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On atomic ideals in some factor rings of $C(X,\mathbb{Z})$

Alireza Olfati

Abstract. A nonzero R-module M is atomic if for each two nonzero elements a,b in M, both cyclic submodules Ra and Rb have nonzero isomorphic submodules. In this article it is shown that for an infinite P-space X, the factor rings $C(X,\mathbb{Z})/C_F(X,\mathbb{Z})$ and $C_c(X)/C_F(X)$ have no atomic ideals. This fact generalizes a result published in paper by A. Mozaffarikhah, E. Momtahan, A. R. Olfati and S. Safaeeyan (2020), which says that for an infinite set X, the factor ring $\mathbb{Z}^X/\mathbb{Z}^{(X)}$ has no atomic ideal. Another result is that for each infinite P-space X, the socle of the factor ring $C_c(X)/C_F(X)$ is always equal to zero. Also, zero-dimensional spaces X are characterized for which $C^F(X,\mathbb{Z})/C_F(X,\mathbb{Z})$ have atomic ideals.

Keywords: P-space; rings of integer-valued continuous functions; functionally countable subalgebra; atomic ideal; socle

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1. Introduction

In this article, by ring, we mean a commutative ring with identity. A submodule N of an R-module M is said to be essential in M and denoted by $N \leq_{\text{ess}} M$ if N intersects each nonzero submodule of M nontrivially. The socle of a module M is the sum of all minimal submodules of M. For every nonzero module M, the socle of M is equal to the intersection of all essential submodules of M.

We assume throughout this paper that any topological space X is Tychonoff. By $C(X)$, we mean the ring of all real valued continuous functions on X. For any $f \in C(X)$, the set $Z(f) := \{x \in X \colon f(x) = 0\}$ is called the zero-set of f. For any Tychonoff space X, we denote by $C(X,\mathbb{Z})$, the set of all integer valued continuous functions on X. The set of all continuous real valued functions on X with countable image is denoted by $C_c(X)$. Clearly $C(X,\mathbb{Z})$ is a proper subset of $C_c(X)$. Equipped with the pointwise addition and multiplication of $\mathbb{R}^X$, the sets $C(X,\mathbb{Z})$ and $C_c(X)$ form two subrings of $C(X)$. The ring of integer-valued continuous functions have been studied in many ways and some outstanding results were achieved by some mathematicians, see for example [1], [6], [9] and [10]. Just recently in [2] an extensive study for the subalgebra $C_c(X)$ has been done. In

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[5] it is shown that the socle of $C(X)$ is equal to the set of all functions $f \in C(X)$ such that $X \setminus Z(f)$ is finite. We denote the socle of $C(X)$ by $C_F(X)$. It is obvious to see that $C_F(X)$ is a subset of $C_c(X)$. In [3], it was shown that the socle of the factor ring $C(X)/C_F(X)$ is always equal to zero. We also denote by $C_F(X,\mathbb{Z})$ the set $C_F(X) \cap C(X,\mathbb{Z})$. The notion of atomic submodules was introduced and studied in [7] as a generalization for minimal submodules. In [8] it is shown that for an arbitrary infinite set X, the factor ring $\mathbb{Z}^X/\mathbb{Z}^{(X)}$ has no atomic ideal. In this note, we generalize this fact to every P-space and show that for an infinite P-space X the factor rings $C(X,\mathbb{Z})/C_F(X,\mathbb{Z})$ and $C_c(X)/C_F(X)$ have no atomic ideals. This result leads to the fact that such factor rings have no uniform ideals, i.e., the socle of them equals zero.

2. Main results

Definition 2.1. A nonzero R-module M is called *atomic* if for each a, b in $M\setminus(0)$, the cyclic submodules Ra and Rb have nonzero isomorphic submodules.

Recall that a uniform module is a nonzero module A such that the intersection of any two nonzero submodules of A is nonzero, or, equivalently, such that every nonzero submodule of A is essential in A. It is obvious that every uniform module is atomic. Since every minimal ideal in a semiprime ring is uniform, we observe that every minimal ideal is atomic but the converse is not true. For example, if X is a zero-dimensional space, $C(X,\mathbb{Z})$ has no minimal ideals, but it has an atomic ideal if and only if X has an isolated point. For every module M, we denote by $\Sigma_a(M)$, the sum of all atomic submodules of M. For example, in [8] it is shown that for every zero-dimensional space X, $\Sigma_a(C(X,\mathbb{Z})) = C_F(X,\mathbb{Z})$ and for every Tychonoff space X, $\Sigma_a(C(X)) = C_F(X)$. For every nonzero module M, the factor module $M/\Sigma_a(M)$ may or may not have any atomic submodules, for example see [8]. But it is shown in [8] that for every infinite set X, the factor ring $\mathbb{Z}^X/\mathbb{Z}^{(X)}$ has no atomic ideals, or equivalently $\Sigma_a(\mathbb{Z}^X/\mathbb{Z}^{(X)})$ is equal to zero. The main objective in the sequal is to extend this result to every P-space and show that the two factor rings $C(X,\mathbb{Z})/C_F(X,\mathbb{Z})$ and $C_c(X)/C_F(X)$ have no atomic ideals. But first we need the following two results.

Proposition 2.2. *If R is a semiprime ring, the ideal I of R is an atomic R-module if and only if I is a uniform ideal.*

Proof: Let I be an atomic ideal of R. Then for every two $a, b \in I \setminus (0)$, there exist two nonzero submodules M of Ra and N of Rb and an isomorphism $\varphi\colon M \to N$. Then for some $r, s \in R$, $\varphi(ra) = sb \neq 0$. Since R is semiprime and φ is a homomorphism, $g\varphi(rf) = \varphi(rgf)$ is nonzero and therefore $fg \neq 0$.

For the converse, suppose that $a, b \in I \setminus (0)$. Since $ab \neq 0$, the cyclic submodule Rab is contained in $Ra \cap Rb$. Thus Ra and Rb have nonzero isomorphic submodules. □

We recall that a topological space X is a P-space if and only if for each countable family of open sets $\{U_n : n \in \mathbb{N}\}$, the subset $\bigcap_{n\in\mathbb{N}} U_n$ is open in X.

Proposition 2.3. *Every infinite P-space X has a countable infinite family $\{C_i : i \in \mathbb{N}\}$ of nonempty clopen subsets, such that $X = \bigcup_{i\in\mathbb{N}} C_i$ and for each pair of distinct $i, j \in \mathbb{N}$, $C_i \cap C_j = \emptyset$.*

Proof: Suppose that X is an infinite P-space. By [4, Exercise 4K.2], X is not pseudocompact. Hence there exists a continuous real valued function $f : X \to \mathbb{R}$ such that $f(X)$ is unbounded and hence infinite. For each $r \in f(X)$, the inverse image $f^{\leftarrow}(r)$ is clopen. Choose a countable subset $\{r_n : n \in \mathbb{N}\}$ of $f(X)$. Since two sets $T = \bigcup_{n\in\mathbb{N}} f^{\leftarrow}(r_n)$ and $C_0 = f^{\leftarrow}(\mathbb{R}) \setminus T$ are open, the subset C_0 is clopen. Now the family $\{C_i : i \in \mathbb{N} \cup \{0\}\}$ is an infinite clopen partition for X. □

Theorem 2.4. *Assume that X is an infinite P-space. Then both the factor rings $C(X,\mathbb{Z})/C_F(X,\mathbb{Z})$ and $C_c(X)/C_F(X)$ have no atomic ideals.*

Proof: Assume that $I/C_F(X,\mathbb{Z})$ is an ideal of $C(X,\mathbb{Z})/C_F(X,\mathbb{Z})$ and argue by contraposition that it is not an atomic ideal. Let $\bar{f} = f + C_F(X,\mathbb{Z})$ and $\bar{g} = g + C_F(X,\mathbb{Z})$ be two nonzero elements of $I/C_F(X,\mathbb{Z})$ such that $\bar{f}\bar{g} \neq 0$. This means that $fg \notin C_F(X,\mathbb{Z})$ and hence $X \setminus Z(fg)$ is infinite. Since $X \setminus Z(fg)$ is an infinite P-space, by Theorem 2.4, there exists a countably infinite clopen partition $\{C_i : i \in \mathbb{N}\}$ of $X \setminus Z(fg)$. From the fact that $X \setminus Z(fg)$ is clopen in X, each C_i is clopen in X. It is easy to see that the sets $T = \bigcup_{i\in\mathbb{N}} C_{2i}$ and $S = \bigcup_{i\in\mathbb{N}} C_{2i-1}$ are two clopen subsets of X. Let χ_T and χ_S be the characteristic functions of T and S, respectively. Define $h = \chi_T f$ and $k = \chi_S g$. Clearly, $\bar{h} = h + C_F(X,\mathbb{Z})$ and $\bar{k} = k + C_F(X,\mathbb{Z})$ are two nonzero elements of the ideal $I/C_F(X,\mathbb{Z})$ and $\bar{h}\bar{k} = 0$, a contradiction.

The proof for the factor ring $C_c(X)/C_F(X)$ can be repeated verbatim without any extra work. □

Since each minimal ideal of a commutative ring is atomic, the following corollary is immediate.

Corollary 2.5. *For each infinite P-space X, we have that the socle of the factor ring $C_c(X)/C_F(X)$ is equal to zero.*

In the sequel we observe a necessary and sufficient condition for a compact zero-dimensional space X such that the factor ring $C(X,\mathbb{Z})/C_F(X,\mathbb{Z})$ has a uniform and hence an atomic ideal. First, we need the following result from [8].

Proposition 2.6. *Let X be a zero-dimensional space. An ideal I of $C(X,\mathbb{Z})$ is atomic if and only if it is a principal ideal generated by the characteristic function of an isolated point of X.*

Corollary 2.7. *Let X be a zero-dimensional space. For the ring $C(X,\mathbb{Z})$,*

$$\Sigma_a(C(X,\mathbb{Z})) = C_F(X,\mathbb{Z}).$$

The set of all isolated points of a topological space X is denoted by $\mathbb{I}(X)$.

Proposition 2.8. *Let X be a compact zero-dimensional space. The factor ring $C(X,\mathbb{Z})/C_F(X,\mathbb{Z})$ has a uniform (or equivalently atomic) ideal if and only if the subset $X \setminus \mathbb{I}(X)$ has an isolated point.*

Proof: Let X be a compact zero-dimensional space. Assume that the factor ring $C(X,\mathbb{Z})/C_F(X,\mathbb{Z})$ has a uniform ideal. Since by [9, Proposition 4.11], the factor ring $C(X,\mathbb{Z})/C_F(X,\mathbb{Z})$ is isomorphic with the ring $C(X\setminus\mathbb{I}(X),\mathbb{Z})$, it follows that the ring $C(X \setminus \mathbb{I}(X),\mathbb{Z})$ has an atomic ideal and hence by Proposition 2.6, the subspace $X \setminus \mathbb{I}(X)$ has an isolated point. □

The set of all continuous integer-valued functions with finite images is denoted by $C^F(X,\mathbb{Z})$. We remind the reader that the Banaschewski compactification of a zero-dimensional Hausdorff space X is a compact Hausdorff space $\beta_0 X$ which contains X as a dense subspace and each continuous real valued function $f\colon X \to \mathbb{R}$ with a finite image has an extension to $\beta_0 X$; see, e.g. [2].

Proposition 2.9. *Let X be an arbitrary zero-dimensional space. The factor ring $C^F(X,\mathbb{Z})/C_F(X,\mathbb{Z})$ has a uniform (or equivalently atomic) ideal if and only if the subset $\beta_0 X \setminus \mathbb{I}(X)$ has an isolated point.*

Proof: For each $f \in C^F(X,\mathbb{Z})$, there is a unique $f^{\beta_0} \in C^F(\beta_0 X,\mathbb{Z}) = C(\beta_0 X,\mathbb{Z})$, such that $f^{\beta_0}|_X = f$ and $f^{\beta_0}(X) = f(X)$. Note that since X is dense in $\beta_0 X$, $\mathbb{I}(\beta_0 X) = \mathbb{I}(X)$. Clearly under the isomorphism $f \to f^{\beta_0}$, $C_F(X,\mathbb{Z})$ is sent to $C_F(\beta_0 X,\mathbb{Z})$. Hence $C^F(X,\mathbb{Z})/C_F(X,\mathbb{Z}) \cong C(\beta_0 X,\mathbb{Z})/C_F(\beta_0 X,\mathbb{Z})$. Now by Proposition 2.8, $C^F(X,\mathbb{Z})/C_F(X,\mathbb{Z})$ has a uniform (or equivalently atomic) ideal if and only if $\beta_0 X \setminus \mathbb{I}(\beta_0 X) = \beta_0 X \setminus \mathbb{I}(X)$ has an isolated point. □

Example 2.10. It is well known that $\beta\mathbb{N}\setminus\mathbb{N}$ has no isolated points. Now Proposition 2.9 implies that $C^F(\mathbb{N},\mathbb{Z})/C_F(\mathbb{N},\mathbb{Z})$ has no uniform (or equivalently atomic) ideals.

With regard to the latter proposition, the interested reader is encouraged to characterize all zero-dimensional topological spaces X for which the factor ring $C(X,\mathbb{Z})/C_F(X,\mathbb{Z})$ has an atomic (equivalently, uniform) ideal, a question which is unsettled yet.

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A. Olfati:
Department of Mathematics, Yasouj University, Daneshjoo St., Yasouj, 7591874934, Iran

E-mail: alireza.olfati@yu.ac.ir

E-mail: olfati.alireza@gmail.com

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