Extreme points of the Besicovitch–Orlicz space of almost periodic functions equipped with the Luxemburg norm

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Abstract. We investigate which points in the unit sphere of the Besicovitch–Orlicz space of almost periodic functions, equipped with the Luxemburg norm, are extreme points. Sufficient conditions for the strict convexity of this space are also given.

Keywords: Besicovitch–Orlicz space; extreme point; strict convexity; almost periodic function

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1. Introduction

It is well known that extreme points, which are connected with strict convexity of the space, are the most basic concepts in the geometric theory of Banach spaces, see [5]. The notion of extreme point plays an important role in functional analysis, convex analysis and optimization. For example, the Krein–Milman theorem which shows that any compact convex set is the convex hull of its extreme point set.

The criteria for extreme points and strict convexity in classical Orlicz spaces (i.e. Banach spaces of which the L^p spaces are a special case) and Musielak–Orlicz spaces (i.e. spaces which are generalization of Lebesgue spaces with variable exponents $L^{p(x)}$) equipped with the Orlicz norm, the Luxemburg norm, and p-Amemiya norm, have been obtained earlier, see for instance [4], [6], [13], [12].

In recent years, some geometrical properties of the Besicovitch–Orlicz space of almost periodic functions have been considered in [1], [3], [8], [10], [11]. However until now, criteria for extreme points in this class of generalized almost periodic function spaces are not given. In this paper, we characterize extreme points of the unit ball of the Besicovitch–Orlicz space of almost periodic functions equipped with the Luxemburg norm.

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2. Preliminaries

In this section, we recall a sequence of definitions and results which will be used in what follows.

Let $\Sigma(\mathbb{R})$ be the $\sigma-$algebra of all Lebesgue-measurable subsets of $\mathbb{R}$, μ the Lebesgue measure on $\mathbb{R}$ and $M(\mathbb{R},\mathbb{C})$ the set of all complex valued Lebesgue measurable functions defined on $\mathbb{R}$.

We denote by $B(\mathbb{X})$ ($S(\mathbb{X})$, respectively) the closed unit ball (the unit sphere, respectively) of a Banach space $(\mathbb{X},\|\cdot\|)$.

A point $x\in S(\mathbb{X})$ is said to be an extreme point of $B(\mathbb{X})$ if it cannot be written as the arithmetic mean $\frac{1}{2}(y+z)$ of two distinct points $y,z\in S(\mathbb{X})$. Namely, if the following implication holds

$$y,z\in S(\mathbb{X}),\qquad x=\frac{y+z}{2}\Rightarrow y=z.$$

The set of all extreme points of $B(\mathbb{X})$ will be denoted by $\mathbf{extr}[B(\mathbb{X})]$. It is well known that if $\mathbf{extr}[B(\mathbb{X})]=S(\mathbb{X})$ then $\mathbb{X}$ is strictly convex (rotund).

2.1 Young functions. To introduce the desired class of almost periodic functions, recall that a function $\varphi\colon\mathbb{R}\to\mathbb{R}^+$ is said to be a Young function if it is even, convex, vanishing only at zero and $\lim_{|x|\to\infty}\varphi(x)=\infty$.

A Young function φ is said to satisfy the Δ_2-condition for large values (we write $\varphi\in\Delta_2$), when there exist constants $k>0$ and $u_0>0$ such that,

$$\varphi(2u)\leq k\varphi(u),\qquad\forall\,|u|\geq u_0.$$

A Young function φ is called strictly convex on $\mathbb{R}$ if

$$\varphi\Big(\frac{u+v}{2}\Big)<\frac{1}{2}(\varphi(u)+\varphi(v)),\qquad\forall\,u,v\in\mathbb{R},\ u\neq v.$$

Let us recall that if φ is strictly convex then it is uniformly convex on any bounded interval, see [4, Proposition 1.4]. Namely, for any $k>0$, and $\varepsilon>0$, there exists $\delta>0$ such

$$\varphi\Big(\frac{u+v}{2}\Big)\leq(1-\delta)\Big(\frac{\varphi(u)+\varphi(v)}{2}\Big)\tag{2.1}$$

for any $u,v\in\mathbb{R}$ satisfying $|u|\leq k$, $|v|\leq k$ and $|u-v|\geq\varepsilon$.

Following [4], an interval $[a,b]$ is called a structural affine interval of a Young function φ, provided that φ is affine on $[a,b]$ and it is not affine on either $[a-\varepsilon,b]$ or $[a,b+\varepsilon]$ for any $\varepsilon>0$.

Let $\{[a_i,b_i]\}_i$ be all the structural affine intervals of φ. We denote $S_\varphi=\mathbb{R}\backslash\big[\bigcup_i\,]a_i,b_i[\,\big]$ the set of strictly convex points of φ. Clearly, if $u,v\in\mathbb{R}$, $\alpha\in]0,1[$

and $\alpha u + (1-\alpha)v \in S_\varphi$, then

$$\varphi(\alpha u + (1-\alpha)v) < \alpha\varphi(u) + (1-\alpha)\varphi(v).$$

2.2 The Besicovitch–Orlicz space of almost periodic functions. We denote by $L^\varphi_{\mathrm{loc}}(\mathbb{R},\mathbb{C})$ the subspace of $M(\mathbb{R},\mathbb{C})$ such that for each bounded interval U there exists $\alpha > 0$ such that

$$\int_U \varphi(\alpha|f(s)|)\,\mathrm{d}s < \infty.$$

When $U = [0,1]$, we get the *Orlicz space* $L^\varphi([0,1],\mathbb{C})$, see [4].

The Besicovitch–Orlicz pseudo modular ϱ_{B^φ} is defined in [7] as follows

$$\begin{aligned} \varrho_{B^\varphi}\colon L^\varphi_{\mathrm{loc}}(\mathbb{R},\mathbb{C}) &\to \overline{\mathbb{R}}^+ \\ f &\mapsto \varlimsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \varphi(|f(t)|)\,\mathrm{d}\mu. \end{aligned}$$

Its associated modular space, called Besicovitch–Orlicz space, is

$$\mathfrak{B}^\varphi(\mathbb{R},\mathbb{C}) = \{f \in L^\varphi_{\mathrm{loc}}(\mathbb{R},\mathbb{C})\colon\ \varrho_{B^\varphi}(\lambda f) < \infty \text{ for some } \lambda > 0\}.$$

This space is endowed with the Luxemburg pseudonorm

$$\|f\|_{B^\varphi} = \inf\Big\{k > 0\colon\ \varrho_{B^\varphi}\Big(\frac{f}{k}\Big) \le 1\Big\}.$$

Let us consider the equivalence relation

$$f \sim_\varphi g \Leftrightarrow \|f-g\|_{B^\varphi} = 0, \qquad \forall\, f,g \in \mathfrak{B}^\varphi(\mathbb{R},\mathbb{C}).$$

We denote by $B^\varphi(\mathbb{R},\mathbb{C}) := \mathfrak{B}^\varphi(\mathbb{R},\mathbb{C})/\sim_\varphi$ the quotient space. Henceforth, we will not distinguish between an element of $\mathfrak{B}^\varphi(\mathbb{R},\mathbb{C})$ and its equivalence class in $B^\varphi(\mathbb{R},\mathbb{C})$.

Endowed with the Luxemburg norm $\|\cdot\|_{B^\varphi}$, $B^\varphi(\mathbb{R},\mathbb{C})$ is a Banach space.

Denote by $\mathrm{Trig}(\mathbb{R},\mathbb{C})$ the linear set of all generalized trigonometric polynomials, i.e.

$$\mathrm{Trig}(\mathbb{R},\mathbb{C}) = \Big\{P(t) = \sum_{j=1}^{n} \alpha_j \exp(i\lambda_j t)\colon\ \lambda_j \in \mathbb{R},\ \alpha_j \in \mathbb{C},\ j \in \mathbb{N}\Big\}.$$

In his celebrated paper [7], T. R. Hillmann has used a similar approach like Besicovitch in [2] to obtain an extension of Besicovitch almost periodic functions in context of Orlicz spaces. Namely, the Besicovitch–Orlicz space of almost periodic functions, denoted by $B^{\varphi}_{\text{a.p.}}(\mathbb{R},\mathbb{C})$, is the closure of $\operatorname{Trig}(\mathbb{R},\mathbb{C})$ in $B^{\varphi}(\mathbb{R},\mathbb{C})$, with respect to the norm $\|\cdot\|_{B^{\varphi}}$. More exactly we define

$$\begin{aligned} B^{\varphi}_{\text{a.p.}}(\mathbb{R},\mathbb{C}) &= \big\{ f \in B^{\varphi}(\mathbb{R},\mathbb{C})\colon \exists\,(P_n)_{n\geq 1} \subset \operatorname{Trig}(\mathbb{R},\mathbb{C}), \\ &\qquad \text{s.t. } \lim_{n\to\infty} \|f - P_n\|_{B^{\varphi}} = 0\big\} \\ &= \big\{ f \in B^{\varphi}(\mathbb{R},\mathbb{C})\colon \exists\,(P_n)_{n\geq 1} \subset \operatorname{Trig}(\mathbb{R},\mathbb{C}), \\ &\qquad \text{s.t. } \forall\, k>0,\ \lim_{n\to\infty} \varrho_{B^{\varphi}}(k(f-P_n)) = 0\big\}. \end{aligned}$$

Remark 1.

(1) If we denote by $AP(\mathbb{R},\mathbb{C})$ the Banach space of almost periodic functions, we have

$$AP(\mathbb{R},\mathbb{C}) \subset B^{\varphi}_{\text{a.p.}}(\mathbb{R},\mathbb{C}).$$

(2) In the particular case where $f \in B^{\varphi}_{\text{a.p.}}(\mathbb{R},\mathbb{C})$ we have

$$\varrho_{B^{\varphi}}(k(f)) < \infty, \qquad \forall\, k > 0.$$

Indeed, if $f \in B^{\varphi}_{\text{a.p.}}(\mathbb{R},\mathbb{C})$ then for any $\varepsilon > 0$ there exists a trigonometric polynomial P_ε such that for any $k > 0$

$$\varrho_{B^{\varphi}}(k(f - P_\varepsilon)) \leq \frac{\varepsilon}{2}.$$

Then using the convexity of φ and the fact that the trigonometric polynomial P_ε is bounded we get

$$\varrho_{B^{\varphi}}(kf) \leq \frac{1}{2}\varrho_{B^{\varphi}}(2k(f-P_\varepsilon)) + \frac{1}{2}\varrho_{B^{\varphi}}(2kP_\varepsilon) < \infty.$$

From [8], we know that when $f \in B^{\varphi}_{\text{a.p.}}(\mathbb{R},\mathbb{C})$ the limit in the expression of $\varrho_{B^{\varphi}}(f)$ exists and is finite, i.e.

$$\varrho_{B^{\varphi}}(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \varphi(|f(t)|)\,\mathrm{d}\mu. \tag{2.2}$$

This fact is very useful in our computations.

T. R. Hillmann in [7] has introduced the subadditive measure $\overline{\mu}_B$ on $\Sigma(\mathbb{R})$ as the following

$$\overline{\mu}_B(A) = \varlimsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \chi_A(t)\,\mathrm{d}\mu = \varlimsup_{T\to\infty} \frac{1}{2T}\,\mu(A\cap[-T,T]), \tag{2.3}$$

where χ_A denotes the characteristic function of A.

It is clear that $\overline{\mu}_B$ is increasing, null on sets with μ-finite measure and it is not σ-additive.

Let us recall that a sequence $(f_n)_{n\geq 1} \subset B^{\varphi}(\mathbb{R}, \mathbb{C})$ is called:

(1) modular convergent to some $f \in B^{\varphi}(\mathbb{R}, \mathbb{C})$ when there exists $\alpha > 0$ such that
$$\lim_{n\to\infty} \varrho_{B^{\varphi}}(\alpha(f_n - f)) = 0.$$

(2) $\overline{\mu}_B$-convergent to a function f when for all $\varepsilon > 0$,
$$\lim_{n\to\infty} \overline{\mu}_B\{t \in \mathbb{R}\colon |f_n(t) - f(t)| \geq \varepsilon\} = 0.$$

In his work [9], M. Morsli showed that if $(f_n)_{n\in\mathbb{N}}$ is modular convergent to some $f \in B^{\varphi}(\mathbb{R}, \mathbb{C})$, it is also $\overline{\mu}_B$-convergent to f. He also gave in [9] a result similar to the usual Lebesgue dominated convergence theorem in the space $B^{\varphi}(\mathbb{R}, \mathbb{C})$, as it can be seen in the following proposition.

Proposition 1 (see [9]). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $B^{\varphi}(\mathbb{R}, \mathbb{C})$. Then if $(f_n)_{n\in\mathbb{N}}$ is $\overline{\mu}_B$-convergent to some $f \in B^{\varphi}(\mathbb{R}, \mathbb{C})$ and there exists $g \in B^{\varphi}_{\text{a.p.}}(\mathbb{R}, \mathbb{C})$ such that $\max(|f_n(x)|, |f(x)|) \leq |g(x)|$ for all $x \in \mathbb{R}$. Then,*
$$\lim_{n\to\infty} \varrho_{B^{\varphi}}(f_n) = \varrho_{B^{\varphi}}(f).$$

The next lemmas will be very useful in the proof of the main result.

Lemma 1 (see [1]). *Let $f \in B^{\varphi}_{\text{a.p.}}(\mathbb{R}, \mathbb{C})$ then:*

(1) *$\|f\|_{B^{\varphi}} \leq 1$ if and only if $\varrho_{B^{\varphi}}(f) \leq 1$,*
(2) *$\|f\|_{B^{\varphi}} = 1$ if and only if $\varrho_{B^{\varphi}}(f) = 1$.*

Lemma 2 (see [8]). *Let $f \in B^{\varphi}_{\text{a.p.}}(\mathbb{R}, \mathbb{C})$ such that $\|f\|_{B^{\varphi}} = a$ with $a > 0$. Then there exists real numbers $0 < \alpha < \beta$ and $\theta \in]0, 1[$ such that $\overline{\mu}_B(G) \geq \theta$, where*
$$G = \{t \in \mathbb{R}\colon \alpha \leq |f(t)| \leq \beta\}.$$

3. Main result

Our first goal in this work is to show that if $f \in B^{\varphi}_{\text{a.p.}}(\mathbb{R}, \mathbb{C})$ then we do not necessarily have $f\chi_A \in B^{\varphi}_{\text{a.p.}}(\mathbb{R}, \mathbb{C})$ for any $A \in \Sigma(\mathbb{R})$.

Lemma 3. *There is a Lebesgue measurable subset A of $\mathbb{R}$ for which the limit*
$$\lim_{T\to\infty} \frac{1}{2T}\,\mu([-T, T] \cap A), \tag{3.1}$$
does not exist.

Proof: Let us note first that if the limit (3.1) exists, it would be the same if T is an integer. So to show this lemma, it is sufficient to find a subset $A \in \Sigma(\mathbb{R})$ such that

$$\lim_{N\to\infty} \frac{1}{N}\,\mu([0,N]\cap A),$$

does not exist.

Since, for $n \geq 1$, there exists $k \in \mathbb{N}$ for which $n \in [3^{2k}, 3^{2k+1}[$, we define the sequences $(u_n)_{n\geq 1}$ and $(A_n)_{n\geq 1}$ as following

$$u_n = \begin{cases} 0 & \text{if } \ n \in [3^{2k}, 3^{2k+1}[\,, \\ 1 & \text{if } \ n \in [3^{2k+1}, 3^{2(k+1)}[\,, \end{cases} \tag{3.2}$$

and

$$A_n = \begin{cases} [n, n+1[& \text{if } \ n \in [3^{2k}, 3^{2k+1}[\,, \\ \emptyset & \text{if } \ n \in [3^{2k+1}, 3^{2(k+1)}[\,. \end{cases}$$

We have $\mu(A_n) = u_n$ for all $n \geq 1$.

Defining $A = \bigcup_{n\geq 1} A_n$ and $S_N = \frac{1}{N}\sum_{n=1}^{N} u_n$, we get

$$\lim_{N\to\infty} \frac{1}{N}\,\mu(A\cap[0,N]) = \lim_{N\to\infty} S_N.$$

The limit $\lim_{N\to\infty} S_N$ does not exist. Indeed, for each $N \geq 1$ we have

$$\begin{aligned} S_{3^{2N}} &= \frac{1}{3^{2^N}} \sum_{n=1}^{3^{2N}} u_n = \frac{1}{3^{2^N}} \sum_{k=0}^{N-1} \Bigg(\sum_{n=3^{2k}}^{3^{2k+1}-1} u_n + \sum_{n=3^{2k+1}}^{3^{2(k+1)}-1} u_n \Bigg) + u_{3^{2N}} \\ &= \frac{1}{3^{2^N}} \sum_{k=0}^{N-1} (3^{2(k+1)} - 3^{2k+1}) = \frac{6}{3^{2^N}} \sum_{n=1}^{N-1} 3^{2k} \\ &= \frac{6}{3^{2^N}} \Big(\frac{3^{2N}-1}{8} \Big). \end{aligned}$$

It follows that $\lim_{N\to\infty} S_{3^{2N}} = \frac{3}{4}$.

In the other hand,

$$\lim_{N\to\infty} S_{3^{2N+1}} = \lim_{N\to\infty} \Big(\frac{1}{3} S_{3^{2N}} + \frac{1}{3^{2N+1}} \Big) = \frac{1}{4}.$$

This ends the proof. $\square$

Remark 2. If we take f the constant function equal to 1, then using (2.2) and Lemma 3, we deduce that there exists $A \in \Sigma(\mathbb{R})$ such that $f\chi_A \notin B^{\varphi}_{\text{a.p.}}(\mathbb{R},\mathbb{C})$.

Let us give another property of $\overline{\mu}_B$.

Lemma 4. *The function* $\overline{\mu}_B \colon \Sigma(\mathbb{R}) \to [0,1]$ *is surjective.*

Proof: (1) Let $A \in \Sigma(\mathbb{R})$.

(a) It is clearly, by the definition of $\overline{\mu}_B$, that if $\mu(A) < \infty$ we have $\overline{\mu}_B(A) = 0$.
(b) Since $\overline{\mu}_B(\mathbb{R}) = 1$, we obtain $\overline{\mu}_B(A) = 1$ when $\mu(A^c) < \infty$.

(2) Let $\beta \in]0,1[$, there exists $\alpha > 0$ such that $\beta = \alpha/(\alpha+1)$. We define

$$A_n = [(\alpha+1)(n-1), (\alpha+1)(n-1)+\alpha], \quad n \in \mathbb{Z}^*, \qquad \text{and} \quad A = \bigcup_{n \in \mathbb{Z}^*} A_n.$$

Then,

$$\overline{\mu}_B(A) = \overline{\lim_{T\to\infty}} \frac{1}{T} \mu\Big([0,T] \cap \Big(\bigcup_{n\geq 1} A_n\Big)\Big) = \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{n\geq 1} \mu(A_n \cap [0,T]).$$

It is easy to show that for any $n \geq 1$,

$$A_n \cap [0,T] = \begin{cases} A_n & \text{if } n \leq \left\lfloor \frac{T+1}{1+\alpha} \right\rfloor, \\ \varphi & \text{if } n > \left\lfloor 1 + \frac{T}{\alpha+1} \right\rfloor, \\ \left[\frac{T+1}{1+\alpha}, T\right] & \text{if } n \in \left[\frac{T+1}{1+\alpha}, 1 + \frac{T}{1+\alpha}\right], \end{cases}$$

where $\lfloor \cdot \rfloor$ denotes the floor function.
Since there is at most one integer in the interval $\left[\frac{T+1}{1+\alpha}, 1+\frac{T}{1+\alpha}\right]$, it follows that there exists $0 \leq \theta \leq 1$ such that

$$\begin{aligned} \overline{\mu}_B(A) &= \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{n\geq 1} \mu(A_n \cap [0,T]) = \overline{\lim_{T\to\infty}} \frac{1}{T} \left(\sum_{n=1}^{\left\lfloor \frac{T+1}{1+\alpha} \right\rfloor} \mu(A_n) + \theta\alpha \right) \\ &= \overline{\lim_{T\to\infty}} \frac{1}{T} \left(\left\lfloor \frac{T+1}{1+\alpha} \right\rfloor \alpha + \theta\alpha \right). \end{aligned}$$

Using the inequality $x-1 < \lfloor x \rfloor \leq x$ for all $x \in \mathbb{R}$, we get $\overline{\mu}_B(A) = \alpha/(\alpha+1) = \beta$. □

In the following we characterize extreme points of the unit ball of $B^{\varphi}_{\text{a.p.}}(\mathbb{R},\mathbb{C})$. We start with some auxiliary lemmas.

Definition 1. A function $f \in B^{\varphi}(\mathbb{R},\mathbb{C})$ is said to be absolutely φ-integrable in $\overline{\mu}_B$ sense, if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that for every measurable subset $A \in \Sigma(\mathbb{R})$ with $\overline{\mu}_B(A) < \delta$ we have

$$\|f\chi_A\|_{B^{\varphi}} \leq \varepsilon.$$

Lemma 5. *Functions $f \in B^{\varphi}_{\text{a.p.}}(\mathbb{R},\mathbb{C})$ are absolutely φ-integrable in the $\overline{\mu}_B$ sense.*

Proof: First, let us show that bounded functions are absolutely φ-integrable in $\overline{\mu}_B$ sense.

Let $\varepsilon > 0$, $A \in \Sigma(\mathbb{R})$ and $f\colon \mathbb{R} \to \mathbb{C}$ be a bounded function. Put $\mathcal{C} = \sup_{t\in\mathbb{R}} |f(t)|$. Here, we exclude for simplicity the trivial case, when $\overline{\mu}_B(A) = 0$. Clearly, we have

$$\|\chi_A\|_{B^\varphi} = \frac{1}{\varphi^{-1}(1/\overline{\mu}_B(A))} \qquad \text{and} \qquad \|f\chi_A\|_{B^\varphi} \le \mathcal{C}\|\chi_A\|_{B^\varphi}.$$

Since the function $t \to (\varphi^{-1}(1/t))^{-1}$ is continuous and increasing on $]\,0,\infty[$, we deduce that there exists $\delta := (\varphi(\mathcal{C}/\varepsilon))^{-1}$ such that $\|f\chi_A\|_{B^\varphi} \le \varepsilon$, whenever $\overline{\mu}_B(A) < \delta$.

Now, let us assume that $f \in B^\varphi_{\text{a.p.}}(\mathbb{R},\mathbb{C})$. There exists a trigonometric polynomial P_ε such that

$$\|f - P_\varepsilon\|_{B^\varphi} \le \frac{\varepsilon}{2}. \tag{3.3}$$

Since P_ε is absolutely φ-integrable in the $\overline{\mu}_B$ sense, there exists $\delta > 0$ such that $\|P_\varepsilon\chi_A\|_{B^\varphi} \le \varepsilon/2$ whenever $\overline{\mu}_B(A) < \delta$. For such δ, we have

$$\|f\chi_A\|_{B^\varphi} \le \|(f - P_\varepsilon)\chi_A\|_{B^\varphi} + \|P_\varepsilon\chi_A\|_{B^\varphi} \le \|f - P_\varepsilon\|_{B^\varphi} + \|P_\varepsilon\chi_A\|_{B^\varphi} \le \varepsilon.$$

This completes the proof of the lemma. □

Lemma 6. *Let f be a function in $B^\varphi_{\text{a.p.}}(\mathbb{R},\mathbb{C})$, then there exists $\delta > 0$ such that*

$$f\chi_{E^c} \in B^\varphi_{\text{a.p.}}(\mathbb{R},\mathbb{C})$$

for any $E \in \Sigma(\mathbb{R})$ with $\overline{\mu}_B(E) < \delta$. Consequently, $f\chi_E \in B^\varphi_{\text{a.p.}}(\mathbb{R},\mathbb{C})$.

Proof: Let $\varepsilon > 0$, there exists a trigonometric polynomial P_ε such that

$$\|f - P_\varepsilon\|_{B^\varphi} \le \frac{\varepsilon}{2}. \tag{3.4}$$

Using Lemma 5, there exists $\delta > 0$ such that $\|P_\varepsilon\chi_E\|_{B^\varphi} \le \varepsilon/2$ for every measurable subset $E \in \Sigma(\mathbb{R})$ with $\overline{\mu}_B(E) < \delta$.

For the above P_ε, E and δ we have

$$\begin{aligned}\|f\chi_{E^c} - P_\varepsilon\|_{B^\varphi} &= \|f\chi_{E^c} - P_\varepsilon\chi_{E^c} - P_\varepsilon\chi_E\|_{B^\varphi}\\ &\le \|(f - P_\varepsilon)\chi_{E^c}\|_{B^\varphi} + \|P_\varepsilon\chi_E\|_{B^\varphi}\\ &\le \|f - P_\varepsilon\|_{B^\varphi} + \|P_\varepsilon\chi_E\|_{B^\varphi} \le \varepsilon.\end{aligned}$$

This shows that $f\chi_{E^c} \in B^\varphi_{\text{a.p.}}(\mathbb{R},\mathbb{C})$. Hence, the space $B^\varphi_{\text{a.p.}}(\mathbb{R},\mathbb{C})$ being linear, we get $f\chi_E \in B^\varphi_{\text{a.p.}}(\mathbb{R},\mathbb{C})$. □

We are now ready to give our principal result.

Theorem 1. *Let $f \in S(B^{\varphi}_{\text{a.p.}}(\mathbb{R},\mathbb{C}))$. We suppose that $\overline{\mu}_B(f^{-1}([a,b])) = 0$ for any structural affine interval $[a,b]$ of φ. Then $f \in \mathbf{extr}[B(B^{\varphi}_{\text{a.p.}}(\mathbb{R},\mathbb{C}))]$ if and only if $\mu(\{t \in \mathbb{R}\colon f(t) \notin S_\varphi\}) = 0$.*

Proof: The proof is inspired by the proof of [8, Theorem 1]) and [4, Theorem 2.1].

Sufficiency: Suppose that there exists $g, h \in S(B^{\varphi}_{\text{a.p.}}(\mathbb{R},\mathbb{C}))$, $g \neq h$, such that $2f = g + h$. By Lemma 1 we have

$$\varrho_{B^\varphi}((g+h)/2) = \varrho_{B^\varphi}(f) = \varrho_{B^\varphi}(g) = \varrho_{B^\varphi}(h) = 1.$$

Since $\|g-h\|_{B^\varphi} \neq 0$, Lemma 2 ensures the existence of constants $0 < \alpha < \beta$ and $\theta \in]\,0,1[$ for which $\overline{\mu}_B(G_1) > \theta$, where

$$G_1 = \{t \in \mathbb{R}\colon \alpha \leq |g(t) - h(t)| \leq \beta\}.$$

Define $M = \varphi^{-1}(2/\overline{\mu}_B(G_1))$, then $M \leq \varphi^{-1}(2/\theta)$.

Denoting $G_2 = \{t \in \mathbb{R}\colon \text{ s.t. } |g(t)| \geq M\}$ we have,

$$1 = \varrho_{B^\varphi}(g) \geq \varrho_{B^\varphi}(g\chi_{G_2}) \geq \varphi(M)\,\overline{\mu}_B(G_2) = 2\,\frac{\overline{\mu}_B(G_2)}{\overline{\mu}_B(G_1)}.$$

Consequently, we have

$$\overline{\mu}_B(G_2) \leq \frac{\overline{\mu}_B(G_1)}{2}. \tag{3.5}$$

Consider now the subset Q of $\mathbb{R}^2$ defined by

$$Q = \Big\{(u,v) \in \mathbb{R}^2\colon\ u, v \in \Big[-\Big(\varphi^{-1}\Big(\frac{2}{\theta}\Big) + \beta\Big), \Big(\varphi^{-1}\Big(\frac{2}{\theta}\Big) + \beta\Big)\Big],$$
$$\Big|u - v\Big| \geq \alpha,\ \frac{u+v}{2} \in S_\varphi\Big\}.$$

Let $F\colon \mathbb{R}^2 \backslash (0,0) \to \mathbb{R}$ be the function defined by

$$F(u,v) = \frac{2\varphi((u+v)/2)}{\varphi(u) + \varphi(v)}.$$

Function F is continuous and $F(u,v) < 1$ for all $u, v \in Q$.

Since Q is compact, there exists $0 < \delta < 1$ such that $\sup_{(u,v)\in Q} F(u,v) = 1 - \delta$.

So we have

$$\varphi\Big(\frac{u+v}{2}\Big) \leq (1-\delta)\frac{\varphi(u) + \varphi(v)}{2}, \qquad \forall\,(u,v) \in Q. \tag{3.6}$$

Let us define $G = (G_1 \cap E)\backslash G_2$, where $E = \{t \in \mathbb{R}\colon \text{ s.t. } f(t) \in S_\varphi\}$. It is clear that for all $t \in G$, $(g(t), h(t)) \in Q$. We have also $\overline{\mu}_B(G) \geq \theta/2$.

Indeed, since $\mu(E^c) = 0$ we have $\overline{\mu}_B(G_1 \cap E) = \overline{\mu}_B(G_1)$. Then sing (3.5) we get

$$\overline{\mu}_B(G) = \overline{\mu}_B((G_1 \cap E)\backslash G_2) \geq \overline{\mu}_B(G_1 \cap E) - \overline{\mu}_B(G_2) \geq \theta - \frac{\theta}{2} = \frac{\theta}{2}. \tag{3.7}$$

We denote $\overline{G} = G \cap [-T, T]$ and $\varrho_T((g+h)/2) = \frac{1}{2T}\int_{-T}^{T} \varphi(|g(t)+h(t)|/2)\,\mathrm{d}\mu$, then by (3.6) we obtain

$$\begin{aligned}
\varrho_T\Big(\frac{g+h}{2}\Big) &= \frac{1}{2T}\int\limits_{\overline{G}} \varphi\Big(\frac{|g(t)+h(t)|}{2}\Big)\,\mathrm{d}\mu + \frac{1}{2T}\int\limits_{\overline{G}^c} \varphi\Big(\frac{|g(t)+h(t)|}{2}\Big)\,\mathrm{d}\mu \\
&\leq (1-\delta)\,\frac{1}{2T}\int\limits_{\overline{G}} \frac{1}{2}[\varphi(|g(t)|) + \varphi(|h(t)|)]\,\mathrm{d}\mu \\
&\quad + \frac{1}{2T}\int\limits_{\overline{G}^c} \frac{1}{2}[\varphi(|g(t)|) + \varphi(|h(t)|)]\,\mathrm{d}\mu \\
&\leq \frac{1}{2T}\int\limits_{-T}^{T} \frac{1}{2}[\varphi(|g(t)|) + \varphi(|h(t)|)]\,\mathrm{d}\mu \\
&\quad - \delta\,\frac{1}{2T}\int\limits_{\overline{G}} \frac{1}{2}[\varphi(|g(t)|) + \varphi(|h(t)|)]\,\mathrm{d}\mu.
\end{aligned}$$

Since φ is an increasing convex function we have

$$\frac{1}{2}(\varphi(|g(t)|) + \varphi(|h(t)|)) \geq \varphi\Big(\frac{|g(t)|+|h(t)|}{2}\Big) \geq \varphi\Big(\frac{|g(t)-h(t)|}{2}\Big).$$

Using the fact that $G \subset G_1$ we obtain

$$\frac{1}{2}(\varrho_T(g) + \varrho_T(h)) - \varrho_T\Big(\frac{g+h}{2}\Big) \geq \delta\varphi\Big(\frac{\alpha}{2}\Big)\frac{\mu(\overline{G})}{2T}.$$

Letting $T \to \infty$, then by (2.2) and (2.3) we get

$$\frac{1}{2}[\varrho_{B^\varphi}(g) + \varrho_{B^\varphi}(h)] - \varrho_{B^\varphi}\Big(\frac{g+h}{2}\Big) \geq \delta\varphi\Big(\frac{\alpha}{2}\Big)\overline{\mu}_B(G).$$

Consequently, by the inequality (3.7), we deduce that

$$\begin{aligned}
1 = \varrho_{B^\varphi}\Big(\frac{g+h}{2}\Big) &\leq \frac{1}{2}[\varrho_{B^\varphi}(g) + \varrho_{B^\varphi}(h)] - \delta\varphi\Big(\frac{\alpha}{2}\Big)\overline{\mu}_B(G) \\
&\leq 1 - \delta\varphi\Big(\frac{\alpha}{2}\Big)\frac{\theta}{2}.
\end{aligned}$$

Which is absurd. Thus we showed that f is an extreme point.

Necessity: Suppose that $\mu(\{t \in \mathbb{R}\colon f(t) \notin S_\varphi\}) > 0$.

Let $\varepsilon > 0$. Since $\mathbb{R} \setminus S_\varphi$ is the union of at most countably many open intervals, there exists an interval $]a, b[$ such that for $\varepsilon > 0$

$$\mu(\{t \in \mathbb{R}\colon f(t) \in]a+\varepsilon, b-\varepsilon[\,\}) > 0,$$

and φ is affine on $[a, b]$. That is,

$$\varphi(u) = ku + \beta \qquad \text{for } u \in [a, b] \text{ with } k \in \mathbb{R}^+ \text{ and } \beta \in \mathbb{R}.$$

We divide the set $H = \{t \in \mathbb{R}\colon f(t) \in]a+\varepsilon, b-\varepsilon[\,\}$ into two sets A and B. Then we define

$$(g(t), h(t)) = \begin{cases} (f(t), f(t)) & \text{if } t \in \mathbb{R}\backslash(A \cup B), \\ (f(t)-\varepsilon, f(t)+\varepsilon) & \text{if } t \in A, \\ (f(t)+\varepsilon, f(t)-\varepsilon) & \text{if } t \in B. \end{cases} \tag{3.8}$$

Then $g \neq h$, $g + h = 2f$ and $g, h \in B^\varphi_{\text{a.p.}}(\mathbb{R}, \mathbb{C})$. Indeed, we have $\overline{\mu}_B(H) = 0$ because $H \subset f^{-1}([a, b])$, then by Lemma 6 we get

$$f\chi_{H^c}, (f-\varepsilon)\chi_A, (f+\varepsilon)\chi_B \in B^\varphi_{\text{a.p.}}(\mathbb{R}, \mathbb{C}).$$

Now, we should show that $\|g\|_{B^\varphi} \leq 1$. We have

$$\begin{aligned} \varrho_T(g) &= \varrho_T(f\chi_{(A\cup B)^c}) + \varrho_T(f\chi_A) + \varrho_T(f\chi_B) \\ &= \varrho_T(f\chi_{(A\cup B)^c}) + \frac{1}{2T} \int\limits_{[-T,T]\cap A} (k|f(t)-\varepsilon| + \beta)\,\mathrm{d}\mu \\ &\quad + \frac{1}{2T} \int\limits_{[-T,T]\cap B} (k|f(t)+\varepsilon| + \beta)\,\mathrm{d}\mu \\ &\leq \varrho_T(f\chi_{(A\cup B)^c}) + \frac{1}{2T} \int\limits_{[-T,T]\cap A} (k|f(t)| + \beta + k\varepsilon)\,\mathrm{d}\mu \\ &\quad + \frac{1}{2T} \int\limits_{[-T,T]\cap B} (k|f(t)| + \beta + k\varepsilon)\,\mathrm{d}\mu \\ &\leq \varrho_T(f\chi_{(A\cup B)^c}) + \varrho_T(f\chi_{(A\cup B)}) + k\,\varepsilon\,\frac{1}{2T}\,\mu\left([-T,T] \cap (A \cup B)\right)\mathrm{d}\mu. \end{aligned}$$

Letting $T \to \infty$, we get

$$\varrho_{B^\varphi}(g) \leq \varrho_{B^\varphi}(f) + k\varepsilon\overline{\mu}_B(H).$$

By applying the hypothesis $\overline{\mu}_B(f^{-1}([a,b])) = 0$ for any structural affine interval $[a,b]$ of φ, we deduce that $\overline{\mu}_B(H) = 0$. Then we get

$$\varrho_{B^\varphi}(g) \leq \varrho_{B^\varphi}(f) = 1.$$

Using same arguments, so by Lemma 1 we get $\|h\|_{B^\varphi} \leq 1$, which completes the proof. $\square$

The following corollary gives sufficient conditions for the strict convexity of the Besicovitch–Orlicz space of almost periodic functions equipped with the Luxemburg norm. Note that the conditions are the same as those given in [8, Theorem 1] when we consider $B^\varphi_{\text{a.p.}}(\mathbb{R},\mathbb{C})$ instead of $\widetilde{B}^\varphi_{\text{a.p.}}(\mathbb{R},\mathbb{C})$. Recall that

$$\begin{aligned}\widetilde{B}^\varphi_{\text{a.p.}}(\mathbb{R},\mathbb{C}) = \{f \in B^\varphi(\mathbb{R},\mathbb{C})\colon \exists\,(P_n)_{n\geq 1} \subset \operatorname{Trig}(\mathbb{R},\mathbb{C}),\\ \exists\, k > 0 \text{ s.t. } \lim_{n\to\infty} \varrho_{B^\varphi}(k(f - P_n)) = 0\}.\end{aligned}$$

Corollary 1. *If φ is strictly convex on $\mathbb{R}$ then $B^\varphi_{\text{a.p.}}(\mathbb{R},\mathbb{C})$ is strictly convex.*

Proof: The hypothesis of strict convexity of φ on $\mathbb{R}$ means that $S_\varphi = \mathbb{R}$ and then by Theorem 1 we get $\mathbf{extr}[B(B^\varphi_{\text{a.p.}}(\mathbb{R},\mathbb{C}))] = S(B^\varphi_{\text{a.p.}}(\mathbb{R},\mathbb{C}))$ and the claim is proved. $\square$

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