

Exponential domination in function spaces

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Abstract. Given a Tychonoff space X and an infinite cardinal κ , we prove that exponential κ -domination in X is equivalent to exponential κ -cofinality of $C_p(X)$. On the other hand, exponential κ -cofinality of X is equivalent to exponential κ -domination in $C_p(X)$. We show that every exponentially κ -cofinal space X has a κ^+ -small diagonal; besides, if X is κ -stable, then $nw(X) \leq \kappa$. In particular, any compact exponentially κ -cofinal space has weight not exceeding κ . We also establish that any exponentially κ -cofinal space X with $l(X) \leq \kappa$ and $t(X) \leq \kappa$ has i -weight not exceeding κ while for any cardinal κ , there exists an exponentially ω -cofinal space X such that $l(X) \geq \kappa$.

Keywords: exponential κ -domination; exponential κ -cofinality; κ -stable space; i -weight; function space; duality; κ^+ -small diagonal

Classification: 54C35, 54C05, 54G20

1. Introduction

It was proved in the paper [6] that a regular space X must have density not exceeding κ if any subset of X of cardinality $(2^\kappa)^+$ is κ -dominated, i.e., contained in the closure of a set $B \subset X$ such that $|B| \leq \kappa$. This makes it natural to study spaces X with exponential κ -domination, in which every set $A \subset X$ with $|A| \leq 2^\kappa$ is κ -dominated. It was established in [6] that spaces with exponential κ -domination have nice categorical properties: they are preserved by continuous images, products with 2^κ -many factors and κ -unions. Any Čech-complete space with exponential κ -domination turns out to have density less than or equal to κ as well as any space X with $\chi(X) \leq \kappa$. It was also shown in [6] that there exist non-separable spaces of countable pseudocharacter with exponential ω -domination.

In this paper we introduce a new property called *exponential κ -cofinality* for any infinite cardinal κ . A space X is exponentially κ -cofinal if for any continuous onto map $f: X \rightarrow Y$ such that $w(Y) \leq 2^\kappa$, there exist continuous surjective maps $g: X \rightarrow Z$ and $h: Z \rightarrow Y$ such that $iw(Z) \leq \kappa$ and $h \circ g = f$. Exponentially κ -cofinal spaces generalize the class of spaces of i -weight not exceeding κ and give global information about a space via its small continuous images. The importance of this class stems from the fact that it is bidual to the class of spaces with

exponential κ -domination: we will prove that a space X features exponential κ -domination if and only if the function space $C_p(X)$ is exponentially κ -cofinal and X is exponentially κ -cofinal if and only if $C_p(X)$ is a space with exponential κ -domination.

The class of exponentially κ -cofinal spaces turned out to have nontrivial relationships with some classical properties which makes it interesting in itself. We will show that the diagonal of any exponentially κ -cofinal space is κ^+ -small. The concept of λ -small diagonal was introduced in [8] (under a different name) where it was proved, among other things, that, under CH, a compact space of countable tightness is metrizable whenever it has an ω_1 -small diagonal. Later it was proved in [9] that the requirement of countable tightness can be omitted in some models of ZFC (Zermelo–Fraenkel set theory). We will show that exponential ω -cofinality of a compact space X implies metrizability of X so this property is strictly stronger than having a small diagonal.

More generally, if X is an exponentially κ -cofinal space which is κ -stable, then $nw(X) \leq \kappa$. It is worth noting that the concept of κ -stability was introduced and studied in [1]. Last, but not least, we show that an exponentially κ -cofinal space X with $l(X) \leq \kappa$ and $t(X) \leq \kappa$ must have i -weight not exceeding κ while exponentially ω -cofinal spaces can have Lindelöf number as large as we wish.

2. Notation and terminology

In this paper all spaces are assumed to be Tychonoff. If X is a space, then $\tau(X)$ is its topology and $\tau(x, X) = \{U \in \tau(X) : x \in U\}$ for any point $x \in X$. If κ is a cardinal, then $[X]^{\leq \kappa} = \{A \subset X : |A| \leq \kappa\}$. The set of reals with its usual topology is denoted by \mathbb{R} and $\mathbb{I} = [0, 1] \subset \mathbb{R}$. A set $B \subset X$ is said to dominate a set $A \subset X$ if $A \subset \overline{B}$. The space X features exponential κ -domination if for any set $A \in [X]^{\leq 2^\kappa}$, there exists $B \in [X]^{\leq \kappa}$ that dominates A .

A family \mathcal{N} of subsets of a space X is a *network* in X if every open subset of X is the union of a subfamily of \mathcal{N} . The *network weight* $nw(X)$ of a space X is the minimal cardinality of a network in X and the density $d(X)$ of the space X is the minimal cardinality of a dense subset of X . The cardinal $w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base of } X\}$ is the *weight* of X . Let $s(X) = \sup\{|D| : D \text{ is a discrete subspace of } X\}$ and $\text{ext}(X) = \sup\{|D| : D \text{ is a closed discrete subspace of } X\}$. The cardinals $s(X)$ and $\text{ext}(X)$ are the *spread* and *extent* of X , respectively. Given spaces X and Y and a continuous map $f : X \rightarrow Y$ say that f is a *condensation* if it is a bijection.

The cardinal $iw(X) = \min\{\kappa : \text{there is a condensation of } X \text{ onto a space of weight less than or equal to } \kappa\}$ is called the *i -weight* of the space X . If $x \in X$, then the cardinal $\psi(x, X) = \min\{|\mathcal{U}| : \mathcal{U} \subset \tau(X) \text{ and } \bigcap \mathcal{U} = \{x\}\}$ is called the

pseudocharacter of x in X and $\psi(X) = \sup\{\psi(x, X) : x \in X\}$ is the *pseudocharacter* of X . Furthermore, a space X is κ -stable if $nw(Y) \leq \kappa$ for any continuous image Y of the space X such that $iw(Y) \leq \kappa$. If μ is a cardinal, then X is said to be a P_μ -space if $\bigcap \mathcal{U} \in \tau(X)$ for any $\mathcal{U} \subset [\tau(X)]^{\leq \mu}$.

For any space X , let $\Delta_X = \{(x, x) : x \in X\} \subset X \times X$ be the diagonal of the space X . The space X has a κ^+ -small diagonal if for any set $A \subset (X \times X) \setminus \Delta_X$ such that $|A| = \kappa^+$, there exists $B \subset A$ such that $|B| = \kappa^+$ and $\bar{B} \cap \Delta_X = \emptyset$. The cardinal $t(X) = \min\{\kappa : \bar{A} = \bigcup\{\bar{B} : B \in [A]^{\leq \kappa}\}$ for every $A \subset X\}$ is the *tightness* of the space X . A space X is κ -monolithic if $nw(\bar{A}) \leq \kappa$ for each $A \in [X]^{\leq \kappa}$. We say that X is a *Lindelöf Σ -space* if there exists a space Y that maps continuously onto X and perfectly onto a second countable space. A set A is *concentrated* around a set $F \subset X$ if $|A \setminus U| < |A|$ whenever $F \subset U \in \tau(X)$; the set A is concentrated around a point $x \in X$ if it is concentrated around $\{x\}$.

The expression $C(X, Y)$ denotes the set of all continuous maps from a space X to a space Y . We follow the usual practice to write $C(X)$ instead of $C(X, \mathbb{R})$. The space $C_p(X)$ is the set $C(X)$ endowed with the pointwise convergence topology. Let $C_{p,0}(X) = X$ and $C_{p,n+1}(X) = C_p(C_{p,n}(X))$ for each natural number n . Given spaces X and Y , if $\varphi : X \rightarrow Y$ is a continuous onto map then its dual map $\varphi^* : C_p(Y) \rightarrow C_p(X)$ is defined by $\varphi^*(f) = f \circ \varphi$ for every $f \in C_p(Y)$. For any $A \subset X$, the restriction map $\pi_A : C_p(X) \rightarrow C_p(A)$ is defined by $\pi_A(f) = f|_A$ for each $f \in C_p(X)$. Given a set $F \subset C_p(X)$ let $e_F(x)(f) = f(x)$ for any $x \in X$ and $f \in F$; then $e_F : X \rightarrow C_p(F)$ is the *reflection map* which coincides with the diagonal product ΔF of the functions from F .

The rest of our notation is standard and follows the book [5]. All relevant information on cardinal invariants can be found in the paper of Hodel [7]. The books [12], [13], [14] contain all necessary facts and notions of C_p -theory.

3. A dual property for exponential κ -domination

We will present a topological property, called exponential κ -cofinality, that is dual to exponential κ -domination with respect to function spaces in the sense that X features exponential κ -domination if and only if $C_p(X)$ is exponentially κ -cofinal. Our purpose is to show that the class of exponentially κ -cofinal spaces is interesting in itself.

Definition 3.1. Given an infinite cardinal κ , we will say that a space X is *exponentially κ -cofinal* if for any continuous onto map $f : X \rightarrow Y$ such that $w(Y) \leq 2^\kappa$, there exist continuous surjective maps $g : X \rightarrow Z$ and $h : Z \rightarrow Y$ such that $iw(Z) \leq \kappa$ and $h \circ g = f$.

The proof of the following proposition is straightforward from the definition. It shows that exponential κ -cofinality is a weakening of the property of having i -weight not exceeding κ . Therefore an important line of study of exponential κ -cofinality is to find nice classes of spaces in which it coincides with $iw \leq \kappa$.

Proposition 3.2. *If $iw(X) \leq \kappa$, then X is an exponentially κ -cofinal space. In particular, any space X with $nw(X) \leq \kappa$ is exponentially κ -cofinal.*

Proposition 3.3. *Assume that X is an exponentially κ -cofinal space and Y is a continuous image of X with $iw(Y) \leq 2^\kappa$. Then $|Y| \leq 2^\kappa$. In particular, if $nw(Y) \leq 2^\kappa$, then $|Y| \leq 2^\kappa$.*

PROOF: Let $f: X \rightarrow Y$ be a continuous onto map. Take a space M and a condensation $u: Y \rightarrow M$ such that $w(M) \leq 2^\kappa$ and choose continuous onto maps $v: X \rightarrow Z$ and $w: Z \rightarrow M$ such that $iw(Z) \leq \kappa$ and $w \circ v = u \circ f$. It follows from $|Z| \leq 2^{iw(Z)} \leq 2^\kappa$ that $|M| \leq 2^\kappa$ and hence $|Y| = |M| \leq 2^\kappa$. \square

Corollary 3.4. *Any discrete exponentially κ -cofinal space has cardinality not exceeding 2^κ .*

PROOF: If X is a discrete exponentially κ -cofinal space and $|X| > 2^\kappa$, then there exists a surjective map $f: X \rightarrow Y \subset \mathbb{I}^{2^\kappa}$ such that $|Y| > 2^\kappa$. Since the map f is continuous and $w(Y) \leq 2^\kappa$, we have a contradiction with Proposition 3.3. \square

Proposition 3.5. *If X is an exponentially κ -cofinal space and a set $Y \subset X$ is C^* -embedded in X , then Y is exponentially κ -cofinal. In particular, if X is a normal exponentially κ -cofinal space, then every closed subspace of X is exponentially κ -cofinal.*

PROOF: Take any continuous onto map $f: Y \rightarrow Y'$ such that $w(Y') \leq \mu = 2^\kappa$. There is no loss of generality to consider that $Y' \subset \mathbb{I}^\mu$ and there is a family $\{f_\alpha: \alpha < \mu\} \subset C_p(X, \mathbb{I})$ such that $f = \Delta\{f_\alpha: \alpha < \mu\}$. Let $u_\alpha: X \rightarrow \mathbb{I}$ be a continuous extension of f_α for every $\alpha < \mu$. Then the diagonal product $u = \Delta\{u_\alpha: \alpha < \mu\}$ maps X continuously onto a space $X' \subset \mathbb{I}^\mu$ and $u|_Y = f$. By exponential κ -cofinality of X , we can find continuous onto maps $v: X \rightarrow Z$ and $w: Z \rightarrow X'$ such that $w \circ v = u$ and $iw(Z) \leq \kappa$. Let $g = v|_Y$ and $h = w|_v(Y)$. Then $h \circ g = f$ and the i -weight of the space $Z' = v(Y)$ does not exceed $iw(Z) \leq \kappa$ so the maps g and h witness that Y is exponentially κ -cofinal. \square

Corollary 3.6. *Assume that X is an exponentially κ -cofinal space and D is a discrete subset of X . If D is C^* -embedded in X , then $|D| \leq 2^\kappa$. In particular, if X is a normal exponentially κ -cofinal space, then $\text{ext}(X) \leq 2^\kappa$.*

PROOF: Just apply Proposition 3.5 and Corollary 3.4. \square

Corollary 3.7. *If κ is an infinite cardinal and X is an exponentially κ -cofinal space, then any discrete family of nonempty open subsets of X has cardinality not exceeding 2^κ .*

PROOF: If \mathcal{U} is a discrete family of nonempty open subsets of the space X and $|\mathcal{U}| > 2^\kappa$, then pick a point $x_U \in U$ for every $U \in \mathcal{U}$. The set $D = \{x_U : U \in \mathcal{U}\}$ is closed, discrete and $|D| = |\mathcal{U}| > 2^\kappa$. Since the set D is C -embedded in X by [13, Fact 5 of Theorem 132], we have a contradiction with Corollary 3.6. \square

Theorem 3.8. *Given an exponentially κ -cofinal space X and a set $A \subset X$ with $nw(A) \leq 2^\kappa$, there exists a continuous map $\varphi: X \rightarrow M$ such that $w(M) \leq \kappa$ and $\varphi|_A$ is an injection. In particular, $iw(A) \leq \kappa$ and $|A| \leq 2^\kappa$.*

PROOF: The restriction mapping $\pi_A: C_p(X) \rightarrow C_p(A)$ is continuous and the set $E = \pi_A(C_p(X))$ is dense in $C_p(A)$, see [3, Proposition 0.4.1]. Observe that $d(E) \leq nw(E) \leq nw(C_p(A)) = nw(A) \leq 2^\kappa$ so we can find a dense subset $D \subset E$ with $|D| \leq 2^\kappa$. The set D is also dense in $C_p(A)$ and hence it separates the points of A . If $h = \Delta(D)$ is the diagonal product of the functions from D , then $h: X \rightarrow \mathbb{R}^D$; let $Y = h(X)$. The map $h: X \rightarrow Y$ is continuous and surjective. Since $w(Y) \leq 2^\kappa$, there exists a space Z and continuous onto maps $p: X \rightarrow Z$ and $q: Z \rightarrow Y$ such that $q \circ p = h$ and $iw(Z) \leq \kappa$. Observe that the map $h|_A$ is injective and hence so is $p|_A$. There exists a condensation $r: Z \rightarrow M$ for some space M such that $w(M) \leq \kappa$. It is immediate that the map $\varphi = r \circ p$ is as promised. \square

Corollary 3.9. *For any exponentially κ -cofinal space X , if $A \subset X$ and $|A| \leq \kappa$, then $|\bar{A}| \leq 2^\kappa$ and $iw(\bar{A}) \leq \kappa$.*

PROOF: Just note that $w(\bar{A}) \leq 2^\kappa$ and apply Theorem 3.8. \square

Corollary 3.10. *Assume that X is a metrizable exponentially κ -cofinal space. Then $w(X) \leq 2^\kappa$ and hence $iw(X) \leq \kappa$.*

PROOF: It follows from Corollary 3.6 that $w(X) = \text{ext}(X) \leq 2^\kappa$ and we can apply Theorem 3.8 to conclude that $iw(X) \leq \kappa$. \square

Corollary 3.11. *If X is an exponentially κ -cofinal space and $s(X) \leq \kappa$, then $iw(X) \leq \kappa$.*

PROOF: Observe that $nw(X) \leq 2^{s(X)} \leq 2^\kappa$, see [7, Theorem 5.3], and apply Theorem 3.8. \square

If $iw(X) \leq \kappa$, then the diagonal of X is easily seen to be a G_κ -set. This is not necessarily the case for an exponentially κ -cofinal space X but we still have an important weaker property in X .

Proposition 3.12. *If X is an exponentially κ -cofinal space, then the diagonal of X is κ^+ -small. In particular, any exponentially ω -cofinal space has a small diagonal.*

PROOF: Take any faithfully indexed set $A = \{z_\alpha = (x_\alpha, y_\alpha) : \alpha < \kappa^+\}$ contained in $(X \times X) \setminus \Delta_X$; then the cardinality of the set $Y = \{x_\alpha, y_\alpha : \alpha < \kappa^+\}$ does not exceed $\kappa^+ \leq 2^\kappa$ and therefore there exists a continuous map $\varphi: X \rightarrow M$ such that $w(M) \leq \kappa$ and $\varphi|Y$ is injective, see Theorem 3.8. This implies that the set $\{(\varphi(x_\alpha), \varphi(y_\alpha)) : \alpha < \kappa^+\}$ is contained in $(M \times M) \setminus \Delta_M$ which, together with $w(M) \leq \kappa$, guarantees that there is a set $E \subset \kappa^+$ such that $|E| = \kappa^+$ and the closure of the set $\{(\varphi(x_\alpha), \varphi(y_\alpha)) : \alpha \in E\}$ in $M \times M$ does not meet Δ_M . Then the closure of the set $B = \{z_\alpha : \alpha \in E\} \subset A$ in $X \times X$ does not meet Δ_X and $|B| = \kappa^+$, i.e., the set B witnesses that X has a κ^+ -small diagonal. \square

Proposition 3.13. *Given an infinite cardinal κ , if X is a P_{2^κ} -space with $\text{ext}(X) \leq 2^\kappa$, then X is exponentially κ -cofinal.*

PROOF: Let $f: X \rightarrow Y$ be a continuous onto map such that $w(Y) \leq 2^\kappa$. Then $f^{-1}(y)$ is open in X being a G_{2^κ} -set for every $y \in Y$. Therefore the partition $\mathcal{P} = \{f^{-1}(y) : y \in Y\}$ is a discrete family of open subsets of X so it follows from $\text{ext}(X) \leq 2^\kappa$ that $|\mathcal{P}| \leq 2^\kappa$ and hence $|Y| \leq 2^\kappa$. If Z is the set Y with the discrete topology and $g(x) = f(x)$ for any $x \in X$, then the map $g: X \rightarrow Z$ is continuous and for the identity map $h: Z \rightarrow Y$, we have $h \circ g = f$. Since there exists an injection of Z into \mathbb{I}^κ , which is automatically continuous, we conclude that the maps g and h witness exponential κ -cofinality of X . \square

Example 3.14. For any cardinal $\kappa > \mathfrak{c}$, consider the set $X = \kappa \cup \{p\}$ where $p \notin \kappa$. All points of κ are isolated in X and a set $U \subset X$ with $p \in U$ is open if and only if $\kappa \setminus U \leq \mathfrak{c}$. It is immediate that X is a $P_\mathfrak{c}$ -space and $l(X) = \text{ext}(X) = \mathfrak{c}$ so X is exponentially ω -cofinal by Proposition 3.13. Therefore the Souslin number of an exponentially ω -cofinal space can be arbitrarily large. This result should be compared with Corollary 3.7.

Recall that we have the equalities $iw(X) = d(C_p(X))$ and $d(X) = iw(C_p(X))$ for any space X , see [10], i.e., the density and i -weight are bidual with respect to the functor C_p . Since exponential κ -domination and exponential κ -cofinality are their respective weakenings, it is natural to expect them to be bidual as well. We will show next that this is, indeed, the case.

Theorem 3.15. *A space X is exponentially κ -cofinal if and only if $C_p(X)$ is a space with exponential κ -domination.*

PROOF: Suppose that X is exponentially κ -cofinal and take a set $A \subset C_p(X)$ with $|A| \leq 2^\kappa$. If $u = \Delta A$ is the diagonal product of the family A , then $u: X \rightarrow \mathbb{R}^A$

and hence the space $Y = u(X)$ has weight not exceeding 2^κ . We can consider that $u: X \rightarrow Y$ and hence the dual map $u^*: C_p(Y) \rightarrow C_p(X)$ is an embedding. Let $Q = u^*(C_p(Y))$ and observe that $A \subset Q$, see [14, Fact 5 of U.086].

There exists a space Z together with continuous onto maps $v: X \rightarrow Z$ and $w: Z \rightarrow Y$ such that $iw(Z) \leq \kappa$ and $w \circ v = u$. The space $C_p(Z)$ has density not exceeding κ , see [10]; since $E = v^*(C_p(Z))$ is homeomorphic to $C_p(Z)$, we can fix a set $B \subset E$ such that $|B| \leq \kappa$ and $E \subset \bar{B}$. It follows from [12, Problem 163] that $Q \subset E$ whence $A \subset Q \subset E \subset \bar{B}$ so the set B witnesses exponential κ -domination in $C_p(X)$.

To prove sufficiency, assume that $C_p(X)$ features exponential κ -domination and take a continuous onto map $u: X \rightarrow Y$ for some space Y of weight not exceeding 2^κ . The dual map $u^*: C_p(Y) \rightarrow C_p(X)$ is an embedding so the density of the set $Q = u^*(C_p(Y))$ is the same as the density of $C_p(Y)$ while $d(C_p(Y)) \leq nw(C_p(Y)) = nw(Y) \leq 2^\kappa$. This makes it possible to take a set $B \subset C_p(X)$ such that $|B| \leq \kappa$ and $Q \subset \bar{B}$. The reflection map $e_{\bar{B}}: X \rightarrow C_p(\bar{B})$ is continuous and the i -weight of space $Z = e_{\bar{B}}(X) \subset C_p(\bar{B})$ does not exceed $iw(C_p(\bar{B})) \leq |B| \leq \kappa$. The dual map $\varphi = e_{\bar{B}}^*: C_p(Z) \rightarrow C_p(X)$ is an embedding and $\bar{B} \subset \varphi(C_p(Z))$ by [14, Fact 5 of U.086]. Therefore $Q \subset \bar{B} \subset \varphi(C_p(Z))$ so we can apply [3, Proposition 0.4.7] to see that there exist continuous onto maps $v: X \rightarrow Z$ and $w: Z \rightarrow Y$ such that $u = w \circ v$. Since $iw(Z) \leq \kappa$, the space Z witnesses that X is exponentially κ -cofinal. □

Theorem 3.16. *Let κ be an infinite cardinal. A space X features exponential κ -domination if and only if $C_p(X)$ is exponentially κ -cofinal.*

PROOF: Assume first that X is a space with exponential κ -domination and we have a continuous onto map $u: C_p(X) \rightarrow Y$ for some space Y with $w(Y) \leq 2^\kappa$. There exists a set $A \subset X$ and a continuous onto map $\varphi: \pi_A(C_p(X)) \rightarrow Y$ such that $|A| \leq 2^\kappa$ and $\varphi \circ \pi_A = u$, see [2, Theorem 1]. By exponential κ -domination of X there exists a set $B \subset X$ such that $|B| \leq \kappa$ and $A \subset \bar{B}$. It is standard that the restriction map $\pi_A: C_p(X) \rightarrow \pi_A(C_p(X))$ factorizes through $\pi_{\bar{B}}(C_p(X))$ so there exists a continuous onto map $w: Z = \pi_{\bar{B}}(C_p(X)) \rightarrow Y$ such that $u = w \circ \pi_{\bar{B}}$. Since $iw(Z) \leq C_p(\bar{B}) \leq |B| \leq \kappa$, we conclude that Z witnesses exponential κ -cofinality of $C_p(X)$.

Now assume that the space $C_p(X)$ is exponentially κ -cofinal and $A \subset X$ is a set of cardinality less than or equal to 2^κ . By Theorem 3.15 the space $C_p C_p(X)$ features exponential κ -domination so there exists a set $E \subset C_p C_p(X)$ such that $|E| \leq \kappa$ and $A \subset \bar{E}$. Here we identify the space X with its canonical copy in $C_p C_p(X)$, see [3, Corollary 0.5.5]. Every continuous real-valued function on $C_p(X)$ depends on countably many coordinates, see [2, Theorem 1], so we can

choose for each $u \in E$ a countable set $B_u \subset X$ such that $u(f) = u(g)$ whenever $f, g \in C_p(X)$ and $f|_{B_u} = g|_{B_u}$.

The set $B = \bigcup\{B_u : u \in E\}$ has cardinality not exceeding κ ; assume that $p \in A \setminus \overline{B}$. There exists a function $f \in C_p(X)$ such that $f(p) = 1$ and $f(\overline{B}) \subset \{0\}$; let $g(x) = 0$ for any $x \in X$. It follows from $p \in \overline{E}$ that there is a function $u \in E$ such that $u(f) > \frac{1}{2}$ and $u(g) < \frac{1}{2}$. However, $f|_B = g|_B$ and hence $f|_{B_u} = g|_{B_u}$ which implies that $u(f) = u(g)$. This contradiction shows that $A \subset \overline{B}$, i.e., the set B witnesses that X is a space with exponential κ -domination. \square

Corollary 3.17. *Given a cardinal $\kappa \geq \omega$, if a space X features exponential κ -domination, then $C_{p,2n}(X)$ is a space with exponential κ -domination and $C_{p,2n+1}(X)$ is exponentially κ -cofinal for all $n \in \omega$.*

PROOF: It follows from Theorem 3.16 that the space $C_p(X)$ must be exponentially κ -cofinal and hence Theorem 3.15 can be applied to see that $C_p C_p(X)$ is a space with exponential κ -domination. Proceeding by induction assume that $C_{p,2n}(X)$ is a space with exponential κ -domination. Then $C_{p,2n+1}(X) = C_p(C_{p,2n}(X))$ is an exponentially κ -cofinal space by Theorem 3.16 which makes it possible to apply Theorem 3.15 again to conclude that

$$C_{p,2n+2}(X) = C_p(C_{p,2n+1}(X))$$

features exponential κ -domination. \square

Corollary 3.18. *If X is an exponentially κ -cofinal space, then $C_{p,2n+1}(X)$ is a space with exponential κ -domination and $C_{p,2n}(X)$ is exponentially κ -cofinal for all $n \in \omega$.*

PROOF: By Theorem 3.15, the space $Y = C_p(X)$ has exponential κ -domination; apply Corollary 3.17 to convince ourselves that $C_{p,2n+1}(X) = C_{p,2n}(Y)$ is a space with exponential κ -domination and $C_{p,2n}(X) = C_{p,2n-1}(Y)$ is an exponentially κ -cofinal space. \square

Theorem 3.19. *Suppose that X is an exponentially κ -cofinal space such that $l(X) \leq \kappa$ and $t(X) \leq \kappa$. Then $iw(X) \leq \kappa$ and hence $|X| \leq 2^\kappa$.*

PROOF: We will prove first that $\psi(X) \leq \kappa$. Striving for contradiction, assume that $p \in X$ and $\psi(p, X) > \kappa$. Take any $x_0 \in X \setminus \{p\}$ and let $G_0 = X$. Proceeding by induction, assume that $\beta < \kappa^+$ and we have a set $\{x_\alpha : \alpha < \beta\} \subset X \setminus \{p\}$ and a family $\{G_\alpha : \alpha < \beta\}$ of closed G_κ -subsets of X with the following properties:

- (1) $\{p, x_\alpha\} \subset G_\alpha$ for all $\alpha < \beta$;
- (2) $G_\alpha \subset G_\gamma$ whenever $\gamma < \alpha < \beta$;
- (3) $\overline{\{x_\gamma : \gamma < \alpha\}} \cap G_\alpha \subset \{p\}$ for every $\alpha < \beta$.

The set $B_\beta = \{p\} \cup \overline{\{x_\alpha : \alpha < \beta\}}$ has cardinality less than or equal to 2^κ by Corollary 3.9 which implies, by Theorem 3.8, that $\psi(p, B_\beta) \leq \kappa$ and hence we can choose a closed G_κ -set Q in the space X such that $Q \cap B_\beta = \{p\}$. Let $G_\beta = \bigcap \{G_\alpha : \alpha < \beta\} \cap Q$; since $\psi(p, X) > \kappa$, we can pick a point $x_\beta \in G_\beta \setminus \{p\}$ completing our inductive construction. Observe that it follows from the properties (1) and (3) that the set $A = \{x_\alpha : \alpha < \kappa^+\}$ is faithfully indexed and hence $|A| = \kappa^+$.

Let $F_\alpha = \overline{\{x_\beta : \alpha \leq \beta < \kappa^+\}} \subset G_\alpha$ for any $\alpha < \kappa^+$; it follows from $l(X) \leq \kappa$ that $\emptyset \neq F = \bigcap \{F_\alpha : \alpha < \kappa^+\}$. If $x \neq p$ and $x \in F$, then $x \in \bar{A}$; since $t(X) \leq \kappa$, there exists $\beta < \kappa^+$ such that $x \in \overline{\{x_\alpha : \alpha < \beta\}}$. Since also $x \in F_\beta \subset G_\beta$, we obtained a contradiction with the property (3). Thus, $F = \{p\}$ and it is standard to deduce from $l(X) \leq \kappa$ that

- (4) the family $\{F_\alpha : \alpha < \kappa^+\}$ is a network at p , i.e., for any set $U \in \tau(p, X)$ there exists $\alpha < \kappa^+$ such that $F_\alpha \subset U$ and hence $\{x_\beta : \alpha < \beta\} \subset U$.

It is an immediate consequence of the property (4) that A is concentrated around the point p and hence the set $D = \{(p, x_\alpha) : \alpha < \kappa^+\} \subset (X \times X) \setminus \Delta_X$ is concentrated around the diagonal Δ_X . Since $|D| = \kappa^+$, the diagonal of X is not κ^+ -small; this contradiction with Proposition 3.12 shows that $\psi(X) \leq \kappa$. Arhangel'skii's inequality $|X| \leq 2^{l(X) \cdot \psi(X) \cdot t(X)} \leq 2^\kappa$ shows that $|X| \leq 2^\kappa$ and therefore $iw(X) \leq \kappa$ by Theorem 3.8. □

Corollary 3.20. *If X is a Lindelöf exponentially ω -cofinal space and $t(X) \leq \omega$, then $iw(X) \leq \omega$.*

Corollary 3.21. *Suppose that X is a space for which $C_p(X)$ features exponential κ -domination while $l(C_p(X)) \leq \kappa$ and $t(C_p(X)) \leq \kappa$. Then $d(C_p(X)) \leq \kappa$.*

PROOF: The space X is exponentially κ -cofinal by Theorem 3.15. Next, observe that $t(X) \leq l(C_p(X)) \leq \kappa$ by [4] and $l(X) \leq t(C_p(X)) \leq \kappa$ by [11] which shows that Theorem 3.19 can be applied to see that $iw(X) \leq \kappa$ whence $d(C_p(X)) = iw(X) \leq \kappa$, see [10]. □

Proposition 3.22. *Assume that X is a κ -stable exponentially κ -cofinal space. Then $nw(X) \leq \kappa$.*

PROOF: If $d(C_p(X)) > \kappa$, then there exists a left-separated set $L \subset C_p(X)$ with $|L| = \kappa^+$. Next, observe that $C_p(X)$ is a space with exponential κ -domination by Theorem 3.15 and hence there exists a set $A \subset C_p(X)$ such that $|A| \leq \kappa$ and $L \subset \bar{A}$. The space $C_p(X)$ is κ -monolithic by [1] so $nw(\bar{A}) \leq \kappa$. Therefore $\kappa^+ \leq hd(L) \leq nw(L) \leq nw(\bar{A}) \leq \kappa$; this contradiction shows that $d(C_p(X)) \leq \kappa$ and therefore $iw(X) = d(C_p(X)) \leq \kappa$. Finally, apply κ -stability of X to conclude that $nw(X) \leq \kappa$. □

Corollary 3.23. *Any exponentially ω -cofinal Lindelöf Σ -space has a countable network.*

PROOF: Observe that any Lindelöf Σ -space is ω -stable by [3, Theorem II.6.21]; Proposition 3.22 does the rest. \square

Corollary 3.24. *Any exponentially ω -cofinal pseudocompact space is compact and metrizable.*

PROOF: Observe that it follows from Corollary II.6.34 of [3] that any pseudocompact space is ω -stable and apply Proposition 3.22. \square

Example 3.25. For any cardinal $\kappa > \mathfrak{c}$, there exists an exponentially ω -cofinal space Y such that $l(Y) \geq \kappa$. To see it, consider the exponentially ω -cofinal space $X = \kappa \cup \{p\}$ from Example 3.14. All points of κ are isolated in X and a set $U \subset X$ with $p \in U$ is open if and only if $\kappa \setminus U \leq \mathfrak{c}$. It is standard to see that $C_p(X)$ is homeomorphic to the $\Sigma_{\mathfrak{c}}$ -product $S = \{x \in \mathbb{R}^{\kappa} : |x^{-1}(\mathbb{R} \setminus \{0\})| \leq \mathfrak{c}\}$ in the space \mathbb{R}^{κ} and therefore S is a space with exponential ω -domination by Theorem 3.15. If we let $u(\alpha) = 1$ for any $\alpha < \kappa$, then we obtain a point $u \in \mathbb{R}^{\kappa}$ such that $u \notin \bar{A}$ for any $A \subset S$ with $|A| < \kappa$. This shows that the space $Z = S \cup \{u\}$ has tightness equal to κ . The union of countably many spaces with exponential ω -domination is easily seen to have exponential ω -domination so Z features exponential ω -domination. It was proved in [4] that $l(C_p(Z)) \geq t(Z) = \kappa$ and therefore $l(C_p(Z)) \geq \kappa$. Finally, observe that $C_p(Z)$ is exponentially ω -cofinal by Theorem 3.16 and hence the space $Y = C_p(Z)$ is as promised.

4. Open questions

The author hopes that this work demonstrates that exponentially κ -cofinal spaces form a class interesting in itself. This class encompasses important new information about function spaces and a new metrization theorem for compact spaces. The most intriguing open question in this topic is whether Lindelöf exponentially ω -cofinal spaces must have countable i -weight.

Question 4.1. Suppose that X is a Lindelöf exponentially ω -cofinal space. Is it true that $iw(X) \leq \omega$?

Question 4.2. Suppose that X is a Lindelöf exponentially ω -cofinal space. Is it true that $\psi(X) \leq \omega$?

Question 4.3. Suppose that X is a Lindelöf exponentially ω -cofinal space. Is it true that $|X| \leq 2^{\mathfrak{c}}$?

Question 4.4. Suppose that X is an exponentially ω -cofinal space of countable character. Is it true that $iw(X) \leq \omega$?

Question 4.5. Suppose that X is an exponentially ω -cofinal Fréchet–Urysohn space. Is it true that $iw(X) \leq \omega$?

Question 4.6. Let X be an exponentially ω -cofinal space with $\psi(X) \leq \omega$. Is it true that $iw(X) \leq \omega$?

Question 4.7. Let X be an exponentially ω -cofinal space with a G_δ -diagonal. Is it true that $iw(X) \leq \omega$?

Question 4.8. Suppose that X is an exponentially ω -cofinal space. Is it true that $\text{ext}(X) \leq \mathfrak{c}$?

Question 4.9. Suppose that X is a space with exponential ω -domination such that $t(X) = l(X) = \omega$. Must X be separable?

Question 4.10. Suppose that X is a Lindelöf Fréchet–Urysohn space featuring exponential ω -domination. Must X be separable?

Question 4.11. Suppose that $C_p(X)$ is a Fréchet–Urysohn space that features exponential ω -domination. Must $C_p(X)$ be separable?

Question 4.12. Suppose that $C_p(X)$ is an exponentially ω -cofinal Fréchet–Urysohn space. Must X be separable?

REFERENCES

- [1] Arkhangel'skiĭ A. V., *Factorization theorems and spaces of functions: stability and monolithism*, Dokl. Akad. Nauk SSSR **265** (1982), no. 5, 1039–1043 (Russian).
- [2] Arkhangel'skiĭ A. V., *Continuous mappings, factorization theorems and spaces of functions*, Trudy Moskov. Mat. Obshch. **47** (1984), 3–21, 246 (Russian).
- [3] Arkhangel'skiĭ A. V., *Topological Function Spaces*, Mathematics and Its Applications (Soviet Series), 78, Kluwer Academic Publishers Group, Dordrecht, 1992.
- [4] Asanov M. O., *On cardinal invariants of spaces of continuous functions*, Sovr. Topologia i Teoria Mnozhestv **2** (1979), 8–12 (Russian).
- [5] Engelking R., *General Topology*, Monografie Matematyczne, 60, PWN—Polish Scientific Publishers, Warsaw, 1977.
- [6] Gruenhagen G., Tkachuk V. V., Wilson R. G., *Domination by small sets versus density*, Topology Appl. **282** (2020), 107306, 10 pages.
- [7] Hodel R. E., *Cardinal Functions. I.*, Handbook of Set-Theoretic Topology, North Holland, Amsterdam, 1984, 1–61.
- [8] Hušek M., *Topological spaces without κ -accessible diagonal*, Comment. Math. Univ. Carolinae **18** (1977), no. 4, 777–788.
- [9] Juhász I., Szentmiklóssy Z., *Convergent free sequences in compact spaces*, Proc. Amer. Math. Soc. **116** (1992), no. 4, 1153–1160.
- [10] Noble N., *The density character of function spaces*, Proc. Amer. Math. Soc. **42** (1974), no. 1, 228–233.

- [11] Pytkeev E. G., *Tightness of spaces of continuous functions*, Uspekhi Mat. Nauk **37** (1982), no. 1(223), 157–158 (Russian).
- [12] Tkachuk V. V., *A C_p -Theory Problem Book*, Topological and Function Spaces, Problem Books in Mathematics, Springer, New York, 2011.
- [13] Tkachuk V. V., *A C_p -Theory Problem Book*, Special Features of Function Spaces, Problem Books in Mathematics, Springer, Cham, 2014.
- [14] Tkachuk V. V., *A C_p -Theory Problem Book*, Compactness in Function Spaces, Problem Books in Mathematics, Springer, Cham, 2015.

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