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On some imaginary triquadratic number fields k with $\mathrm{Cl}_2(k) \simeq (2,4)$ or $(2,2,2)$

Abdelmalek Azizi,
Mohamed Mahmoud Chems-Eddin, Abdelkader Zekhnini

Abstract. Let d be a square free integer and $L_d := \mathbb{Q}(\zeta_8, \sqrt{d})$. In the present work we determine all the fields L_d such that the 2-class group, $\mathrm{Cl}_2(L_d)$, of L_d is of type $(2,4)$ or $(2,2,2)$.

Keywords: 2-group rank; 2-class group; imaginary triquadratic number fields

Classification: 11R11, 11R16, 11R18, 11R27, 11R29

1. Introduction

Let k be a number field and $\mathrm{Cl}_2(k)$ its 2-class group, that is, the 2-Sylow subgroup of its ideal class group $\mathrm{Cl}(k)$. The problem of determining the structure of $\mathrm{Cl}_2(k)$ is one of the most interesting problems of algebraic number theory, accordingly many mathematicians treated this problem for some number fields of degree 2, 4. For example, in [18] using binary quadratic forms theory, P. Kaplan determined the 2-class group of some quadratic number fields. The authors of [11], [12] used genus theory and class field theory to characterize those imaginary quadratic number fields, k, with 2-class group of type $(2,2^m)$ or $(2,2,2)$ and the 2-rank of the class group of its Hilbert 2-class field equal to 2. In [5], using units and the 2-part of the class number of subextensions of k, the authors determined the 2-class group of some real biquadratic number fields $k = \mathbb{Q}(\sqrt{m}, \sqrt{d})$ with d be an odd square free integer. Using similar techniques, the paper [7] characterizes all the fields $k = \mathbb{Q}(i, \sqrt{d})$ such that $\mathrm{Cl}_2(k)$ is of type $(2,4)$ or $(2,2,2)$ (here $(a_1, \ldots, a_r)$ denotes the direct sum of cyclic groups of order a_i for $i = 1, \ldots, r$). Whenever k is an imaginary multiquadratic number field, this problem is strongly related to the units of k and the class number of the 2-part of the class numbers of its subextensions as we will see later. This paper is, actually, a continuation and extension of our earlier work [4], in which we determined the rank of the 2-class group of all fields of the form $L_d := \mathbb{Q}(\zeta_8, \sqrt{d})$ with d being a positive square free integer and moreover we determined all fields L_d for which the 2-class

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group, $\mathrm{Cl}_2(L_d)$, is of type $(2,2)$. In this work, we are interested in determining all positive square free integers d satisfying $\mathrm{Cl}_2(L_d)$ is of type $(2,4)$ or $(2,2,2)$.

Notations

The next notations will be used for the rest of this article:

- d: a positive odd square free integer;
- L_d: $\mathbb{Q}(\zeta_8, \sqrt{d})$;
- $\mathrm{Cl}_2(L)$: the 2-class group of some number field L;
- $h(L)$: the class number of the number field L;
- $h_2(L)$: the 2-class number of the number field L;
- $h_2(m)$: the 2-class number of the quadratic field $\mathbb{Q}(\sqrt{m})$;
- E_L: the unit group of any number field L;
- ε_m: the fundamental unit of $\mathbb{Q}(\sqrt{m})$;
- W_L: the set of roots of unity contained in a field L;
- ω_L: the cardinality of W_L;
- L^+: the maximal real subfield of an imaginary number field L;
- Q_L: the Hasse's index, that is, $[E_L : W_L E_{L^+}]$, if L/L^+ is CM;
- $Q(L/k)$: the unit index of a V_4-extension L/k;
- $q(L) := \left[E_L : \prod_i E_{k_i}\right]$ with k_i the quadratic subfields of L.

2. Preliminaries

Let us first collect some results and definitions that we will need in the sequel. Recall that a field K is said to be a CM-field if it is a totally complex quadratic extension of a totally real number field. Note also that a V_4-extension K/k (i.e., a normal extension of number fields with $\mathrm{Gal}(K/k) = V_4$, where V_4 is the Klein four-group) is called V_4-extension of CM-fields if exactly two of its three quadratic subextensions are CM-fields. Let us next recall the class number formula for a V_4-extension of CM-fields:

Proposition 1 ([20]). *Let L/K be a V_4-extension of* CM-*fields, then*

$$h(L) = \frac{Q_L}{Q_{K_1} Q_{K_2}} \cdot \frac{\omega_L}{\omega_{K_1}\omega_{K_2}} \cdot \frac{h(K_1)h(K_2)h(L^+)}{h(K)^2}.$$

Here K_1, K_2, L^+ are the three subextensions of L/K, with K_1 and K_2 are CM-fields.

The following class number formula for a multiquadratic number field is usually attributed to S. Kuroda [21], but it goes back to G. Herglotz, see [16] (cf. [13, page 27]).

Proposition 2. *Let K be a multiquadratic number field of degree 2^n, $n \in \mathbb{N}$, and k_i the $s = 2^n - 1$ quadratic subfields of K. Then*

$$h(K) = \frac{1}{2^v}\left[E_K : \prod_{i=1}^{s} E_{k_i}\right]\prod_{i=1}^{s} h(k_i),$$

with

$$v = \begin{cases} n(2^{n-1}-1) & \text{if } K \text{ is real,} \\ (n-1)(2^{n-2}-1) + 2^{n-1} - 1 & \text{if } K \text{ is imaginary.} \end{cases}$$

Continue with the next formula called Kuroda's class number formula for a V_4-extension K/k.

Proposition 3 ([22], page 247)**.** *Let K/k be a V_4-extension. Then we have:*

$$h(K) = \begin{cases} \frac{1}{4} \cdot Q(K/k) \cdot \prod_{i=1}^{3} h(k_i) & \text{if } k = \mathbb{Q} \text{ and } K \text{ is real,} \\ \frac{1}{2} \cdot Q(K/k) \cdot \prod_{i=1}^{3} h(k_i) & \text{if } k = \mathbb{Q} \text{ and } K \text{ is imaginary,} \\ \frac{1}{4} \cdot Q(K/k) \cdot \prod_{i=1}^{3} h(k_i)/h(k)^2 & \text{if } k \text{ is an imaginary quadratic} \\ & \text{extension of } \mathbb{Q}. \end{cases}$$

Here k_i are the 3 subextensions of K/k.

3. Fields L_d for which $\mathrm{Cl}_2(L_d)$ is of type $(2,4)$

Our goal in this section is to determine all fields L_d for which $\mathrm{Cl}_2(L_d) \simeq (2,4)$. Recall first the definition of the rational biquadratic residue symbol:

For a prime $p \equiv 1 \pmod 4$ and a quadratic residue $a \pmod p$, $\left(\frac{a}{p}\right)_4$ will denote the rational biquadratic residue symbol defined by

$$\left(\frac{a}{p}\right)_4 = \pm 1 \equiv a^{(p-1)/4} \pmod p.$$

Moreover, for an integer $a \equiv 1 \pmod 8$, the symbol $\left(\frac{a}{p}\right)_4$ is defined by

$$\left(\frac{a}{p}\right)_4 = 1 \quad \text{if } \equiv 1 \pmod{16}, \qquad \left(\frac{a}{p}\right)_4 = -1 \quad \text{if } a \equiv 9 \pmod{16}.$$

It turns out that $\left(\frac{a}{p}\right)_4 = (-1)^{(a-1)/8}$.

In all this section, let p, p_i, q and q_i be prime integers such that $p \equiv p_i \equiv 1 \pmod 4$ and $q \equiv q_i \equiv 3 \pmod 4$ with $i \in \mathbb{N}^*$. The following lemma is proved in our earlier paper [4, Theorem 5.6].

Lemma 1. *The rank of the 2-class group $\mathrm{Cl}_2(L_d)$ of L_d equals 2 if and only if d takes one of the following forms:*

1. $d = q_1q_2$ *with* $q_1 \equiv q_2 \equiv 3 \pmod 8$.
2. $d = p_1p_2$ *with* $p_1 \equiv p_2 \equiv 5 \pmod 8$.
3. $d = q_1q_2$ *with* $q_1 \equiv 3 \pmod 8$ *and* $q_2 \equiv 7 \pmod 8$.
4. $d = pq$ *with* $p \equiv 5 \pmod 8$ *and* $q \equiv 7 \pmod 8$.
5. $d = p$ *with* $p \equiv 1 \pmod 8$ *and* $\left[p \equiv 9 \pmod{16} \text{ or } \left(\frac{2}{p}\right)_4 \neq \left(\frac{p}{2}\right)_4\right]$.

We need also the following result.

Lemma 2. *Let* $d = p_1p_2$, *where* p_1, p_2 *are two rational primes such that* $p_i \equiv 5 \pmod 8$. *Then* $h_2(L_d)$ *is divisible by* 16.

Proof: We have $L_d = \mathbb{Q}(\sqrt{p_1p_2}, \sqrt{2}, i)$ is an imaginary multiquadratic number field of degree 2^3. So by Proposition 2 we have

$$h(L_d) = \frac{1}{2^5} q(L_d) h(p_1p_2) h(-p_1p_2) h(2p_1p_2) h(-2p_1p_2) h(2) h(-2) h(-1).$$

It is known that $h_2(2)$, $h_2(-2)$ and $h_2(-1)$ are equal to 1, and by [18, page 350] $h_2(-2p_1p_2) = 4$. On the other hand, [8, Theorem 2.2] implies that $q(L_d)$ is a power of 2. So by passing to the 2-part in the above equation, we get

$$h_2(L_d) = \frac{1}{2^3} q(L_d) h_2(p_1p_2) h_2(-p_1p_2) h_2(2p_1p_2). \tag{1}$$

Note that:

- By [15, Corollary 18.4], $h_2(p_1p_2)$ is divisible by 2.
- By [18, pages 348–349] (Propositions B_1' and B_4'), $h_2(-p_1p_2)$ is divisible by 8.
- By [15, Corollaries 18.4, 19.7 and 19.8], $h_2(2p_1p_2)$ is divisible by 4.

On one hand, $\zeta_8 = (1+i)/\sqrt{2} \in E_{L_d}$. On the other hand, letting k_i be the quadratic subfields of L_d, one gets easily

$$\prod_i E_{k_i} = \langle i, \varepsilon_2, \varepsilon_{-2}, \varepsilon_{p_1p_2}, \varepsilon_{2p_1p_2}, \varepsilon_{-p_1p_2}, \varepsilon_{-2p_1p_2} \rangle.$$

So the 8th root of unity $\zeta_8 \notin \prod_i E_{k_i}$. Thus $\overline{1}$ and $\overline{\zeta_8}$ are two distinct cosets in the quotient $E_{L_d}/\prod_i E_{k_i}$. Thus, $q(L_d)$ is divisible by 2. It follows by the equality (1) above that $h_2(L_d)$ is divisible by $(1/2^3)\cdot 2 \cdot 2 \cdot 8 \cdot 4 = 16$ as we wished to prove. $\square$

Example 1. Let $d = p_1p_2$ be as in the above lemma. Using PARI/GP calculator version 2.9.4 (64bit), December 20, 2017, for $5 \leq p_i \leq 200$, $i = 1, 2$, we could not find a field L_d such that $h_2(L_d) = 16$. We have the following examples.

1. For $p_1 = 13$ and $p_2 = 5$, we have $h_2(L_{13\cdot 5}) = 32$.
2. For $p_1 = 37$ and $p_2 = 53$, we have $h_2(L_{37\cdot 53}) = 64$.

Remark 1. With hypothesis and notations of Lemma 2, we find in [8] a unit group of L_d.

Lemma 3 ([9], Lemma 2)**.** *Let $d \equiv 1 \pmod 4$ be a positive square free integer and $\varepsilon_d = x + y\sqrt{d}$ be the fundamental unit of $\mathbb{Q}(\sqrt{d})$. Assume $N(\varepsilon_d) = 1$, then:*

1. *$x+1$ and $x-1$ are not squares in $\mathbb{N}$, i.e., $2\varepsilon_d$ is not a square in $\mathbb{Q}(\sqrt{d})$.*
2. *For all prime p dividing d, $p(x+1)$ and $p(x-1)$ are not squares in $\mathbb{N}$.*

Lemma 4. *Let $d = q_1q_2$, with $q_1 \equiv q_2 \equiv 3 \pmod 8$ two primes such that $\left(\frac{q_1}{q_2}\right) = 1$. Then we have $E_{L_d} = \langle \zeta_8, \varepsilon_2, \varepsilon_{q_1q_2}, \sqrt{\varepsilon_{2q_1q_2}}\rangle$, and thus $q(L_d) = 4$.*

Proof: As $q_1q_2 \equiv 1 \pmod 8$, we claim that the unit $\varepsilon_{q_1q_2}$ can be written as $\varepsilon_{q_1q_2} = a + b\sqrt{q_1q_2}$, where a and b are two integers. Indeed, suppose that $\varepsilon_{q_1q_2} = (\alpha + \beta\sqrt{d})/2$ where α, β are two integers. Since $N(\varepsilon_{q_1q_2}) = 1$, one deduces that $\alpha^2 - 4 = \beta^2 d$, hence $\alpha^2 - 4 \equiv \beta^2 \pmod 8$. On the other hand, if we suppose that α and β are odd, then $\alpha^2 \equiv \beta^2 \equiv 1 \pmod 8$, but this implies the contradiction $-3 \equiv 1 \pmod 8$. Thus α and β are even and our claim is established.

It is known that $N(\varepsilon_{q_1q_2}) = 1$, so by the unique factorization in $\mathbb{Z}$ and Lemma 3 one gets

$$(1): \begin{cases} a+1 = 2q_1b_1^2, \\ a-1 = 2q_2b_2^2, \end{cases} \quad \text{or} \quad (2): \begin{cases} a-1 = 2q_1b_1^2, \\ a+1 = 2q_2b_2^2, \end{cases}$$

for some integers b_1 and b_2 such that $b = 2b_1b_2$.

If the system (2) holds, then

$$-1 = \left(\frac{2q_1}{q_2}\right) = \left(\frac{a-1}{q_2}\right) = \left(\frac{a+1-2}{q_2}\right) = \left(\frac{-2}{q_2}\right) = 1.$$

This is absurd. Therefore $\begin{cases} a+1 = 2q_1b_1^2, \\ a-1 = 2q_2b_2^2. \end{cases}$ Thus, $\sqrt{\varepsilon_{q_1q_2}} = b_1\sqrt{q_1} + b_2\sqrt{q_2}$. So ε_d is not a square in L_d^+.

We have ε_{2d} has a positive norm. Put $\varepsilon_{2d} = x + y\sqrt{2q_1q_2}$. We similarly show that $\sqrt{2\varepsilon_{2d}} = y_1 + y_2\sqrt{2q_1q_2} \in L_d^+$ for some integers y_1 and y_2. Therefore, ε_{2d} is a square in L_d^+ since $\sqrt{2} \in L_d^+$. As ε_2 has a negative norm, then using the algorithm described in [25] (or in [3, page 113]) we have $\{\varepsilon_2, \varepsilon_{q_1q_2}, \sqrt{\varepsilon_{2q_1q_2}}\}$ is a fundamental system of units of L_d^+. Hence the result follows easily by [2, Proposition 2]. □

To continue, consider the following parameters:

- For a rational prime p such that $p \equiv 1 \pmod 8$, set $p = u^2 - 2v^2$ where u and v are two positive integers such that $u \equiv 1 \pmod 8$ (for the existence of u and v, see [23]).

- For two primes q_1 and q_2 such that $q_1 \equiv q_2 \equiv 3 \pmod 8$, $\left(\dfrac{q_1}{q_2}\right) = 1$, there exist five integers X, Y, k, l and m such that $2q_2 = k^2X^2 + 2lXY + 2mY^2$ and $q_1 = l^2 - 2k^2m$ (cf. [18, page 356]).

Theorem 1. *Let d be an odd positive square free integer. Then $\mathrm{Cl}_2(L_d) \simeq (2,4)$ if and only if d takes one of the two following forms:*

1. $d = p \equiv 9 \pmod{16}$ *is a prime such that* $\left(\dfrac{2}{p}\right)_4 \neq \left(\dfrac{p}{2}\right)_4$ *and* $\left(\dfrac{u}{p}\right)_4 = -1$.
2. $d = q_1q_2$, *with* $q_1 \equiv q_2 \equiv 3 \pmod 8$ *primes such that* $\left(\dfrac{q_1}{q_2}\right)_4 = 1$ *and* $\left(\dfrac{-2}{|k^2X + lY|}\right) = -1$.

Proof: It suffices to determine for which forms of d appearing in Lemma 1, we have $h_2(L_d) = 8$. Let us firstly eliminate some cases.

- By [4, Propositions 5.13 and 5.14], $h_2(L_d) \neq 8$ whenever d takes forms in the third and the fourth item of Lemma 1.
- The form of d in the second item of Lemma 1 is also eliminated by Lemma 2.
- Note that if $p \equiv 1 \pmod{16}$ and $\left(\dfrac{2}{p}\right)_4 \neq \left(\dfrac{p}{2}\right)_4$ then by [4, Theorem 5.7], $h_2(L_p) \neq 8$.

Note that by results in the beginning of Section 3, $p \equiv 9 \pmod{16}$ implies that $\left(\dfrac{p}{2}\right)_4 = -1$. Hence, by Lemma 1, we have to check the following cases.

(I) $d = p$ is a prime such that $\left(\dfrac{2}{p}\right)_4 \neq \left(\dfrac{p}{2}\right)_4$.

(II) $d = p$ is a prime such that $p \equiv 9 \pmod{16}$ and $\left(\dfrac{2}{p}\right)_4 \neq \left(\dfrac{p}{2}\right)_4$.

(III) $d = q_1q_2$ for two primes q_1 and q_2 such that $q_1 \equiv q_2 \equiv 3 \pmod 8$.

Let $p \equiv 1 \pmod 8$ be a prime. Set $L_p^+ = \mathbb{Q}(\sqrt{2}, \sqrt{p})$, $K = \mathbb{Q}(\sqrt{2}, i)$ and $K' = \mathbb{Q}(\sqrt{2}, \sqrt{-p})$. By applying Proposition 1 to the extension $L_p/\mathbb{Q}(\sqrt{2})$, we have

$$h(L_p) = \frac{Q_{L_p}}{Q_KQ_{K'}} \frac{\omega_{L_p}}{\omega_K\omega_{K'}} \frac{h(L_p^+)h(K)h(K')}{h(\mathbb{Q}(\sqrt{2}))^2}.$$

We have $h(\mathbb{Q}(\sqrt{2})) = h(K) = 1$. By [6, Théorème 3], $Q_{L_p} = 1$ and by [4, Lemma 2.5] $Q_K = 1$. Since $\omega_{L_p} = \omega_K = 8$ and $\omega_{K'} = 2$, then by passing to the 2-part in the above equality we get

$$h_2(L_p) = \frac{1}{2Q_{K'}} h_2(L_p^+)h_2(K')\cdot \tag{2}$$

As ε_2 has a negative norm, so by the item (2) of Section 3 of [2] we obtain that $E_{K'} = \langle -1, \varepsilon_2\rangle$. This in turn implies that $q(K') = Q_{K'} = 1$. From which we infer, by Proposition 2, that $h_2(K') = \frac{1}{2}\cdot 1\cdot h_2(2)h_2(-p)h_2(-2p) = \frac{1}{2}h_2(-p)h_2(-2p)$. It follows, by the equality (2), that

$$h_2(L_p) = \frac{1}{4}h_2(L_p^+)h_2(-p)h_2(-2p)\cdot \tag{3}$$

Note that from [14, Theorem 2] and the proof of [14, Theorem 1], one deduces easily that $\left(\frac{2}{p}\right)_4 \neq \left(\frac{p}{2}\right)_4$.

$\circ$ Suppose that d takes the form (3). So 8 divides $h_2(-p)$. Note also that, by [17, page 596], $h_2(-2p)$ is divisible by 4, and by [19, Theorem 2] $h_2(L_p^+)$ is even. It follows by the equality (3) that 16 divides $h_2(L_p)$. So this case is eliminated.

$\circ$ Suppose that d takes the form (3). Thus $\left(\frac{2}{p}\right)_4$ and $\left(\frac{p}{2}\right)_4 = -1$. So, by [19, Theorem 2], $h_2(L_p^+) = 1$. Therefore, by the equality (3) and the note below it, we have $h_2(L_p) = h_2(-2p)$.

Keep the notations of [17, page 601] and let $p = a^2 + b^2 = 2e^2 - d^2$, with $e \geq 1$. By using notations in the proof of [14, Theorem 1] and [17, page 601] we easily deduce that $\left(\frac{2}{p}\right)_4 = (-1)^{b/4}$, so $b \equiv 0 (\mathrm{mod}\ 8)$. Therefore, by [17, Théroème 3], $h_2(-2p) \equiv 0(\mathrm{mod}\ 8)$. It follows, by [23, Theorem 2], $h_2(-2p) = 8$ if and only if $\left(\frac{u}{p}\right)_4 = -1$. So the first item of our theorem.

$\circ$ Suppose that d takes the third form (3). Without loss of generality we may assume that $\left(\frac{q_1}{q_2}\right) = 1$. By Proposition 2 we have

$$h(L_d) = \frac{1}{2^5}q(L_d)h(q_1q_2)h(-q_1q_2)h(2q_1q_2)h(-2q_1q_2)h(2)h(-2)h(-1).$$

By [15, Corollary 18.4] $h_2(q_1q_2) = 1$ and by [18, pages 345, 354] we have $h_2(2q_1q_2) = 2$ and $h_2(-q_1q_2) = 4$, respectively. It is known that $h(2) = h(-2) = h(-1) = 1$. Since by Lemma 4 $q(L_d) = 4$, then by passing to the 2-part in the above equality we get

$$h_2(L_d) = \frac{1}{2^5}\cdot 4\cdot 1\cdot 4\cdot 2\cdot h_2(-2q_1q_2) = h_2(-2q_1q_2).$$

Therefore, by [18, page 357] and Lemma 1, $\mathrm{Cl}_2(L_d) = (2,4)$ if and only if $h_2(-2q_1q_2) = 8$ which is equivalent to $\left(\frac{-2}{|k^2X + lY|}\right) = -1$. This achieves the proof. $\square$

Let l be a positive integer. For a finite abelian group G, its 2^l-rank is defined as $r_{2^l}(G) = \dim_{\mathbb{F}_2}(2^{l-1}G/2^lG)$. Or equivalently looking at the decomposition of the group G into cyclic groups as $G = \prod_i C_{n_i}$, the 2^l-rank of G equals the number of n_i's divisible by 2^l.

Corollary 1. *Let p be a rational prime such that $p \equiv 1 \pmod 8$ and $\left(\frac{2}{p}\right)_4 = \left(\frac{p}{2}\right)_4 = -1$, then the 8-rank of $\mathrm{Cl}_2(L_p)$ equals 1.*

PROOF: By [6, Théorème 10], we deduce that the 4-rank of $\mathrm{Cl}_2(L_p)$ equals 1, and by the proof of Theorem 1, $h_2(L_p)$ is divisible by 16, so the result follows. □

From the proof of Theorem 1 we get

Corollary 2. *Let q_1 and q_2 be two rational primes such that $q_1 \equiv q_2 \equiv 3 \pmod 8$, then $h_2(L_{q_1q_2}) = h_2(-2q_1q_2)$.*

Example 2. For all the examples below, we used PARI/GP calculator version 2.9.4 (64bit), December 20, 2017.

1. For $p = 89$, $u = 17$ and $v = 10$, we have $p = u^2 - 2v^2$, $\left(\frac{2}{p}\right)_4 = -\left(\frac{u}{p}\right)_4 = 1$ and $\mathrm{Cl}_2(L_{89}) \simeq (2, 4)$.
2. For $q_1 = 11$, $q_2 = 19$, $k = 1$, $l = 3$, $m = -1$, $X = 4$ and $Y = 1$, we have $q_1 = l^2 - 2k^2m$, $2q_2 = k^2X^2 + 2lXY + 2mY^2$, $\left(\frac{-2}{|k^2X + lY|}\right) = \left(\frac{-2}{7}\right) = -1$ and $\mathrm{Cl}_2(L_{11\cdot 19}) \simeq (2, 4)$.

4. Fields L_d for which $\mathrm{Cl}_2(L_d)$ is of type $(2, 2, 2)$

In this section we determine all fields L_d such that $\mathrm{Cl}_2(L_d) \simeq (2, 2, 2)$. Keep the above notations: p, p_i, q and q_i are prime integers satisfying $p \equiv p_i \equiv 1 \pmod 4$ and $q \equiv q_i \equiv 3 \pmod 4$ with $i \in \mathbb{N}^*$. From Section 4 of [4], it is easy to deduce the following result:

Lemma 5. *The rank of the 2-class group $\mathrm{Cl}_2(L_d)$ of L_d equals 3 if and only if d takes one of the following forms.*

1. *$d = p$ with $p \equiv 1 \pmod 8$ and $\left(\frac{2}{p}\right)_4 = \left(\frac{p}{2}\right)_4 = 1$.*
2. *$d = q_1q_2$ with $q_1 \equiv q_2 \equiv 7 \pmod 8$.*
3. *$d = qp$ with $q \equiv 3 \pmod 8$ and $p \equiv 1 \pmod 8$ and $\left(\frac{2}{p}\right)_4 = -1$.*
4. *$d = p_1p_2$ with $p_1 \equiv 5 \pmod 8$, $p_2 \equiv 1 \pmod 8$ and $\left(\frac{2}{p}\right)_4 \neq \left(\frac{p}{2}\right)_4$.*
5. *$d = q_1q_2p$ with $q_1 \equiv q_2 \equiv 3 \pmod 8$ and $p \equiv 5 \pmod 8$.*
6. *$d = qp_1p_2$ with $q \equiv 3 \pmod 8$ and $p_1 \equiv p_2 \equiv 5 \pmod 8$.*

We will need the following lemmas.

Lemma 6. *Let $d = q_1q_2$, with $q_1 \equiv q_2 \equiv 7 (\bmod 8)$ two primes such that $\left(\frac{q_1}{q_2}\right) = 1$. Then we have $E_{L_d} = \langle \zeta_8, \varepsilon_2, \varepsilon_{q_1q_2}, \sqrt{\varepsilon_{2q_1q_2}}$ or $\sqrt{\varepsilon_{q_1q_2}\varepsilon_{2q_1q_2}}\rangle$, and thus $q(L_d) = 4$.*

Proof: Similar to the proof of Lemma 4. □

Lemma 7.

1. *Let $d = p$ for a prime $p \equiv 1 (\bmod 8)$ such that $\left(\frac{2}{p}\right)_4 = \left(\frac{p}{2}\right)_4 = 1$. Then, $h_2(L_p) \equiv 0 (\bmod 16)$, and thus $\mathrm{Cl}_2(L_p)$ is not elementary.*
2. *Let $d = q_1q_2$ with q_1 and q_2 two rational primes such that $q_1 \equiv q_2 \equiv 7 (\bmod 8)$. Then, $h_2(L_{q_1q_2}) \equiv 0 (\bmod 32)$, and thus $\mathrm{Cl}_2(L_{q_1q_2})$ is not elementary.*
3. *Let $d = p_1p_2$ with $p_1 \equiv 5 (\bmod 8)$, $p_2 \equiv 1 (\bmod 8)$ two primes such that $\left(\frac{p_2}{p_1}\right) = 1$. Then, $h_2(L_{p_1p_2}) \equiv 0 (\bmod 32)$.*

Proof: 1. By equality (3) in the proof of Theorem 1 we have

$$h_2(L_p) = \frac{1}{4} h_2(L_p^+) h_2(-p) h_2(-2p).$$

Note that by [17, page 596], $h_2(-2p)$ is divisible by 4. Since for a prime $p' \equiv 1 (\bmod 8)$, $h_2(-p') = 4$ if and only if $\left(\frac{2}{p'}\right)_4 \neq \left(\frac{p'}{2}\right)_4$ (see the proof of Theorem 1), then $h_2(-p)$ is divisible by 8 (in fact $\left(\frac{2}{p}\right)_4 = \left(\frac{p}{2}\right)_4$ and by [15, Corollaries 18.4 and 19.6] $h_2(-p)$ is divisible by 4). Thus, $h_2(L_p)$ is divisible by $8 \cdot h_2(L_p^+)$. By [19, Theorem 2], $h_2(L_p^+)$ is even. From which we infer that $h_2(L_p)$ is divisible by 16. So we have the first item.

2. By Proposition 2 we have

$$h(L_{q_1q_2}) = \frac{1}{2^5} q(L_{q_1q_2}) h(q_1q_2) h(-q_1q_2) h(2q_1q_2) h(-2q_1q_2) h(2) h(-2) h(-1) \cdot$$

On one hand, by [18, page 345], $h_2(2q_1q_2)$ is divisible by 4, by [18, pages 354, 356], $h_2(-q_1q_2)$ and $h_2(-2q_1q_2)$ are both divisible by 8, and by [15, Corollary 18.4], $h_2(q_1q_2) = 1$. On the other hand, by Lemma 6, $q(L_{q_1q_2}) = 4$. From all these results, it follows by passing to the 2-part in the above equality that $h_2(L_d)$ is divisible by $(1/2^5) \cdot 4 \cdot 8 \cdot 4 \cdot 8 = 32$. And the second item follows.

3. We proceed as in the proof the second item. □

Example 3. Let $d = p$ be as in the first item of the above lemma. Using PARI/GP calculator version 2.9.4 (64bit), December 20, 2017, for $3 \leq p \leq 10^4$, we could not find a field L_d such that $h_2(L_d) = 16$. We have the following examples.

1. For $p = 113$, we have $\left(\frac{2}{p}\right)_4 = \left(\frac{p}{2}\right)_4 = 1$ and $h_2(L_{113}) = 64$.
2. For $p = 337$, we have $\left(\frac{2}{p}\right)_4 = \left(\frac{p}{2}\right)_4 = 1$ and $h_2(L_{337}) = 32$.
3. For $q_1 = 7$ and $q_2 = 31$, we have $h_2(L_{7\cdot 31}) = 32$.
4. For $q_1 = 7$ and $q_2 = 23$, we have $h_2(L_{7\cdot 23}) = 64$.

Lemma 8.

1. *If* $d = q_1q_2p$ *with* $q_1 \equiv q_2 \equiv 3 (\mathrm{mod}\ 8)$ *and* $p \equiv 5 (\mathrm{mod}\ 8)$, *then* $h_2(L_{q_1q_2p}) \equiv 0 (\mathrm{mod}\ 32)$. *So* $\mathrm{Cl}_2(L_{q_1q_2p})$ *is not elementary.*
2. *If* $d = qp_1p_2$ *with* $q \equiv 3\,(\mathrm{mod}\ 8)$ *and* $p_1 \equiv p_2 \equiv 5\,(\mathrm{mod}\ 8)$, *then* $h_2(L_{qp_1p_2}) \equiv 0 (\mathrm{mod}\ 32)$. *So* $\mathrm{Cl}_2(L_{qp_1p_2})$ *is not elementary.*

Proof: 1. Consider the following diagram (Figure 1 below):

$$\begin{tikzcd}[arrows=dash]
 & L_d = \mathbb{Q}(\sqrt{q_1q_2p}, \sqrt{2}, i) \arrow[dl] \arrow[d] \arrow[dr] & \\
K_2 = \mathbb{Q}(\sqrt{q_1q_2p}, i) \arrow[dr] & K_1 = \mathbb{Q}(\sqrt{2}, i) \arrow[d] & K_3 = \mathbb{Q}(\sqrt{2q_1q_2p}, i) \arrow[dl] \\
 & \mathbb{Q}(i) &
\end{tikzcd}$$

Figure 1. $\dfrac{L_{q_1q_2p}}{\mathbb{Q}(i)}$.

By applying Proposition 3 to the V_4-extension $L_{q_1q_2p}/\mathbb{Q}(i)$ we get:

$$h(L_d) = \frac{1}{4} Q\Big(\frac{L_d}{\mathbb{Q}(i)}\Big) \frac{h(K_1)h(K_2)h(K_3)}{h(\mathbb{Q}(i))^2}.$$

It is known that $h(K_1) = h(\mathbb{Q}(i)) = 1$ (in fact, $K_1 = \mathbb{Q}(\zeta_8)$), so

$$h(L_d) = \frac{1}{4} Q\Big(\frac{L_d}{\mathbb{Q}(i)}\Big) h(K_2)h(K_3). \tag{4}$$

Note that by [24, Proposition 2], the ranks of the 2-class groups of K_2 and K_3 equal 2 and 3, respectively. The author of [1] determined all fields $\mathbb{Q}(\sqrt{d}, i)$ for which the 2-class group is of type $(2, 2)$. By checking all the results of this last reference, we deduce that 2-class group of K_2 is not of type $(2, 2)$. So 8 divides the class number of K_2.

On the other hand, by [7, Théorème 5.3] the 2-class group of K_3 is not of type $(2, 2, 2)$. So the class number of K_2 is divisible by 16. Hence by equality (4) $h(L_d)$ is divisible by $\frac{1}{4} \cdot 8 \cdot 16 = 32$. So the first item follows.

2. We similarly prove the second item by using [7, Théorème 5.3] and [24, Proposition 2]. $\square$

Example 4. Let $d = qp_1p_2$ be as in the second item of the above Lemma. Using PARI/GP calculator version 2.9.4 (64bit), December 20, 2017, for $3 \leq q, p_i \leq 100$, we could not find a field L_d such that $h_2(L_d) = 32$. We have the following examples.

1. For $q_1 = 3$, $q_2 = 11$ and $p = 5$, we have $h_2(L_{3\cdot 11\cdot 5}) = 32$.
2. For $q_1 = 3$, $q_2 = 11$ and $p = 13$, we have $h_2(L_{3\cdot 11\cdot 13}) = 128$.
3. For $q = 3$, $p_1 = 5$ and $p_2 = 13$, we have $h_2(L_{3\cdot 5\cdot 13}) = 64$.
4. For $q = 3$, $p_1 = 5$ and $p_2 = 29$, we have $h_2(L_{3\cdot 5\cdot 29}) = 128$.

Lemma 9. *Let $d = qp$ with $q \equiv 3 \pmod 8$, $p \equiv 1 \pmod 8$ and $\left(\frac{p}{q}\right) = -1$. Then $E_{L_d} = \langle \zeta_8, \varepsilon_2, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}\,\rangle$, and thus $q(L_d) = 8$.*

Proof: We proceed as in the proof of Lemma 4. □

Now we are able to state the main theorem of this subsection.

Theorem 2. *Let d be a square free integer, then $\mathrm{Cl}_2(L_d) \simeq (2,2,2)$ if and only if d takes one of the two following forms:*

1. $d = p_1p_2$ *with* $p_1 \equiv 5 \pmod 8$, $p_2 \equiv 1 \pmod 8$, $\left(\frac{2}{p_2}\right)_4 \neq \left(\frac{p_2}{2}\right)_4$ *and* $\left(\frac{p_2}{p_1}\right) = -1$.
2. $d = qp$ *with* $q \equiv 3 \pmod 8$, $p \equiv 1 \pmod 8$, $\left(\frac{2}{p}\right)_4 = -1$ *and* $\left(\frac{p}{q}\right) = -1$.

Proof: It suffices to determine for which forms of d appearing in Lemma 5, we have $h_2(L_d) = 8$. Let us start, as above, by eliminating certain inconvenient cases.

- The forms of d in the first and the second items of Lemma 5 are eliminated by Lemma 7.
- The forms of d in the two last items of Lemma 5 are eliminated by Lemma 8.

It follows that it suffices to check the two following cases:

(I) $d = p_1p_2$ with $p_1 \equiv 5 \pmod 8$, $p_2 \equiv 1 \pmod 8$ and $\left(\frac{2}{p_2}\right)_4 \neq \left(\frac{p_2}{2}\right)_4$.

(II) $d = qp$ with $q \equiv 3 \pmod 8$ and $p \equiv 1 \pmod 8$ and $\left(\frac{2}{p}\right)_4 = -1$.

∘ Suppose that d takes the form I, then [10, Theorem 4.2] and the last assertion of Lemma 7 give the first item.

∘ Suppose now that d takes the form II. Consider the following diagram (Figure 2 below):

$$\begin{tikzcd}[column sep=small]
 & L_d = \mathbb{Q}(\sqrt{2}, \sqrt{d}, i) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
L_d^+ = \mathbb{Q}(\sqrt{d}, \sqrt{2}) \arrow[dr, dash] & K_2 = \mathbb{Q}(\sqrt{2}, i) \arrow[d, dash] & K_1 = \mathbb{Q}(\sqrt{2}, \sqrt{-d}) \arrow[dl, dash] \\
 & \mathbb{Q}(\sqrt{2}) &
\end{tikzcd}$$

FIGURE 2. $\dfrac{L_d}{\mathbb{Q}(\sqrt{2})}$.

By Proposition 1 we have

$$h(L_d) = \frac{Q_{L_d}}{Q_{K_1}Q_{K_2}} \frac{\omega_{L_d}}{\omega_{K_1}\omega_{K_2}} \frac{h(L_d^+)h(K_1)h(K_2)}{h(\mathbb{Q}(\sqrt{2}))^2}. \tag{5}$$

Note that $h(K_2) = h(\mathbb{Q}(\sqrt{2})) = 1$. We have, $d\varepsilon_2$ is not a square in $\mathbb{Q}(\sqrt{2})$. Otherwise we will get for some α in $\mathbb{Q}(\sqrt{2})$, $d\varepsilon_2 = \alpha^2$, then $N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(d\varepsilon_2) = -d^2 = N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(\alpha)^2$, which is false. It follows by [2, Proposition 3], that $\{\varepsilon_2\}$ is a fundamental system of units of K_1, i.e., $Q_{K_1} = 1$. Since $Q_{K_2} = 1$ (cf. [4, Lemma 2.5]), $\omega_{L_d} = \omega_{K_2} = 8$ and $\omega_{K_1} = 2$, then by passing to the 2-part in the equality (5), we get $h_2(L_d) = \frac{1}{2}Q_{L_d}h_2(L_d^+)h_2(K_1)$. By Proposition 3, we have $h_2(K_1) = \frac{1}{2}h_2(-2d)h_2(-d)h_2(2)$. So

$$h_2(L_d) = \frac{1}{4}Q_{L_d}h_2(L_d^+)h_2(-2d)h_2(-d). \tag{6}$$

Note that, by Lemma 9, if $\left(\frac{p}{q}\right) = -1$, then $Q_{L_d} = 1$. Note also that,

(1) from [5, Théorème 2] and its proof, we deduce that $h_2(L_d^+)$ is divisible by 4 and $h_2(L_d^+) = 4$ if and only if $\left(\frac{p}{q}\right) = -1$,
(2) from [15, Corollaries 18.5 and 19.6], we deduce that $h_2(-d)$ is even and $h_2(-d) = 2$ if and only if $\left(\frac{p}{q}\right) = -1$,
(3) from [15, Corollaries 18.5 and 19.6] and [18, page 353], we deduce that $h_2(-2d)$ is divisible by 4 and $h_2(-2d) = 4$ if and only if $\left(\frac{p}{q}\right) = -1$.

Hence plugging all of these results into equality (6), one gets the second item, which completes the proof. $\square$

Example 5. For all the examples below, we used PARI/GP calculator version 2.9.4 (64bit), December 20, 2017.

1. Let $p_1 = 29$ and $p_2 = 17$. We have $p_2 \equiv 1 (\mathrm{mod}\ 8)$, $\left(\frac{2}{p_2}\right)_4 \neq \left(\frac{p_2}{2}\right)_4$, $\left(\frac{p_2}{p_1}\right) = -1$ and $\mathrm{Cl}_2(L_{29\cdot 17}) \simeq (2,2,2)$.
2. Let $q = 11$ and $p = 17$. We have $\left(\frac{2}{p}\right)_4 = -1$, $\left(\frac{p}{q}\right) = -1$ and $\mathrm{Cl}_2(L_{11\cdot 17}) \simeq (2,2,2)$.

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A. Azizi, M. M. Chems-Eddin:
Mohammed First University, Mathematics Department, Sciences Faculty, Mohammed V avenue, P. O. Box 524, Oujda 60040, Morocco

E-mail: abdelmalekazizi@yahoo.fr

E-mail: 2m.chemseddin@gmail.com

A. Zekhnini:
Mohammed First University, Mathematics Department,
Pluridisciplinary Faculty, B. P. 300, Selouane, Nador 62700, Morocco

E-mail: zekha1@yahoo.fr

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