Remarks on WDC sets

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Abstract. We study WDC sets, which form a substantial generalization of sets with positive reach and still admit the definition of curvature measures. Main results concern WDC sets $A \subset \mathbb{R}^2$. We prove that, for such A, the distance function $d_A = \operatorname{dist}(\cdot, A)$ is a "DC aura" for A, which implies that each closed locally WDC set in $\mathbb{R}^2$ is a WDC set. Another consequence is that compact WDC subsets of $\mathbb{R}^2$ form a Borel subset of the space of all compact sets.

Keywords: distance function; WDC set; DC function; DC aura; Borel complexity

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1. Introduction

In [10] (cf. also [8], [7] and [11]), the authors introduced the class of WDC sets which form a substantial generalization of sets with positive reach and still admit the definition of curvature measures. The following question naturally arises, see [8, Question 2, page 829] and [7, 10.4.3].

Question. Is the distance function $d_A = \operatorname{dist}(\cdot, A)$ of each WDC set $A \subset \mathbb{R}^d$ a DC aura for A (see Definition 2.3)?

We answer this question positively in the case $d = 2$ (Theorem 3.3 below); it remains open for $d \geq 3$. The proof is based on a characterization (proved in [11]) of closed locally WDC sets in $\mathbb{R}^2$ and the main result of [12] which asserts that

(1.1) $\quad d_A$ is a DC function if $A \subset \mathbb{R}^2$ is a graph of a *DC* function $g\colon \mathbb{R} \to \mathbb{R}$.

Recall that a function is called DC, if it is the difference of two convex functions. Let us note that each set A as in (1.1) is a WDC set (it is easy to show that the function $(x, y) \mapsto |g(x) - y|$ is a DC aura for $\operatorname{graph} g$, cf. the proof of [11, Proposition 6.6]).

Theorem 3.3 easily implies that each closed locally WDC set in $\mathbb{R}^2$ is WDC. Further, we use Theorem 3.3 to prove that compact WDC subsets of $\mathbb{R}^2$ form a Borel subset of the space of all compact sets of $\mathbb{R}^2$ (Theorem 4.1 (i)). The importance of this result is the fact that it suggests that (at least in $\mathbb{R}^2$) a theory

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of point processes on the space of compact WDC sets (analogous to the concept of point processes on the space of sets with positive reach introduced in [16]) can be build.

Concerning the compact WDC subsets of $\mathbb{R}^d$ for $d > 2$, we are able to prove only a weaker fact that they form an analytic set (Theorem 4.1 (ii)) which is not probably sufficient for the above mentioned application.

2. Preliminaries

2.1 Basic definitions. The symbol $\mathbb{Q}$ denotes the set of all rational numbers. In any vector space V, we use the symbol 0 for the zero element. We denote by $B(x,r)$ ($U(x,r)$) the closed (open) ball with centre x and radius r. The boundary and the interior of a set M are denoted by ∂M and $\mathrm{int} M$, respectively. A mapping is called K-Lipschitz if it is Lipschitz with a (not necessarily minimal) constant $K \geq 0$.

The metric space of all real-valued continuous functions on a compact K (equipped with the usual supremum metric $\varrho_{\sup}$) will be denoted $C(K)$.

In the Euclidean space $\mathbb{R}^d$, the norm is denoted by $|\cdot|$ and the scalar product by $\langle \cdot, \cdot \rangle$. By S^{d-1} we denote the unit sphere in $\mathbb{R}^d$.

The distance function from a set $A \subset \mathbb{R}^d$ is $d_A := \mathrm{dist}\,(\cdot, A)$ and the metric projection of $z \in \mathbb{R}^d$ to A is $\Pi_A(z) := \{a \in A \colon \mathrm{dist}\,(z, A) = |z - a|\}$.

2.2 DC functions. Let f be a real function defined on an open convex set $C \subset \mathbb{R}^d$. Then we say that f is a *DC function*, if it is the difference of two convex functions. Special DC functions are semiconvex and semiconcave functions. Namely, f is a *semiconvex* (*semiconcave*, respectively) function, if there exist $a > 0$ and a convex function g on C such that

$$f(x) = g(x) - a|x|^2 \quad (f(x) = a|x|^2 - g(x), \ \text{ respectively}), \qquad x \in C.$$

We will use the following well-known properties of DC functions.

Lemma 2.1. *Let C be an open convex subset of $\mathbb{R}^d$. Then the following assertions hold:*

- (i) *If $f \colon C \to \mathbb{R}$ and $g \colon C \to \mathbb{R}$ are DC, then for each $a \in \mathbb{R}$, $b \in \mathbb{R}$ the functions $|f|$, $af + bg$, $\max(f, g)$ and $\min(f, g)$ are DC.*
- (ii) *Each locally DC function $f \colon C \to \mathbb{R}$ is DC.*
- (iii) *Each DC function $f \colon C \to \mathbb{R}$ is Lipschitz on each compact convex set $Z \subset C$.*

(iv) *Let* $f_i\colon C \to \mathbb{R}$, $i = 1, \dots, m$, *be DC functions. Let* $f\colon C \to \mathbb{R}$ *be a continuous function such that* $f(x) \in \{f_1(x), \dots, f_m(x)\}$ *for each* $x \in C$. *Then* f *is DC on* C.

Proof: Property (i) follows easily from definitions, see e.g. [14, page 84]. Property (ii) was proved in [9]. Property (iii) easily follows from the local Lipschitzness of convex functions. Assertion (iv) is a special case of [15, Lemma 4.8.] ("Mixing lemma"). □

It is well-known (cf. [12]) that if $\emptyset \neq A \subset \mathbb{R}^d$ is closed, then d_A need not be DC; however (see, e.g., [2, Proposition 2.2.2]),

(2.1) d_A is locally semiconcave (and so locally DC) on $\mathbb{R}^d \setminus A$.

2.3 Clarke generalized gradient. If $U \subset \mathbb{R}^d$ is an open set, $f\colon U \to \mathbb{R}$ is locally Lipschitz and $x \in U$, we denote by $\partial_C f(x)$ the *generalized gradient of* f *at* x, which can be defined as the closed convex hull of all limits $\lim_{i\to\infty} f'(x_i)$ such that $x_i \to x$ and $f'(x_i)$ exists for all $i \in \mathbb{N}$ (see [3, Theorem 2.5.1]; $\partial_C f(x)$ is also called *Clarke subdifferential of* f *at* x in the literature). Since we identify $(\mathbb{R}^d)^*$ with $\mathbb{R}^d$ in the standard way, we sometimes consider $\partial_C f(x)$ as a subset of $\mathbb{R}^d$. We will repeatedly use the fact that the mapping $x \mapsto \partial_C f(x)$ is upper semicontinuous and, hence (see [3, Theorem 2.1.5]),

(2.2) $$v \in \partial_C f(x) \qquad \text{whenever } \; x_i \to x, \; v_i \in \partial_C f(x_i) \; \text{ and } \; v_i \to v.$$

We also use that $|u| \leq K$ whenever $u \in \partial_C f(x)$ and f is K-Lipschitz on a neighbourhood of x. Obviously,

(2.3) $$\partial_C(\alpha f)(x) = \alpha \partial_C f(x).$$

Recall that

(2.4) $$f^0(x, v) := \limsup_{y\to x, t\to 0+} \frac{f(y + tv) - f(y)}{t}$$

and (see [3])

(2.5) $$f^0(x, v) = \sup\{\langle v, \nu\rangle \colon \nu \in \partial_C f(x)\}.$$

We will need the following simple lemma.

Lemma 2.2. *Let* f *be a Lipschitz function on an open set* $G \subset \mathbb{R}^d$, $x \in G$ *and* $\varepsilon > 0$.

(i) *If* $\operatorname{dist}(0, \partial_C f(x)) \geq 2\varepsilon$, *then*

$$\exists\, v \in S^{d-1},\ \varrho > 0\ \forall\, y \in U(x, \varrho),\ 0 < \alpha < \varrho\colon \frac{f(y + \alpha v) - f(y)}{\alpha} \leq -\varepsilon. \tag{2.6}$$

(ii) *If* (2.6) *holds, then* $\operatorname{dist}(0, \partial_C f(x)) \geq \varepsilon$.

PROOF: (i) Let $\operatorname{dist}(0, \partial_C f(x)) \geq 2\varepsilon$. Since $\partial_C f(x)$ is convex, there exists (see e.g. [4, Theorem 1.5.]) $v \in S^{d-1}$ such that

$$\operatorname{dist}(0, \partial_C f(x)) = -\sup\{\langle v, \nu\rangle : \nu \in \partial_C f(x)\}.$$

So, by (2.5), $f^0(x, v) \leq -2\varepsilon$ and thus (2.4) implies (2.6).

(ii) If (2.6) holds, choose corresponding $v \in S^{d-1}$ and $\varrho > 0$. Then $f^0(x, v) \leq -\varepsilon$ by (2.4). Consequently, by (2.5), $-|\nu| \leq \langle v, \nu\rangle \leq -\varepsilon$ for each $\nu \in \partial_C f(x)$ and so $\operatorname{dist}(0, \partial_C f(x)) \geq \varepsilon$. □

2.4 WDC sets. WDC sets (see the definition below) which provide a natural generalization of sets with positive reach were defined in [10] using Fu's notion of an "aura" of a set (see, e.g., [7] for more information). Note that the notion of a DC aura was defined in [10] and [8] by a formally different but equivalent way (cf. [11, Remark 2.12 (v)]).

Definition 2.3 (cf. [11], Definitions 2.8, 2.10)**.** Let $U \subset \mathbb{R}^d$ be open and $f\colon U \to \mathbb{R}$ be locally Lipschitz. A number $c \in \mathbb{R}$ is called a *weakly regular value* of f if whenever $x_i \to x$, $f(x_i) > c = f(x)$ and $u_i \in \partial_C f(x_i)$ for all $i \in \mathbb{N}$ then $\liminf_i |u_i| > 0$.

A set $A \subset \mathbb{R}^d$ is called *WDC* if there exists a DC function $f\colon \mathbb{R}^d \to [0, \infty)$ such that $A = f^{-1}(0)$ and 0 is a weakly regular value of f. In such a case, we call f a *DC aura* (for A).

A set $A \subset \mathbb{R}^d$ is called locally WDC if for any point $a \in A$ there exists a WDC set $A^* \subset \mathbb{R}^d$ that agrees with A on an open neighbourhood of a.

(Note that a weakly regular value of f need not be in the range of f, and so $\emptyset$ is clearly a WDC set by our definition. Also, unlike WDC sets which are always closed, locally WDC sets need not to be closed.)

Note that a set $A \subset \mathbb{R}^d$ has a positive reach at each point if and only if there exists a DC aura for A which is even semiconvex, see [1].

3. Distance function of a WDC set in $\mathbb{R}^2$ is a DC aura

First we present (slightly formally rewritten) [11, Definition 7.9].

Definition 3.1. (i) A set $S \subset \mathbb{R}^2$ will be called a *basic open DC sector* (of radius r) if $S = U(0, r) \cap \{(u, v) \in \mathbb{R}^2 : u \in (-\omega, \omega),\ v > f(u)\}$, where

$0 < r < \omega$ and f is a DC function on $(-\omega, \omega)$ such that $f(0) = 0$, $R(u) := \sqrt{u^2 + f^2(u)}$ is strictly increasing on $[0, \omega)$ and strictly decreasing on $(-\omega, 0]$.

By an *open DC sector* (of radius r) we mean an image $\gamma(S)$ of a basic open DC sector S (of radius r) under a rotation around the origin γ.

(ii) A set of the form $\gamma(\{(u,v) \in \mathbb{R}^2 : u \in [0, \omega),\ g(u) \le v \le f(u)\}) \cap U(0, r)$, where γ is a rotation around the origin, $0 < r < \omega$ and $f, g : \mathbb{R} \to \mathbb{R}$ are DC functions such that $g \le f$ on $[0, \omega)$, $f(0) = g(0) = f'_+(0) = g'_+(0) = 0$ and the functions $R_f(u) := \sqrt{u^2 + f^2(u)}$, $R_g(u) := \sqrt{u^2 + g^2(u)}$ are strictly increasing on $[0, \omega)$, will be called a *degenerated closed DC sector* (of radius r).

We will use the following complete characterization of closed locally WDC sets in $\mathbb{R}^2$ (see [11, Theorem 8.14]).

Theorem PRZ. *Let M be a closed subset of $\mathbb{R}^2$. Then M is a locally WDC set if and only if for each $x \in \partial M$ there is $\varrho > 0$ such that one of the following conditions holds:*

(i) $M \cap U(x, \varrho) = \{x\}$,

(ii) *there is a degenerated closed DC sector C of radius ϱ such that*

$$M \cap U(x, \varrho) = x + C,$$

(iii) *there are pairwise disjoint open DC sectors $C_1, \dots, C_k$ of radius ϱ such that*

$$U(x, \varrho) \setminus M = \bigcup_{i=1}^{k} (x + C_i). \tag{3.1}$$

Lemma 3.2. *Let f be an L-Lipschitz function on $\mathbb{R}$. Denote $d := \operatorname{dist}(\cdot, \operatorname{graph} f)$. Then $|\xi_2| \ge 1/\sqrt{L^2 + 1}$ whenever $\xi = (\xi_1, \xi_2) \in \partial_C d(x)$ and $x \in \mathbb{R}^2 \setminus \operatorname{graph} f$.*

Proof: Pick $x \in \mathbb{R}^2 \setminus \operatorname{graph} f$. Without any loss of generality we can assume that $x = 0$. We will assume that $f(0) < 0$; the case $f(0) > 0$ is quite analogous. Denote $r := d(0)$ and $P := \Pi_{\operatorname{graph} f}(0)$. Set $g(u) := -\sqrt{r^2 - u^2}$, $u \in [-r, r]$. Clearly $f \le g$ on $[-r, r]$ and $(u, v) \in P$ if and only if $f(u) = g(u) = v$. We will show that

$$|u| \le \frac{Lr}{\sqrt{1 + L^2}} \qquad \text{whenever } (u, v) \in P. \tag{3.2}$$

To this end, suppose $(u, v) \in P$. If $u > 0$, then

$$L \ge \frac{f(t) - f(u)}{t - u} \ge \frac{g(t) - g(u)}{t - u} \qquad \text{for each } \ 0 < t < u,$$

and consequently $L \geq g'_-(u)$. Therefore $u < r$ and $L \geq u(r^2-u^2)^{-1/2}$. Analogously considering $g'_+(u)$, we obtain for $u < 0$ that $u > -r$ and $u(r^2-u^2)^{-1/2} \geq -L$. In both cases we have $L \geq |u|(r^2-u^2)^{-1/2}$ and an elementary computation gives (3.2).

Using (3.2) we obtain that if $(u,v) \in P$ then

$$v = g(u) \leq -\sqrt{r^2 - \Big(\frac{Lr}{\sqrt{1+L^2}}\Big)^2} = -\frac{r}{\sqrt{1+L^2}}. \tag{3.3}$$

By [6, Lemma 4.2] and (3.3) we obtain

$$\partial_C d(0) = \operatorname{conv}\Big\{\frac{1}{r}(-u,-v)\colon (u,v) \in P\Big\} \subset \mathbb{R} \times \Big[\frac{1}{\sqrt{L^2+1}}, \infty\Big)$$

and the assertion of the lemma follows. □

Theorem 3.3. *Let $M \neq \emptyset$ be a closed locally WDC set in $\mathbb{R}^2$. Then the distance function d_M is a DC aura for M. In particular, M is a WDC set.*

PROOF: Denote $d := d_M$. For each $x \in \partial M$ choose $\varrho = \varrho(x)$ by Theorem PRZ. We will prove that

(a) distance d is DC on $U(x, \varrho/3)$,
(b) there is $\varepsilon = \varepsilon(x) > 0$ such that $|\xi| \geq \varepsilon$ whenever $y \in U(0,\varrho/3) \setminus M$ and $\xi \in \partial_C d(y)$.

Without any loss of generality we can assume that $x = 0$.

If Case (i) from Theorem PRZ holds, then $d(y) = |y|$, $y \in U(0,\varrho/3)$, and so d is convex and therefore DC on $U(0,\varrho/3)$. Similarly, condition (b) holds as well, since if $y \in U(0,\varrho/3) \setminus M$ and $\xi \in \partial_C d(y)$ then $\xi = y/|y|$ and so $|\xi| = 1$.

If Case (ii) from Theorem PRZ holds, we know that $M \cap U(0,\varrho)$ is a degenerated closed DC sector C of radius ϱ. Let γ, f, g and ω be as in Definition 3.1. Without any loss of generality we may assume that γ is the identity map.

By Lemma 2.1 (iii) we can choose $L > 0$ such that both f and g are L-Lipschitz on $[0,\varrho]$ and define

$$\tilde{f}(u) := \begin{cases} f(u) & \text{if } 0 \leq u \leq \varrho, \\ f(\varrho) & \text{if } \varrho < u, \\ 2Lu & \text{if } u < 0, \end{cases} \qquad \text{and} \qquad \tilde{g}(u) := \begin{cases} g(u) & \text{if } 0 \leq u \leq \varrho, \\ g(\varrho) & \text{if } \varrho < u, \\ -2Lu & \text{if } u < 0. \end{cases}$$

It is easy to see that both $\tilde{f}$ and $\tilde{g}$ are $2L$-Lipschitz and they are DC by Lemma 2.1 (iv).

Put

$$M_0 := \{(u,v) \in \mathbb{R}^2 \colon u \geq 0,\ \tilde{g}(u) \leq v \leq \tilde{f}(u)\},$$

$$M_1 := \{(u,v) \in \mathbb{R}^2 \colon u \geq 0,\ \tilde{f}(u) < v\} \cup \Big\{(u,v) \in \mathbb{R}^2 \colon u < 0,\ -\frac{u}{2L} < v\Big\},$$
$$M_2 := \{(u,v) \in \mathbb{R}^2 \colon u \geq 0,\ \tilde{g}(u) > v\} \cup \Big\{(u,v) \in \mathbb{R}^2 \colon u < 0,\ \frac{u}{2L} > v\Big\}$$

and

$$M_3 := \Big\{(u,v) \in \mathbb{R}^2 \colon \frac{u}{L} < v < -\frac{u}{L}\Big\}.$$

Clearly $\mathbb{R}^2 = M_0 \cup M_1 \cup M_2 \cup M_3$ and M_1, M_2, M_3 are open.

Set $\tilde{d} := \operatorname{dist}(\cdot, M_0)$ and for each $y \in \mathbb{R}^2$ define

$$d_0(y) := 0, \quad d_1(y) := \operatorname{dist}(y, \operatorname{graph} \tilde{f}), \quad d_2(y) := \operatorname{dist}(y, \operatorname{graph} \tilde{g}), \quad d_3(y) := |y|.$$

Functions d_1 and d_2 are DC on $\mathbb{R}^2$ by (1.1), d_0 and d_3 are convex and therefore DC on $\mathbb{R}^2$.

Using (for $K = 1/L, -1/L, 1/(2L), -1/(2L)$) the facts that the lines with the slopes K and $-1/K$ are orthogonal and $M_0 \subset \{(u,v) \colon u \geq 0,\ -Lu \leq v \leq Lu\}$, easy geometrical observations show that

$$\tilde{d}(y) = d_i(y) \qquad \text{if } \ y \in M_i,\ 0 \leq i \leq 3, \tag{3.4}$$

and so Lemma 2.1 (iv) implies that $\tilde{d}$ is DC.

Now pick an arbitrary $y \in \mathbb{R}^2 \setminus M_0 = M_1 \cup M_2 \cup M_3$ and choose $\xi = (\xi_1, \xi_2) \in \partial_C \tilde{d}(y)$. Using (3.4), we obtain that if $y \in M_3$ then $\xi = y/|y|$ and so $|\xi| = 1$. Using Lemma 3.2, we obtain that if $y \in M_1 \cup M_2$, then $|\xi| \geq |\xi_2| \geq 1/\sqrt{4L^2+1}$.

Now, since $d = \tilde{d}$ on $U(0, \varrho/3)$ both (a) and (b) follow.

It remains to prove (a) and (b) if Case (iii) from Theorem PRZ holds. Let C_i, $i = 1, \ldots, k$, be the open DC sectors as in (iii). Denote $A_i := \mathbb{R}^2 \setminus C_i$ and define $\delta_i := \operatorname{dist}(\cdot, A_i)$, $i = 1, \ldots, k$.

Note that, for $y \in U(0, \varrho/3)$, one has

$$d(y) = \begin{cases} \delta_i(y) & \text{if } \ y \in C_i, \\ 0 & \text{if } \ y \in M. \end{cases}$$

Therefore (by Lemma 2.1 (iv)) it is enough to prove that (a) and (b) hold with d and M being replaced by δ_i and A_i, respectively, $i = 1, \ldots, k$. Fix some $i \in \{1, \ldots, k\}$. Without any loss of generality we can assume that C_i is a basic open DC sector of radius ϱ with corresponding f_i and ω_i. Now define

$$\tilde{f}_i(u) := \begin{cases} f_i(u) & \text{if } \ u \in [-\varrho, \varrho], \\ f_i(-\varrho) & \text{if } \ u < -\varrho, \\ f_i(\varrho) & \text{if } \ u > \varrho. \end{cases}$$

Then $\tilde{f}_i$ is Lipschitz and DC on $\mathbb{R}$. Put $\tilde{d}_i(y) = \operatorname{dist}(y, \operatorname{graph} \tilde{f}_i)$. Then $\tilde{d}_i$ is DC by (1.1) and 0 is a weakly regular value of $\tilde{d}_i$ by Lemma 3.2. And since $d_i = \tilde{d}_i$ on $U(0, \varrho/3)$ we are done.

Since d is locally DC on $\mathbb{R}^2 \setminus M$ by (2.1) and on the interior of M (trivially), (a) implies that d is locally DC and so DC by Lemma 2.1 (ii). Further, (b) immediately implies that 0 is a weakly regular value of d and thus $d = d_M$ is an aura for M. □

Remark 3.4. By Theorem 3.3, in $\mathbb{R}^2$ closed locally WDC sets and WDC sets coincide. This gives a partial answer to the part of [10, Problem 10.2] which asks whether the same is true in each $\mathbb{R}^d$.

4. Complexity of the system of WDC sets

In the following, we will work in each moment in an $\mathbb{R}^d$ with a fixed d, and so for simplicity we will use the notation, in which the dependence on d is usually omitted.

The space of all nonempty compact subsets of $\mathbb{R}^d$ equipped with the usual Hausdorff metric ϱ_H is denoted by $\mathcal{K}$. It is well-known (see, e.g., [13, Proposition 2.4.15 and Corollary 2.4.16]) that $\mathcal{K}$ is a separable complete metric space. For a closed set $M \subset \mathbb{R}^d$, we set $\mathcal{K}(M) := \{K \in \mathcal{K} : K \subset M\}$, which is clearly a closed subspace of $\mathcal{K}$. The set of all nonempty compact WDC subsets of $M \subset \mathbb{R}^d$ will be denoted by $\mathrm{WDC}(M)$.

In this section, we will prove the following theorem.

Theorem 4.1. (i) $\mathrm{WDC}(\mathbb{R}^2)$ *is an* $F_{\sigma\delta\sigma}$ *subset of* $\mathcal{K}(\mathbb{R}^2)$.
(ii) *Set* $\mathrm{WDC}(\mathbb{R}^d)$ *is an analytic subset of* $\mathcal{K}(\mathbb{R}^d)$ *for each* $d \in \mathbb{N}$.

Before the proof of this theorem, we introduce some spaces, make a number of observations, and prove a technical lemma.

First observe that $\mathrm{WDC}(\mathbb{R}^d) = \bigcup_{n=1}^{\infty} \mathrm{WDC}(B(0,n))$ and so, to prove Theorem 4.1, it is sufficient to prove that for each $r > 0$,

$$\text{for } d = 2 \ (d \in \mathbb{N}, \text{ respectively}),\ \mathrm{WDC}(B(0,r)) \text{ is an } F_{\sigma\delta\sigma} \text{ (analytic, respectively) subset of } \mathcal{K}(B(0,r)). \tag{4.1}$$

Further observe that it is sufficient to prove (4.1) for $r = 1$. Indeed, denoting $H(x) := x/r$, $x \in \mathbb{R}^d$, it is obvious that $H^* : K \mapsto H(K)$ gives a homeomorphism of $\mathcal{K}(B(0,r))$ onto $\mathcal{K}(B(0,1))$ and $H^*(\mathrm{WDC}(B(0,r)) = \mathrm{WDC}(B(0,1))$ (clearly f is an aura for K if and only if $f \circ H^{-1}$ is an aura for $H^*(K)$).

To prove (4.1) for $r = 1$, we will consider the space X of all 1-Lipschitz functions $f : B(0,4) \to [0,4]$ such that $f \geq 1$ on $B(0,4) \setminus U(0,3)$, equipped with the supremum metric $\varrho_{\sup}$. Obviously, X is a closed subspace of $C(B(0,4))$ and so it is a separable complete metric space.

The motivation for introducing X is the fact that

$$\text{(4.2)} \qquad \text{if } K \in \mathcal{K}(B(0,1)), \text{ then } f_K := d_K \restriction_{B(0,4)} \in X.$$

Since we are interested in $K \in \mathrm{WDC}(B(0,1))$, we define also two subspaces of X:

$$A := \{f \in X\colon\ 0 \text{ is a weakly regular value of } f|_{U(0,4)}\},$$
$$D := \{f \in X\colon f = g - h \text{ for some convex Lipschitz functions } g, h \text{ on } B(0,4)\}.$$

Their complexity is closely related to the complexity of $\mathrm{WDC}(B(0,1))$, as the following lemma indicates.

Lemma 4.2. *Let $\emptyset \neq K \subset B(0,1) \subset \mathbb{R}^d$ be compact. Then:*

(i) *K is WDC if and only if there is a function $g \in D \cap A$ such that $K = g^{-1}(0)$.*

(ii) *If $d = 2$, then K is WDC if and only if $f_K := d_K \restriction_{B(0,4)} \in D \cap A$.*

Proof: (i) Suppose first that K is WDC and f is an aura for K. Using Lemma 2.1 (iii), we can choose $\alpha > 0$ so small that the function αf is 1-Lipschitz on $B(0,4)$ and $0 \le \alpha f(x) \le 4$ for $x \in B(0,4)$. Set

$$h(x) := \max(|x| - 2, \alpha f(x)), \quad x \in \mathbb{R}^d, \qquad \text{and} \qquad g := h \restriction_{B(0,4)}.$$

Then clearly $K = g^{-1}(0)$. Since h is DC on $\mathbb{R}^d$ by Lemma 2.1 (i), we obtain $g \in D$ by Lemma 2.1 (iii). Finally, $g \in A$ since $g = \alpha f$ on $U(0,2)$.

Conversely, suppose that $K = g^{-1}(0)$ for some $g \in A \cap D$ and set

$$f(x) := \begin{cases} \min(g(x), 1), & \text{if } x \in U(0,4), \\ 1, & \text{otherwise.} \end{cases}$$

Since f is DC on $U(0,4)$ by Lemma 2.1 (i) and $f = 1$ on $\mathbb{R}^d \setminus B(0,3)$, we see that f is locally DC and so DC by Lemma 2.1 (ii). Since 0 is clearly a weakly regular value of f, we obtain that f is an aura for K.

(ii) If K is WDC, first note that $f_K \in X$ (see (4.2)). Since d_K is an aura for K by Theorem 3.3, we obtain immediately that $f_K \in A$, and also $f_K \in D$ by Lemma 2.1 (iii).

If $f_K \in A \cap D$, then K is WDC by (i). □

For the application of Lemma 4.2 (ii) we need the simple fact that

$$\text{(4.3)} \quad \Psi\colon K \mapsto f_K, \quad K \in \mathcal{K}(B(0,1)), \qquad \text{is a continuous mapping into } X.$$

Indeed, if $K_1, K_2 \in \mathcal{K}(B(0,1))$ with $\varrho_H(K_1,K_2) < \varepsilon$ and $x \in B(0,4)$, then clearly $d_{K_1}(x) < d_{K_2}(x) + \varepsilon$, $d_{K_2}(x) < d_{K_1}(x) + \varepsilon$, and consequently $\varrho_{\sup}(f_{K_1}, f_{K_2}) \le \varepsilon$.

Further observe that

$$D \text{ is an } F_\sigma \text{ subset of } X. \tag{4.4}$$

To prove it for each $n \in \mathbb{N}$ set

$$C_n := \{g \in C(B(0,4))\colon\ g \text{ is convex } n\text{-Lipschitz and } |g(x)| \le 4n{+}4,\ x \in B(0,4)\}.$$

Now observe that if $f \in D$ then we can choose $n \in \mathbb{N}$ and convex n-Lipschitz functions g, h such that $f = g - h$, $g(0) = 0$ and consequently $\|g\| \le 4n$, $\|h\| \le 4n + 4$, and so g, $h \in C_n$. Consequently, $D = X \cap \bigcup_{n=1}^{\infty} (C_n - C_n)$. Each C_n is clearly closed in $C(B(0,4))$ and so it is compact in $C(B(0,4))$ by the Arzelà–Ascoli theorem. Consequently also $C_n - C_n = \sigma(C_n \times C_n)$, where σ is the continuous mapping $\sigma\colon (g,h) \mapsto g - h$, is compact, and (4.4) follows.

The most technical part of the proof of Theorem 4.1 is to show that A is an $F_{\sigma\delta\sigma}$ subset of X. To prove it, we need some lemmas.

Lemma 4.3. *Let $f \in X$. Then $f \in A$ if and only if*

$$\exists\, 0 < \varepsilon\ \forall x \in f^{-1}(0,\varepsilon),\ \nu \in \partial_C f(x)\colon |\nu| \ge \varepsilon. \tag{4.5}$$

Proof: If (4.5) holds, then we easily obtain $f \in A$ directly from the definition of a weakly regular value.

To prove the opposite implication, suppose that $f \in A$ and (4.5) does not hold. Then there exist points $x_n \in f^{-1}(0, 1/n)$, $n \in \mathbb{N}$, and $\nu_n \in \partial_C f(x_n)$ such that $|\nu_n| < 1/n$. Choose a subsequence $x_{n_k} \to x \in B(0,4)$. Since $0 \le f(x_{n_k}) < 1/n_k$, we have $f(x_{n_k}) \to f(x) = 0$, and consequently $x \in U(0,4)$. Since $\nu_{n_k} \to 0$, we obtain that 0 is not a weakly regular value of $f|_{U(0,4)}$, which contradicts $f \in A$. □

Denote $\mathbb{Q}^* := \mathbb{Q} \cap (0,1)$ and for every $\varepsilon \in \mathbb{Q}^*$ and $d \in \mathbb{N}$ pick a finite set $\mathfrak{S}^d_\varepsilon \subset S^{d-1}$ such that for every $v \in S^{d-1}$ there is some $\nu \in \mathfrak{S}^d_\varepsilon$ satisfying $|v - \nu| < \varepsilon$.

Lemma 4.4. *Let f be a function from X. Then $f \in A$ if and only if*

$$\begin{aligned} &\exists \varepsilon \in \mathbb{Q}^*\ \forall p,q \in \mathbb{Q}^*,\ 0 < p < q < \varepsilon\ \exists \varrho \in \mathbb{Q}^*\ \forall x \in U(0,4)\colon (f(x) \notin (p,q) \\ &\quad \vee\ \exists \nu \in \mathfrak{S}^d_\varepsilon\ \forall y \in U(x,\varrho),\ 0 < \alpha < \varrho\colon\ f(y + \alpha\nu) - f(y) \le -\varepsilon\alpha). \end{aligned} \tag{4.6}$$

Proof: First suppose that (4.6) holds and choose $\varepsilon \in \mathbb{Q}^*$ by (4.6). We will show that

$$\forall x \in f^{-1}(0,\varepsilon),\ \nu \in \partial_C f(x)\colon |\nu| \ge \varepsilon. \tag{4.7}$$

To this end, consider an arbitrary $x \in f^{-1}(0,\varepsilon)$ and choose $p, q \in \mathbb{Q}^*$ such that $0 < p < q < \varepsilon$ and $f(x) \in (p,q)$. Choose $\varrho \in \mathbb{Q}^*$ which exists for ε, p, q by (4.6). So there exists $\nu \in \mathfrak{S}^d_\varepsilon$ such that

$$\forall\, y \in U(x,\varrho),\ 0 < \alpha < \varrho\colon f(y+\alpha v) - f(y) \le -\varepsilon\alpha.$$

Therefore Lemma 2.2 (ii) gives that $|\nu| \ge \varepsilon$ for each $\nu \in \partial_C f(x)$. Thus (4.7) holds and so $f \in A$ by Lemma 4.3.

Now suppose $f \in A$. Using (4.5), we can choose $\varepsilon \in \mathbb{Q}^*$ such that

$$\forall\, x \in f^{-1}(0,\varepsilon),\ \nu \in \partial_C f(x)\colon |\nu| \ge 4\varepsilon. \tag{4.8}$$

To prove (4.6), consider arbitrary $p, q \in \mathbb{Q}^*$, $0 < p < q < \varepsilon$. Using Lemma 2.2 (i), we easily obtain that for each $z \in K \coloneqq f^{-1}([p,q])$ there exist $\varrho(z) > 0$ and $v(z) \in S^{d-1}$ such that

$$\forall\, y \in U(z,\varrho(z)),\ 0 < \alpha < \varrho(z)\colon f(y+\alpha v(z)) - f(y) \le -2\varepsilon\alpha. \tag{4.9}$$

Choose $\varrho \in \mathbb{Q}^*$ as a Lebesgue number, see [5], of the open covering $\{U(z,\varrho(z))\}_{z\in K}$ of the compact K. For an arbitrary $x \in U(0,4)$, either $f(x) \notin (p,q)$ or $x \in K$. In the second case, by the definition of Lebesgue number, there exists $z \in K$ such that $U(x,\varrho) \subset U(z,\varrho(z))$. Then clearly $\varrho < \varrho(z)$ and so (4.9) implies

$$\forall\, y \in U(x,\varrho),\ 0 < \alpha < \varrho\colon f(y+\alpha v(z)) - f(y) \le -2\varepsilon\alpha. \tag{4.10}$$

By the choice of $\mathfrak{S}^d_\varepsilon$ there is some $\nu \in \mathfrak{S}^d_\varepsilon$ such that $|v(z) - \nu| < \varepsilon$. By (4.10) for each $y \in U(x,\varrho)$ and $0 < \alpha < \varrho$,

$$f(y+\alpha v(z)) - f(y) \le -2\varepsilon\alpha.$$

Consequently, using 1-Lipschitzness of $f \in X$, we obtain

$$\begin{aligned} f(y+\alpha\nu) - f(y) &\le f(y+\alpha v(z)) - f(y) + |f(y+\alpha\nu) - f(y+\alpha v(z))| \\ &\le f(y+\alpha v(z)) - f(y) + |\nu - v(z)|\alpha \\ &\le -2\varepsilon\alpha + \varepsilon\alpha = -\varepsilon\alpha, \end{aligned}$$

and so (4.6) holds. □

Corollary 4.5. *The set A is an $F_{\sigma\delta\sigma}$ subset of X.*

Proof: For each quadruple $y \in \mathbb{R}^d$, $\nu \in S^{d-1}$, $\alpha > 0$, $\varepsilon > 0$ we set

$$C(y,\nu,\alpha,\varepsilon) \coloneqq \{f \in X\colon f(y+\alpha\nu) - f(y) \le -\varepsilon\alpha\}.$$

(Of course, we have $C(y,\nu,\alpha,\varepsilon)=\emptyset$ if $y\notin U(0,4)$ or $y+\alpha\nu\notin U(0,4)$.) Further, for each triple $x\in U(0,4)$, $0<p<q$, we set

$$D(x,p,q):=\{f\in X\colon f(x)\notin (p,q)\}.$$

It is easy to see that both $C(y,\nu,\alpha,\varepsilon)$ and $D(x,p,q)$ are always closed subsets of X. It is easy to see that Lemma 4.4 is equivalent to

$$A=\bigcup_{\varepsilon\in\mathbb{Q}^*}\ \bigcap_{\substack{p,q\in\mathbb{Q}^*,\\ 0<p<q<\varepsilon}}\ \bigcup_{\varrho\in\mathbb{Q}^*}\ \bigcap_{x\in U(0,4)}\left(D(x,p,q)\ \cup \bigcup_{\nu\in\mathfrak{S}^d_\varepsilon}\ \bigcap_{\substack{y\in U(x,\varrho),\\ 0<\alpha<\varrho}} C(y,\nu,\alpha,\varepsilon)\right).$$

Therefore, since $\mathbb{Q}^*$ is countable and each $\mathfrak{S}^d_\varepsilon$ is finite, we obtain that A is an $F_{\sigma\delta\sigma}$ subset of X. □

The proof of Theorem 4.1: We know that it is sufficient to prove (4.1) for $r=1$.

Suppose $d=2$. Then Lemma 4.2 (ii) gives that $WDC(B(0,1))=\psi^{-1}(A\cap D)$, where $\psi\colon \mathcal{K}(B(0,4))\to X$ is the continuous mapping from (4.3). Since $A\cap D$ is an $F_{\sigma\delta\sigma}$ subset of X by Corollary 4.5 and (4.4), we obtain (4.1) for $r=1$ and $d=2$, and thus also assertion (i) of Theorem 4.1.

To prove assertion (ii) of Theorem 4.1, it is sufficient to prove that (in each $\mathbb{R}^d$) $WDC(B(0,1))$ is an analytic subset of $\mathcal{K}(B(0,1))$. To this end, consider the following subset S of $\mathcal{K}(B(0,1))\times X$:

$$S:=\{(K,f)\in\mathcal{K}(B(0,1))\times X\colon f^{-1}(0)=K,\ f\in A\cap D\}.$$

By Lemma 4.2 (i), $WDC(B(0,1))=\pi_1(S)$ (where $\pi_1(K,f):=K$) and so it is sufficient to prove that S is Borel. Denoting

$$Z:=\{(K,f)\in\mathcal{K}(B(0,1))\times X\colon\ K=f^{-1}(0),\ f\in X\},$$

we have $S=Z\cap(\mathcal{K}(B(0,1))\times(A\cap D))$. So, since $A\cap D$ is Borel by Corollary 4.5 and (4.4), to prove that S is Borel, it is sufficient to show that Z is Borel in $\mathcal{K}(B(0,1))\times X$. To this end, denote for each $n\in\mathbb{N}$

$$P_n:=\Big\{(K,f)\in\mathcal{K}(B(0,1))\times X\colon\ \exists\, x\in K\colon f(x)\geq\frac{1}{n}\Big\},$$
$$Q_n:=\Big\{(K,f)\in\mathcal{K}(B(0,1))\times X\colon\ \exists\, x\in B(0,4)\colon \operatorname{dist}(x,K)\geq\frac{1}{n},\ f(x)=0\Big\}.$$

Since clearly

$$Z=(\mathcal{K}(B(0,1))\times X)\setminus\left(\bigcup_{n=1}^\infty P_n\cup\bigcup_{n=1}^\infty Q_n\right),$$

it is sufficient to prove that all P_n and Q_n are closed.

So suppose that $(K_i, f_i) \in \mathcal{K}(B(0,1)) \times X,\ i = 1, 2, \dots, (K, f) \in \mathcal{K}(B(0,1)) \times X$, $\varrho_H(K_i, K) \to 0$ and $\varrho_{\sup}(f_i, f) \to 0$.

First suppose that $n \in \mathbb{N}$ and all $(K_i, f_i) \in P_n$. Choose $x_i \in K_i$ with $f_i(x_i) \geq 1/n$. Choose a convergent subsequence $x_{i_j} \to x \in \mathbb{R}^d$. It is easy to see that $x \in K$. Since $|f_{i_j}(x_{i_j}) - f(x_{i_j})| \to 0$ and $f(x_{i_j}) \to f(x)$, we obtain $f_{i_j}(x_{i_j}) \to f(x)$, and consequently $f(x) \geq 1/n$. Thus $(K, f) \in P_n$ and therefore P_n is closed.

Second, suppose that $n \in \mathbb{N}$ and all $(K_i, f_i) \in Q_n$. Choose $x_i \in B(0,4)$ such that $\operatorname{dist}(x_i, K_i) \geq 1/n$ and $f_i(x_i) = 0$. Choose a convergent subsequence $x_{i_j} \to x \in B(0,4)$. Since $|f_{i_j}(x_{i_j}) - f(x_{i_j})| \to 0$ and $f(x_{i_j}) \to f(x)$, we obtain $f(x) = 0$. Now consider an arbitrary $y \in K$ and choose a sequence $y_j \in K_{i_j}$ with $y_j \to y$. Since $|x_{i_j} - y_j| \geq 1/n$ and $x_{i_j} \to x$, we obtain that $|y - x| \geq 1/n$ and consequently $\operatorname{dist}(x, K) \geq 1/n$. Thus $(K, f) \in Q_n$ and therefore Q_n is closed. $\square$

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