-inertial algorithm by taking $\alpha_n = 0$ in Algorithm 3.1.

Example 4.1. Let $H_1 = H_2 = \mathbb{R}^3$ and $H_3 = \mathbb{R}^5$. Let C be an Euclidean ball defined by

$$C := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \colon \|x\|_2 \le 4\}$$

and Q is the half-space defined by

$$Q := \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 \colon \langle y, b\rangle \le 0\}$$

where $b = (-1, 2, -3)$. We define the bifunction $F\colon C \times C \to \mathbb{R}$ by $F(x, u) = -\frac{1}{2}x^2 + \frac{1}{2}u^2$, $\phi\colon C \to H_1$ by $\phi(x) = x$ and $U_1\colon C \to \mathbb{R}\cup\{\infty\}$ by $U_1(x) = \frac{1}{2}x^2$. It is easy to see that

$$T^F_{r_n}(x) = \frac{x}{3r_n + 1}, \qquad \forall\, x \in C.$$

Also, we define the bifunction $G\colon Q \times Q \to \mathbb{R}$ by $G(y, v) = -3y^2 + 2yv + v^2$, $\varphi\colon Q \to H_2$ by $\varphi(y) = 2y$ and $U_2\colon Q \to \mathbb{R}\cup\{\infty\}$ by $U_2(y) = y^2$. Then

$$T^G_{r_n}(y) = \frac{y}{6r_n + 1}, \qquad \forall\, y \in Q.$$

Let $M_1\colon H_1 \to H_1$ be defined by $M_1(x_1, x_2, x_3) = (-2x_1, -3x_2, x_3)$ and $f_1\colon H_1 \to H_1$ be defined by $f_1(x_1, x_2, x_3) = (2x_1, 2x_2, 2x_3)$. After a simple calculation, we see that

$$J^{M_1}_{\lambda}(I - \lambda f_1)x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1-2\lambda}{1-3\lambda} & 0 \\ 0 & 0 & \frac{1-2\lambda}{1+\lambda} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

for $x = (x_1, x_2, x_3) \in H_1$. Also, let $M_2\colon H_2 \to H_2$ be defined by $M_2(y_1, y_2, y_3) = (4y_1, 4y_2, 2y_3)$ and $f_2\colon H_2 \to H_2$ be defined by $f_2(y_1, y_2, y_3) = (y_1 - y_2, 3y_2, y_3/4)$.

Then we get

$$J_\lambda^{M_2}(I-\lambda f_2)y = \begin{pmatrix} \frac{1-\lambda}{1+4\lambda} & \frac{\lambda}{1+4\lambda} & 0 \\ 0 & \frac{1-3\lambda}{1+4\lambda} & 0 \\ 0 & 0 & \frac{4-\lambda}{4(1+2\lambda)} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

for $y = (y_1, y_2, y_3) \in H_2$. The bounded linear operators $A\colon \mathbb{R}^3 \to \mathbb{R}^5$ and $B\colon \mathbb{R}^3 \to \mathbb{R}^5$ are defined by

$$A(x_1,x_2,x_3) = \left(2x_1 + x_2 - x_3, 4x_1 - x_2 + 3x_3, \frac{x_1 - x_2}{3}, x_2 + x_3, x_2\right)$$

and

$$B(y_1,y_2,y_3) = \left(\frac{y_1+y_2}{2}, y_2, 3(y_1+y_3), 2y_1 - 3y_2 + 4y_3, y_1 + y_2 + y_3\right).$$

Now for each $i, j \in \mathbb{N}$, let $S_i\colon H_1 \to H_1$ and $T_j\colon H_2 \to H_2$ be defined by

$$S_i x = \begin{cases} 0, & \text{if } x < 0, \\ \frac{x}{i+1}, & \text{if } x \geq 0, \end{cases}$$

and

$$T_j y = \begin{cases} \frac{|3y|}{2j+4}, & \text{if } 3y < j+2, \\ 1, & \text{if } 3y \geq j+2. \end{cases}$$

It is not difficult to see that S_i and T_j are quasi-nonexpansive mappings. For each $n \in \mathbb{N}$, $i = j \geq 0$, define

$$\beta_{n,i} = \begin{cases} 0, & \text{if } n < i, \\ 1 - \frac{n}{n+1}\left(\sum\limits_{k=1}^{\infty} \frac{1}{2^k}\right), & \text{if } n = i, \\ \frac{1}{2^{i+1}}\left(\frac{n}{n+1}\right), & \text{if } n > i, \end{cases} \tag{4.1}$$

and $\delta_{n,j} = \beta_{n,i}$. It is easy to see that $\lim_{n\to\infty} \beta_{n,i} = 1/2^{i+1}$, and $\sum_{i=0}^{\infty} \beta_{n,i} = 1 = \sum_{j=0}^{\infty} \delta_{n,j}$.

We choose $\lambda = \theta = 1/4$, $r_n = 2n/(4n+1)$, $\xi_n = 1/(n+1)^p$, $\omega_n = 1/(n+1)^{2p}$, and take different choices of p as follow:

Choice (i): $p = 0.3$. Choice (ii): $p = 0.5$. Choice (iii): $p = 1$.

We define the contraction mappings $h_i\colon H_i \to H_i$ by $h_i(w) = w/(4i)$ for all $w \in H_i$, $i = 1, 2$. It is easy to verify that all conditions (C1)–(C4) are satisfied and Algorithm 3.1 becomes:

$$
\begin{aligned}
(\bar{x}_n, \bar{y}_n) &= (x_n, y_n) + \alpha_n(x_n - x_{n-1}, y_n - y_{n-1}),\\
w_n &= J_\lambda^{M_1}(I - \lambda f_1)(\bar{x}_n - \gamma_n A^*(A\bar{x}_n - B\bar{y}_n)),\\
z_n &= J_\lambda^{M_2}(I - \lambda f_2)(\bar{y}_n + \gamma_n B^*(A\bar{x}_n - B\bar{y}_n)),\\
u_n &= \frac{w_n}{3r_n + 1},\\
v_n &= \frac{z_n}{6r_n + 1},\\
x_{n+1} &= \xi_n h_1(x_n) + (1 - \xi_n)\Bigg[\frac{n}{2n+2}\Bigg(u_n - S_n u_n\\
&\qquad + \sum_{i=1}^{n-1} \frac{1}{2^i}(S_i u_n - S_n u_n)\Bigg) + S_n u_n\Bigg],\\
y_{n+1} &= \xi_n h_2(y_n) + (1 - \xi_n)\Bigg[\frac{n}{2n+2}\Bigg(v_n - T_n v_n\\
&\qquad + \sum_{j=1}^{n-1} \frac{1}{2^j}(T_j v_n - T_n v_n)\Bigg) + T_n v_n\Bigg],
\end{aligned} \tag{4.2}
$$

where α_n and γ_n are as defined in (3.2) and (3.3), respectively. Using

$$\frac{\|x_{n+1} - x_n\|^2 + \|y_{n+1} - y_n\|^2}{\|x_2 - x_1\|^2 + \|y_2 - y_1\|^2} < 10^{-4}$$

as the stopping criterion, we consider various values of the initial points (x_0, y_0) and (x_1, y_1) as follows:

Case I: $x_0 = (2, 2, 0)$, $y_0 = (2, 1, 1)$,
$x_1 = (1, 1, 1)$, $y_1 = (-1, 1, 2)$,

Case II: $x_0 = \left(\frac{1}{2}, 0, 1\right)$, $y_0 = (1, 1, 2)$,
$x_1 = \left(\frac{1}{2}, 2, -3\right)$, $y_1 = (2, 0, 2)$,

Case III: $x_0 = (-2, 2, 2)$, $y_0 = (3, 1, 2)$,
$x_1 = \left(\frac{2}{5}, \frac{1}{4}, \frac{1}{10}\right)$, $y_1 = (1, -2, 0)$,

Case IV: $x_0 = (1, 1, 1)$, $y_0 = (1, -4, 2)$,
$x_1 = (2, 1, 2)$, $y_1 = (-1, 2, 4)$.

The numerical results are reported in Table 1 and Figures 1–4.

Remark 4.2. From the example above, we conclude that

(1) The inertial Algorithm 3.1 performs better than the non-inertial algorithm, i.e., when $\alpha_n = 0$.

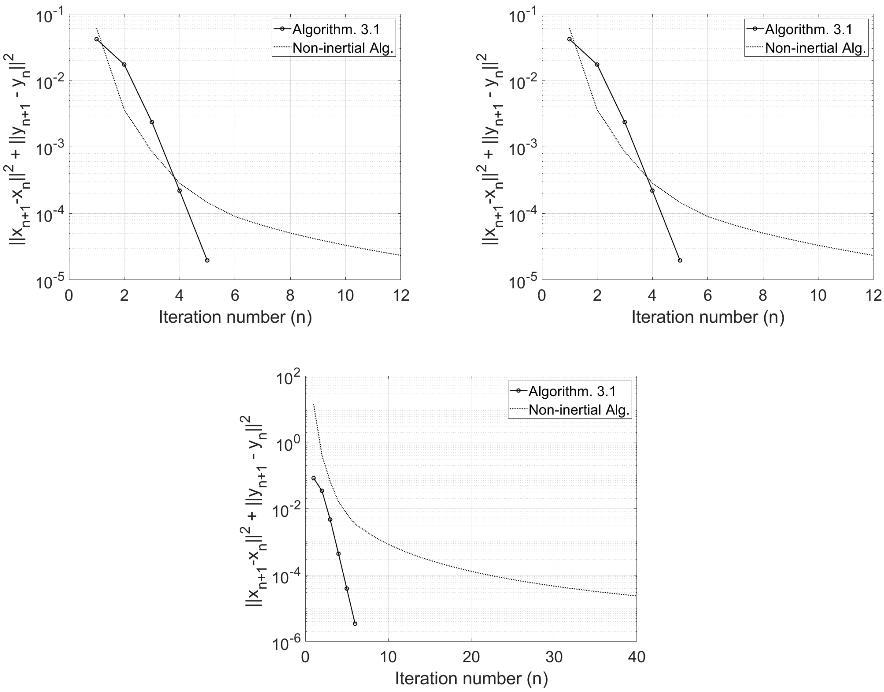

FIGURE 1. Example 4.1 Case I:
Top Left: Choice I; Top Right: Choice II; Bottom: Choice III.

(2) It is evident that the change in the initial data does not have any significant change in the number of iterations nor the cpu-time taken for computation.

Case		Choice I		Choice II		Choice III	
		Alg. 3.1	Non-in.	Alg. 3.1	Non-in.	Alg. 3.1	Non-in.
I	No. Iteration	5	12	4	14	6	40
	Time (Sec.)	0.0078	0.0294	0.0021	0.0503	0.0056	1.2390
II	No. Iteration	5	13	6	13	7	40
	Time (Sec.)	0.0043	0.0974	0.0023	0.0635	0.0049	1.0164
III	No. Iteration	5	13	5	13	7	40
	Time (Sec.)	0.0180	0.6764	0.0852	0.1597	0.1870	0.7870
IV	No. Iteration	5	13	5	18	9	57
	Time (Sec.)	0.0099	0.0941	0.0213	0.0579	0.1464	1.0794

TABLE 1. Table showing computation results for Example 4.1.

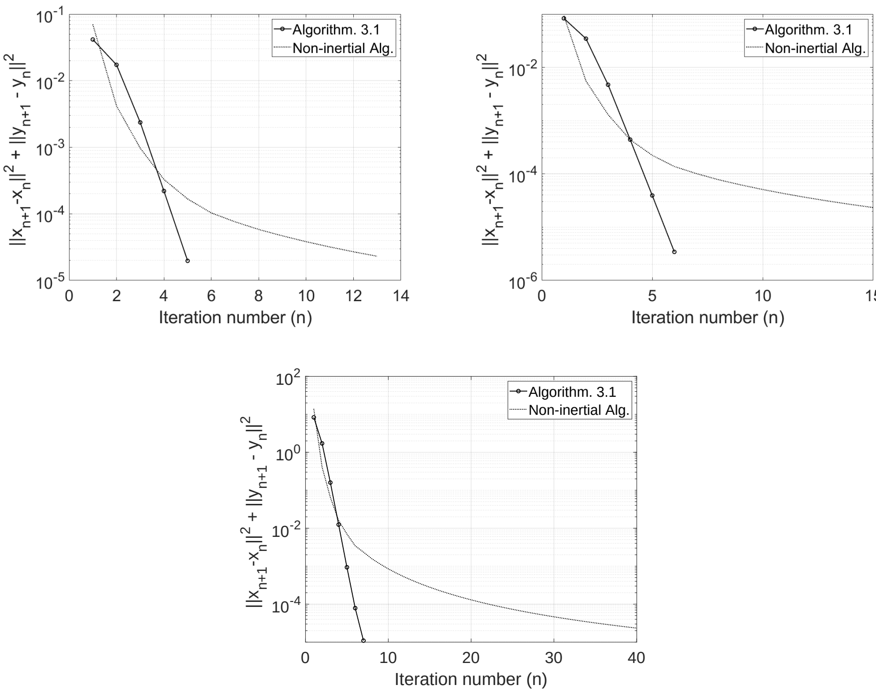

Figure 2. Example 4.1 Case II:
Top Left: Choice I; Top Right: Choice II; Bottom: Choice III.

Acknowledgement. The authors thank the referee of this paper whose valuable comments and suggestions have improved the presentation of the paper.

References

[1] Attouch H., Bolte J., Redont P., Soubeyran A., *Alternating proximal algorithms for weakly coupled convex minimization problems. Applications to dynamical games and PDE's*, J. Convex Anal. **15** (2008), no. 3, 485–506.

[2] Attouch H., Cabot A., Frankel P., Peypouquet J., *Alternating proximal algorithms for linearly constrained variational inequalities: application to domain decomposition for PDE's*, Nonlinear Anal. **74** (2011), no. 18, 7455–7473.

[3] Attouch H., Czarnecki M. O., *Asymptotic control and stabilization of nonlinear oscillators with non-isolated equilibria*, J. Differential Equations **179** (2002), no. 1, 278–310.

[4] Attouch H., Goudou X., Redont P., *The heavy ball with friction. I. The continuous dynamical system: global exploration of the local minima of a real-valued function by asymptotic analysis of a dissipative dynamical system*, Commun. Contemp. Math. **2** (2000), no. 1, 1–34.

[5] Attouch H., Peypouquet J., Redont P., *A dynamical approach to an inertial forward–backward algorithm for convex minimization*, SIAM J. Optim. **24**, (2014), no. 1, 232–256.

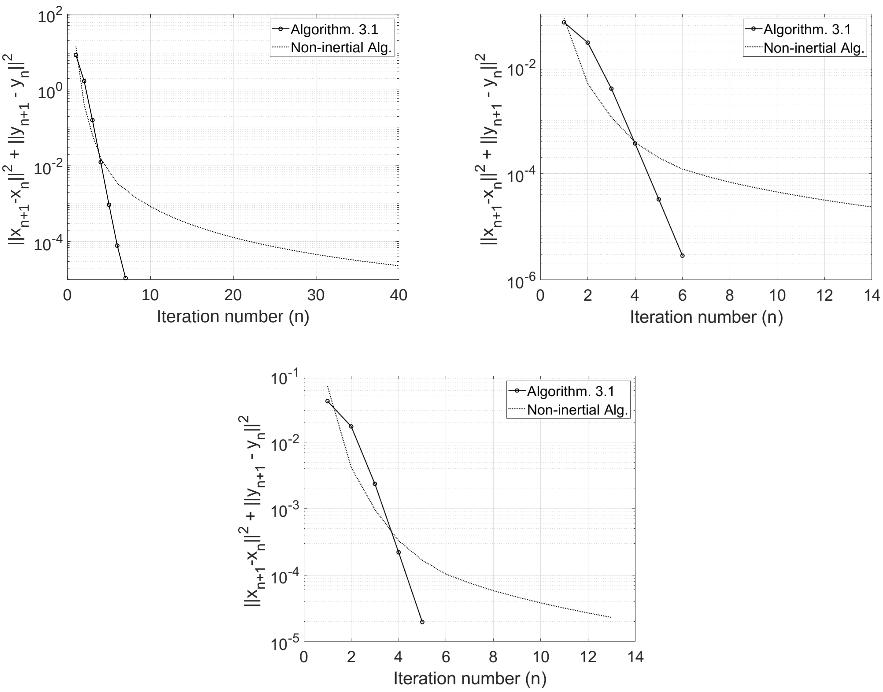

Figure 3. Example 4.1 Case III:
Top Left: Choice I; Top Right: Choice II; Bottom: Choice III.

[6] Blum E., Oettli W., *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), no. 1–4, 123–145.

[7] Boţ R. I., Csetnek E. R., *A hybrid proximal-extragradient algorithm with inertial effects*, Numer. Funct. Anal. Optim. **36** (2015), no. 8, 951–963.

[8] Boţ R. I., Csetnek E. R., *An inertial forward-backward-forward primal-dual splitting algorithm for solving monotone inclusion problems*, Numer. Algorithms **71** (2016), no. 3, 519–540.

[9] Boţ R. I., Csetnek E. R., Hendrich C., *Inertial Douglas–Rachford splitting for monotone inclusion problems*, Appl. Math. Comput. **256** (2015), 472–487.

[10] Boţ R. I., Csetnek E. R., László S. C., *An inertial forward-backward algorithm for the minimization of the sum of two nonconvex functions*, EURO J. Comput. Optim. **4** (2016), no. 1, 3–25.

[11] Byrne C. L., Moudafi A., *Extensions of the CQ algorithm for the split feasibility and split equality problems*, J. Nonlinear Convex Anal. **18** (2017), no. 8, 1485–1496.

[12] Censor Y., *Parallel application of block-iterative methods in medical imaging and radiation therapy*, Math. Programming **42** (1988), no. 2, (Ser. B), 307–325.

[13] Censor Y., Bortfeld T., Martin B., Trofimov A., *A unified approach for inversion problems in intensity-modulated radiation therapy*, Phys. Med. Biol. **51** (2006), no. 10, 2353–2365.

[14] Chang S.-S., Wang L., Wang X. R., Wang G., *General split equality equilibrium problems with application to split optimization problems*, J. Optim. Theory Appl. **166** (2015), no. 2, 377–390.

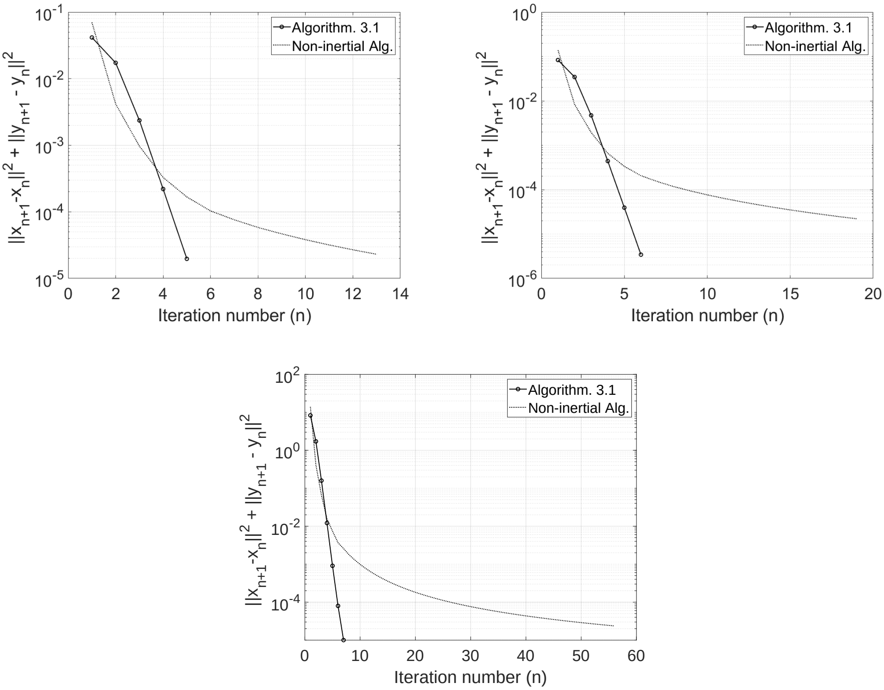

FIGURE 4. Example 4.1 Case IV:
Top Left: Choice I; Top Right: Choice II; Bottom: Choice III.

[15] Chen C., Chan R. H., Ma S., Yang J., *Inertial proximal ADMM for linearly constrained separable convex optimization*, SIAM J. Imaging Sci. **8** (2015), no. 4, 2239–2267.

[16] Cholamjiak W., Pholasa N., Suantai S., *A modified inertial shrinking projection method for solving inclusion problems and quasi-nonexpansive multivalued mappings*, Comput. Appl. Math. **37** (2018), no. 5, 5750–5774.

[17] Chuang C.-S., *Hybrid inertial proximal algorithm for the split variational inclusion problem in Hilbert spaces with applications*, Optimization **66** (2017), no. 5, 777–792.

[18] Dong Q.-L., Lu Y.-Y., Yang J., *The extragradient algorithm with inertial effects for solving the variational inequality*, Optimization **65** (2016), no. 12, 2217–2226.

[19] Guo H., He H., Chen R., *Strong convergence theorems for the split equality variational inclusion problem and fixed point problem in Hilbert spaces*, Fixed Point Theory Appl. (2015), 2015:223, 18 pages.

[20] He Z., *The split equilibrium problem and its convergence algorithms*, J. Inequal. Appl. (2012), 2012:162, 15 pages.

[21] Jolaoso L. O., Abass H. A., Mewomo O. T., *A viscosity-proximal gradient method with inertial extrapolation for solving certain minimization problems in Hilbert space*, Arch. Math. (Brno) **55** (2019), no. 3, 167–194.

[22] Jolaoso L. O., Alakoya T. O., Taiwo A., Mewomo O. T., *A parallel combination extragradient method with Armijo line searching for finding common solutions of finite families of equilibrium and fixed point problems*, Rend. Circ. Mat. Palermo (2) **69** (2019), 711–735.

[23] Jolaoso L. O., Ogbuisi F. U., Mewomo O. T., *An iterative method for solving minimization, variational inequality and fixed point problems in reflexive Banach spaces*, Adv. Pure Appl. Math. **9** (2018), no. 3, 167–184.
[24] Jolaoso L. O., Oyewole K. O., Okeke C. C., Mewomo O. T., *A unified algorithm for solving split generalized mixed equilibrium problem, and for finding fixed point of nonspreading mapping in Hilbert spaces*, Demonstr. Math. **51** (2018), no. 1, 211–232.
[25] Jolaoso L. O., Taiwo A., Alakoya T. O., Mewomo O. T., *A self adaptive inertial subgradient extragradient algorithm for variational inequality and common fixed point of multivalued mappings in Hilbert spaces*, Demonstr. Math. **52** (2019), no. 1, 183–203.
[26] Kazmi K. R., Rizvi S. H., *An iterative method for split variational inclusion problem and fixed point problem for a nonexpansive mapping*, Optim. Lett. **8** (2014), no. 3, 1113–1124.
[27] Latif A., Eslamian M., *Split equality problem with equilibrium problem, variational inequality problem, and fixed point problem of nonexpansive semigroups*, J. Nonlinear Sci. Appl. **10** (2017), no. 6, 3217–3230.
[28] Lemaire B., *Which fixed point does the iteration method select?*, Recent Advances in Optimization, Trier, 1996, Lecture Notes in Econom. and Math. Systems, 452, Springer, Berlin, 1997, pages 154–157.
[29] Li S., Li L., Cao L., He X., Yue X., *Hybrid extragradient method for generalized mixed equilibrium problem and fixed point problems in Hilbert space*, Fixed Point Theory Appl. (2013), 2013:240, 13 pages.
[30] Lin L.-J., Chen Y.-D., Chuang C.-S., *Solutions for a variational inclusion problem with applications to multiple sets split feasibility problems*, Fixed Point Theory Appl. (2013), 2013:333, 21 pages.
[31] López G., Martín-Márquez V., Wang F., Xu H.-K., *Solving the split feasibility problem without prior knowledge of matrix norm*, Inverse Problems **28** (2012), no. 8, 085004, 18 pages.
[32] Ma Z., Wang L., Chang S.-S., Duan W., *Convergence theorems for split equality mixed equilibrium problems with applications*, Fixed Point Theory Appl. (2015), 2015:31, 18 pages.
[33] Maingé P.-E., *Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **325** (2007), no. 1, 469–479.
[34] Maingé P.-E., *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization*, Set-Valued Anal. **16** (2008), no. 7–8, 899–912.
[35] Marino G., Xu H.-K., *Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces*, J. Math. Anal. Appl. **329** (2007), no. 1, 336–346.
[36] Matinez-Yanes C., Xu H.-K., *Strong convergence of the CQ method for fixed point iteration processes*, Nonlinear Anal. **64** (2006), no. 11, 2400–2411.
[37] Mewomo O. T., Ogbuisi F. U., *Convergence analysis of an iterative method for solving multiple-set split feasibility problems in certain Banach spaces*, Quaest. Math. **41** (2018), no. 1, 129–148.
[38] Moudafi A., *A note on the split common fixed-point problem for quasi-nonexpansive operators*, Nonlinear Anal. **74** (2011), no. 12, 4083–4087.
[39] Moudafi A., *Split monotone variational inclusions*, J. Optim. Theory Appl. **150** (2011), no. 2, 275–283.
[40] Moudafi A., *Alternating CQ-algorithms for convex feasibility and split fixed-point problems*, J. Nonlinear Convex Anal. **15** (2014), no. 4, 809–818.
[41] Moudafi A., Al-Shemas E., *Simultaneous iterative methods for split equality problems and applications*, Trans. Math. Program. Appl. **1** (2013), 1–11.
[42] Ochs P., Brox T., Pock T., *iPiasco: inertial proximal algorithm for strongly convex optimization*, J. Math. Imaging Vision **53** (2015), no. 2, 171–181.
[43] Rahaman M., Liou Y.-C., Ahmad R., Ahmad I., *Convergence theorems for split equality generalized mixed equilibrium problems for demi-contractive mappings*, J. Inequal. Appl. (2015), 2015:418, 25 pages.

[44] Rockafellar R. T., *Monotone operators and the proximal point algorithm*, SIAM J. Control. Optim. **14** (1976), no. 5, 877–898.

[45] Shehu Y., Mewomo O. T., *Further investigation into split common fixed point problem for demicontractive operators*, Acta Math. Sin. (Engl. Ser.) **32** (2016), no. 11, 1357–1376.

[46] Shehu Y., Mewomo O. T., Ogbuisi F. U., *Further investigation into approximation of a common solution of fixed point problems and split feasibility problems*, Acta Math. Sci. Ser. B (Engl. Ed.) **36** (2016), no. 3, 913–930.

[47] Shukla R., Pant R., *Approximating solution of split equality and equilibrium problems by viscosity approximation algorithms*, Comput. Appl. Math. **37** (2018), no. 4, 5293–5314.

[48] Taiwo A., Jolaoso L. O., Mewomo O. T., *A modified Halpern algorithm for approximating a common solution of split equality convex minimization problem and fixed point problem in uniformly convex Banach spaces*, Comput. Appl. Math. **38** (2019), no. 2, Paper No. 77, 28 pages.

[49] Taiwo A., Jolaoso L. O., Mewomo O. T., *Parallel hybrid algorithm for solving pseudomonotone equilibrium and split common fixed point problems*, Bull. Malays. Math. Sci. Soc. **43** (2020), no. 2, 1893–1918.

[50] Thong D. V., Hieu D. V., *An inertial method for solving split common fixed point problems*, J. Fixed Point Theory Appl. **19** (2017), no. 4, 3029–3051.

[51] Thong D. V., Hieu D. V., *Inertial subgradient extragradient algorithms with line-search process for solving variational inequality problems and fixed point problems*, Numer. Algorithms **80** (2019), no. 4, 1283–1307.

[52] Zegeye H., Shahzad N., *Convergence of Mann's type iteration method for generalized asymptotically nonexpansive mappings*, Comput. Math. Appl. **62** (2011), no. 11, 4007–4014.

[53] Zhao J., *Solving split equality fixed-point problem of quasi-nonexpanive mappings without prior knowledge of operators norms*, Optimization **64** (2015), no. 12, 2619–2630.

[54] Zhao J., He S., *Strong convergence of the viscosity approximation process for the split common fixed-point problem of quasi-nonexpansive mappings*, J. Appl. Math. **2012** (2012), Art. ID 438023, 12 pages.

[55] Zhao J., Wang S., *Viscosity approximation methods for the split equality common fixed point problem of quasi-nonexpansive operators*, Acta Math. Sci. Ser. B (Engl. Ed.) **36** (2016), no. 5, 1474–1486.

[56] Zhao J., Yang Q., *A simple projection method for solving the multiple-sets split feasibility problem*, Inverse Probl. Sci. Eng. **21** (2013), no. 3, 537–546.

L. O. Jolaoso:
School of Mathematics, Statistics and Computer Science,
University of KwaZulu-Natal, Private Bag X 54001, 4000 Durban,
South Africa

E-mail: 216074984@stu.ukzn.ac.za

E-mail: lateefjolaoso89@gmail.com

O. T. Mewomo:
School of Mathematics, Statistics and Computer Science,
University of KwaZulu-Natal, Private Bag X 54001, 4000 Durban,
South Africa

E-mail: mewomoo@ukzn.ac.za

(*Received* January 28, 2019, *revised* August 26, 2019)