

A note on the height of the third Stiefel–Whitney class of the canonical vector bundles over some Grassmann manifolds

EUDOVÍT BALKO

Abstract. We compute the height of the third Stiefel–Whitney characteristic class of the canonical bundles over some infinite classes of Grassmann manifolds of five dimensional vector subspaces of real vector spaces.

Keywords: Grassmann manifold; height of Stiefel–Whitney characteristic class

Classification: 57N65, 57R20

1. Introduction and results

Let X be a topological space. The height of a cohomology class $x \in \tilde{H}^*(X; R)$ is defined as

$$\text{ht}(x) = \sup\{n \in \mathbb{Z} : x^n \neq 0\}.$$

The knowledge of the height of the first and second Stiefel–Whitney classes of the canonical vector bundle over a real Grassmann manifold $G_k(\mathbb{R}^{n+k})$ was used to compute the cup-length of the Grassmann manifold (see R. E. Stong [4] and S. Dutta and S. S. Khare [2]), the cup-length (for a topological space X) being defined as

$$\text{cup}_{\mathbb{Z}_2}(X) = \sup\{r : \exists x_1, \dots, x_r \in \tilde{H}^*(X; \mathbb{Z}_2) \ni x_1 \cup \dots \cup x_r \neq 0\}.$$

This work expands the known results about the height of the third Stiefel–Whitney class of the canonical bundle over the Grassmann manifold $G_4(\mathbb{R}^{n+4})$, denoted w_3 , which was computed in [1, Table 1].

In Section 3 we prove

Theorem 1.1. *Let $s \geq 4$. The height of w_3 in $H^*(G_5(\mathbb{R}^{n+5}); \mathbb{Z}_2)$ is*

$$\text{ht}(w_3) = \begin{cases} 2^s, & \text{if } n + 5 = 2^s + 3, \\ 2^s + 3, & \text{if } n + 5 = 2^s + 4, \\ 2^s + 6, & \text{if } n + 5 = 2^s + 5, \\ 2^s + 7, & \text{if } n + 5 = 2^s + t, \ t = 6, 7, 8. \end{cases}$$

DOI 10.14712/1213-7243.2021.012

Ludovít Balko was supported by the Slovak Research and Development Agency under the contract No. APVV-16-0053.

Although this is only a partial result, in the near future we plan to expand this result and find values of the height of the third Stiefel–Whitney class of the canonical bundle of $G_5(\mathbb{R}^{n+5})$ for all n . Using the height, we expect to find the cup-length of further classes of Grassmann manifolds.

The obtained results are computed using the method developed by R. E. Stong in [4] which we describe in the following section.

2. Method of computation

In his work [4], R. E. Stong introduced a method for convenient computation in the cohomology ring of the Grassmann manifold $G_k(\mathbb{R}^{n+k})$, the space of all k -dimensional subspaces of \mathbb{R}^{n+k}

$$H^*(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \dots, w_k, \bar{w}_1, \dots, \bar{w}_n] / \{w \cdot \bar{w} = 1\},$$

where w_i and \bar{w}_i are Stiefel–Whitney classes of the canonical k -plane bundle γ_k over the Grassmann manifold and the dual Stiefel–Whitney classes, respectively. Similarly, w and \bar{w} are total Stiefel–Whitney classes of the canonical k -plane bundle over the Grassmann manifold and its dual, respectively.

The total real flag manifold $\text{Flag}(\mathbb{R}^m)$ is a space consisting of ordered m -tuples (V_1, \dots, V_m) of mutually orthogonal one-dimensional vector subspaces $V_i \subset \mathbb{R}^m$. There are m canonical line bundles l_1, \dots, l_m over $\text{Flag}(\mathbb{R}^m)$. The total space of l_i consists of ordered pairs $((V_1, \dots, V_m), v_i)$, where $(V_1, \dots, V_m) \in \text{Flag}(\mathbb{R}^m)$ and $v_i \in V_i$. The cohomology of total flag manifold is

$$H^*(\text{Flag}(\mathbb{R}^m); \mathbb{Z}_2) = \mathbb{Z}_2[e_1, \dots, e_m] / \left\{ \prod_{i=1}^m (1 + e_i) = 1 \right\},$$

where $e_i = w_1(l_i)$ is the first Stiefel–Whitney class of the canonical line bundle l_i .

The map $\pi: \text{Flag}(\mathbb{R}^{n+k}) \rightarrow G_k(\mathbb{R}^{n+k})$ given by $(V_1, \dots, V_{n+k}) \mapsto V_1 \oplus V_2 \oplus \dots \oplus V_k$ induces injective homomorphism on cohomology,

$$\pi^*: H^*(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2) \rightarrow H^*(\text{Flag}(\mathbb{R}^{n+k}); \mathbb{Z}_2).$$

Moreover

$$\pi^*(w) = \prod_{i=1}^k (1 + e_i), \quad \pi^*(\bar{w}) = \prod_{i=k+1}^{n+k} (1 + e_i).$$

Details can be found in [4].

The actual computation in the cohomology ring of Grassmann manifold is based on the following observations.

Proposition 2.1 ([4, Observation, page 106]). *The value of the class $u \in H^*(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2)$ on the fundamental class of $G_k(\mathbb{R}^{n+k})$ is the same as the value of*

$$\pi^*(u)e_1^{k-1}e_2^{k-2} \cdots e_{k-1}e_{k+1}^{n-1}e_{k+2}^{n-2} \cdots e_{n+k-1}$$

on the fundamental class of $\text{Flag}(\mathbb{R}^{n+k})$.

Proposition 2.2 ([4, Corollary, page 106]). *The nonzero monomials in $H^{\text{top}}(\text{Flag}(\mathbb{R}^m); \mathbb{Z}_2)$ are precisely those of the form*

$$e_{\sigma(1)}^{m-1}e_{\sigma(2)}^{m-2} \cdots e_{\sigma(i)}^{m-i} \cdots e_{\sigma(m)}^0,$$

where σ is any permutation of the elements of the set $\{1, 2, \dots, m\}$.

3. Proof of Theorem 1.1

The image of w_3 by π^* in $H^*(\text{Flag}(\mathbb{R}^{n+5}); \mathbb{Z}_2)$ is the sum

$$\pi^*(w_3) = \sum_{1 \leq p < q < r \leq 5} e_p e_q e_r.$$

Case 1: Let $n + 5 = 2^s + 3$, $n \geq 4$ and consider the product

$$\pi^*(w_3^{2^s})e_1^4e_2^3e_3^2e_4 \cdot M = \left(\sum_{1 \leq p < q < r \leq 5} e_p^{2^s} e_q^{2^s} e_r^{2^s} \right) e_1^4e_2^3e_3^2e_4 \cdot M,$$

where $M = e_6^{2^s-3} \cdots e_{2^s+2}$. It is easy to see that all but one of the monomials in the product contain exponents greater or equal to $2^s + 3$ and such monomials will necessarily contribute as zero by Proposition 2.2. The only possibly nonzero monomial in the product is $e_1^4e_2^3e_3^{2^s+2}e_4^{2^s+1}e_5^{2^s} \cdot M$. To get this monomial into the form from Proposition 2.2, we need to multiply it with $e_1^{2^s-5}e_2^{2^s-5}$ or $e_1^{2^s-6}e_2^{2^s-4}$. It can be readily verified that the image $\pi^*(w_1^2w_2^{2^s-6})$ contains only the latter of the two and we have

$$\pi^*(w_3^{2^s} w_1^2 w_2^{2^s-6})e_1^4e_2^3e_3^2e_4 \cdot M \neq 0.$$

This proves that the height of w_3 is at least 2^s and we immediately have

$$\pi^*(w_3^{2^s+1}x)e_1^4e_2^3e_3^2e_4 \cdot M = 0$$

for any class x such that $w_3^{2^s+1}x$ is in the top dimension. The Proposition 2.1 then implies that $w_3^{2^s+1}x = 0$ for any class x such that $w_3^{2^s+1}x$ is in the top dimension and by duality [3, Theorem 11.10] $w_3^{2^s+1} = 0$.

Case 2: Let $n + 5 = 2^s + 4$, $n \geq 4$ and $M = e_6^{2^s-2} \cdots e_{2^s+3}$. Computing

$$\pi^*(w_3^{2^s+3})e_1^4e_2^3e_3^2e_4 = \pi^*(w_3^{2^s})\pi^*(w_3^2)\pi^*(w_3)e_1^4e_2^3e_3^2e_4,$$

we are left with only the following three classes with a chance to be nonzero:

$$e_1^7 e_2^6 e_3^{2^s+3} e_4^{2^s+1} e_5^{2^s+2}, \quad e_1^4 e_2^3 e_3^{2^s+2} e_4^{2^s+1} e_5^{2^s+3}, \quad e_1^4 e_2^3 e_3^{2^s+2} e_4^{2^s+3} e_5^{2^s+1}.$$

Consider now the class $w_1^2 w_2^{2^s-8}$. The image $\pi^*(w_1^2 w_2^{2^s-8})$ contains the monomial $e_1^{2^s-8} e_2^{2^s-6}$ and this is the only monomial in $\pi^*(w_1^2 w_2^{2^s-8})$ such that

$$\pi^*(w_3^{2^s+3} w_1^2 w_2^{2^s-8}) e_1^4 e_2^3 e_3^2 e_4 \cdot M$$

is nonzero in the top dimension.

There is at least one exponent greater or equal to $2^s + 4$ in each monomial of the product

$$\begin{aligned} \pi^*(w_3^{2^s+4}) e_1^4 e_2^3 e_3^2 e_4 &= \pi^*(w_3^{2^s}) \pi^*(w_3^4) e_1^4 e_2^3 e_3^2 e_4 \\ &= \left(\sum_{1 \leq p < q < r \leq 5} e_p^{2^s} e_q^{2^s} e_r^{2^s} \right) \left(\sum_{1 \leq p < q < r \leq 5} e_p^4 e_q^4 e_r^4 \right) \pi^*(w_3^{2^s}). \end{aligned}$$

It follows that $\pi^*(w_3^{2^s+4} x) e_1^4 e_2^3 e_3^2 e_4 \cdot M = 0$ for any x such that $w_3^{2^s+4} x$ is in the top dimension. Using Proposition 2.1 and duality [3, Theorem 11.10] as in the first case gives $w_3^{2^s+4} = 0$.

Case 3: Let $n + 5 = 2^s + 5$, $n \geq 4$. Multiplying

$$\pi^*(w_3^{2^s+6}) = \pi^*(w_3^{2^s}) \pi^*(w_3^4) \pi^*(w_3^2),$$

we find that the only surviving monomial is

$$e_1^{10} e_2^9 e_3^{2^s+2} e_4^{2^s+3} e_5^{2^s+4}.$$

In a way similar to previous cases, we find that for the class $w_1^2 w_2^{2^s-10}$ the value of $\pi^*(w_3^{2^s+6} w_1^2 w_2^{2^s-10}) e_1^4 e_2^3 e_3^2 e_4 \cdot M$, with $M = e_6^{2^s-1} \cdots e_{2^s+4}$, is nonzero and that $\pi^*(w_3^{2^s+7} x) e_1^4 e_2^3 e_3^2 e_4 \cdot M = 0$ for any x with $w_3^{2^s+7} x$ in maximal dimension. Then $w_3^{2^s+7} = 0$ by Proposition 2.1 and duality [3, Theorem 11.10].

Case 4: An easy computation similar to the computations above shows that $w_3^{2^s+8} = 0$ in $G_5(\mathbb{R}^{n+5})$ for $n+5 = 2^s+8$. Let $n+5 = 2^s+6$. The possibly nonzero monomials in $\pi^*(w_3^{2^s+7}) e_1^4 e_2^3 e_3^2 e_4$ can be divided into classes according the exponents at e_1, \dots, e_5 , see Table 1. For example, the monomial $e_1^{2^s+4} e_2^{10} e_3^9 e_4^{2^s+5} e_5^{2^s+3}$ belongs to the monomial class B .

We say that a monomial p in e_1, \dots, e_5 is complementary to a monomial class X , if for some monomial $q \in X$ the monomial $q \cdot p$ contains all mutually different exponents from the set $\{2^s + 1, 2^s + 2, 2^s + 3, 2^s + 4, 2^s + 5\}$. For the case of the monomial class B , the complementary monomials are $e_i^{2^s-8} e_j^{2^s-8}$ and $e_i^{2^s-7} e_j^{2^s-9}$.

We consider now the class $w_1^{2^s-4}w_2^{2^{s-1}-6}$. There are only even exponents in monomials of $\pi^*(w_1^{2^s-4}w_2^{2^{s-1}-6})$ so all the complementary monomials for all of the monomial classes of $\pi^*(w_3^{2^s+7})e_1^4e_2^3e_3e_4$ that are monomials of $\pi^*(w_1^{2^s-4}w_2^{2^{s-1}-6})$ are reduced to a set of monomials given in 'Significant complementary monomials' column in the Table 1.

Monomial class	No.	Significant compl. monomials
$A = \{8, 11, 2^s + 3, 2^s + 4, 2^s + 5\}$	9	$e_i^{2^s-6}e_j^{2^s-10}$
$B = \{9, 10, 2^s + 3, 2^s + 4, 2^s + 5\}$	10	$e_i^{2^s-8}e_j^{2^s-8}$
$C = \{9, 11, 2^s + 2, 2^s + 4, 2^s + 5\}$	5	$e_i^{2^s-8}e_j^{2^s-8}$
$D = \{9, 11, 2^s + 3, 2^s + 3, 2^s + 5\}$	4	
$E = \{9, 11, 2^s + 3, 2^s + 4, 2^s + 4\}$	3	
$F = \{10, 11, 2^s + 1, 2^s + 4, 2^s + 5\}$	3	$e_i^{2^s-8}e_j^{2^s-8}, e_i^{2^s-8}e_j^{2^s-10}e_k^2$
$G = \{10, 11, 2^s + 2, 2^s + 3, 2^s + 5\}$	3	$e_i^{2^s-6}e_j^{2^s-10}, e_i^{2^s-8}e_j^{2^s-10}e_k^2$
$H = \{10, 10, 2^s + 3, 2^s + 3, 2^s + 5\}$	2	
$I = \{10, 11, 2^s + 2, 2^s + 4, 2^s + 4\}$	2	
$J = \{10, 10, 2^s + 2, 2^s + 4, 2^s + 5\}$	3	
$K = \{10, 11, 2^s + 3, 2^s + 3, 2^s + 4\}$	1	$e_i^{2^s-8}e_j^{2^s-10}e_k^2$
$L = \{10, 10, 2^s + 3, 2^s + 4, 2^s + 4\}$	1	

TABLE 1. List of all monomial classes of $\pi^*(w_3^{2^s+7})e_1^4e_2^3e_3e_4$ with exponents from a given set with number of monomials in a class and significant complementary monomials from $\pi^*(w_1^{2^s-4}w_2^{2^{s-1}-6})$.

We now investigate the remaining significant complementary monomials. In the monomial class A we have $e_i^{2^s-6}e_j^{2^s-10} = e_i^{2^{s-1}}e_j^{2^{s-1}-4}e_i^{2^{s-1}-6}e_j^{2^{s-1}-6}$. The part $e_i^{2^{s-1}-6}e_j^{2^{s-1}-6}$ comes from $\pi^*(w_2^{2^{s-1}-6})$ with nonzero coefficient. The remaining part $e_i^{2^{s-1}}e_j^{2^{s-1}-4}$ comes from $\pi^*(w_1^{2^s-4})$ with coefficient

$$\binom{2^s - 4}{2^{s-1}} = \binom{2^s - 4}{2^{s-1} - 4} \equiv 1 \pmod{2}.$$

Similar argument for $e_i^{2^s-8}e_j^{2^s-8}$ gives

$$e_i^{2^s-8}e_j^{2^s-8} = e_i^{2^{s-1}-2}e_j^{2^{s-1}-2}e_i^{2^{s-1}-6}e_j^{2^{s-1}-6}$$

with coefficient at $e_i^{2^{s-1}-2}e_j^{2^{s-1}-2}$ equal to $\binom{2^s-4}{2^{s-1}-2} = 0$ in $\pi^*(w_1^{2^s-4})$.

The last of the complementary monomial is $e_i^{2^s-8}e_j^{2^s-10}e_k^2$ and it can be written as

$$e_i^{2^{s-1}-2}e_j^{2^{s-1}-2}e_i^{2^{s-1}-8}e_j^{2^{s-1}-8}e_i^2e_k^2$$

or

$$e_i^{2^{s-1}}e_j^{2^{s-1}-4}e_i^{2^{s-1}-8}e_j^{2^{s-1}-8}e_i^2e_k^2.$$

Factors $e_i^{2^{s-1}-8}e_j^{2^{s-1}-8}e_i^2e_k^2$ and $e_i^{2^{s-1}-8}e_j^{2^{s-1}-8}e_i^2e_k^2$ come from $\pi^*(w_2^{2^{s-1}-6}) = \pi^*(w_2^{2^{s-1}-8})\pi^*(w_2^2)$ and the remaining factors in $\pi^*(w_1^{2^s-4})$ were addressed above.

From that we conclude that there are odd number of nonzero monomials in $\pi^*(w_3^{2^s+7}w_1^{2^s-4}w_2^{2^{s-1}-6})e_1^4e_2^3e_3e_4 \cdot M$ with $M = e_6^{2^s} \cdots e_{2^s+5}$, namely, 9 from the class A , 3 from the class F , 2 times 3 from the class G and 1 from the class K . Therefore $w_3^{2^s+7}$ is nonzero in $G_5(\mathbb{R}^{2^s+6})$ and we proved that the height of w_3 in $G_5(\mathbb{R}^{2^s+6})$ is 2^s+7 . Due to the fact that the height is nondecreasing as a function of n , the height of w_3 in $G_5(\mathbb{R}^{n+5})$ for $n = 6, 7, 8$ is also 2^s+7 .

REFERENCES

- [1] Balko L., Lörinc J., *On the cup-length of certain classes of flag manifolds*, Acta Math. Hungar. **159** (2019), no. 2, 638–652.
- [2] Dutta S., Khare S. S., *On second Stiefel–Whitney class of Grassmann manifolds and cup-length*, J. Indian Math. Soc. (N.S.) **69** (2002), no. 1–4, 237–251.
- [3] Milnor J. W., Stasheff J. D., *Characteristic Classes*, Annals of Mathematics Studies, 76, Princeton University Press, Princeton, University of Tokyo Press, Tokyo, 1974.
- [4] Stong R. E., *Cup products in Grassmannians*, Topology Appl. **13** (1982), no. 1, 103–113.

L. Balko:

DEPARTMENT OF ALGEBRA AND GEOMETRY, FACULTY OF MATHEMATICS,
PHYSICS AND INFORMATICS, COMENIUS UNIVERSITY, MLYNSKÁ DOLINA F1,
842 48 BRATISLAVA 4, SLOVAKIA

E-mail: ludovit.balko@gmail.com

(Received June 19, 2019, revised April 23, 2020)