Harmonic deformability of planar curves

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Dedicated to the memory of Professor Ivan Netuka

Abstract. We study the formerly established concept of deformation of a planar curve and clarify its applicability and range. We present several applications on classical curves.

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1. Introduction

Let Ω be an open subset of $\mathbb{R}^2$. It is well known that every harmonic function $h\colon \Omega \to \mathbb{R}$ has the following mean value property. For every $z_0 = (x_0, y_0) \in \Omega$ and every disk $\{z \in \mathbb{R}^2 \colon |z - z_0| \le r\}$, $r \ge 0$, that is contained in Ω, it holds that

$$\frac{1}{2\,\pi}\int_0^{2\,\pi} h(x_0 + r\cos t, y_0 + r\sin t)\,\mathrm{d}t = h(x_0, y_0).$$

This means that h has the same mean value over all concentric circles with center (x_0, y_0). This fact is the basic example of a harmonically deformed curve (here a circle). The general framework is as follows.

Let I be an interval in $\mathbb{R}$,

$$I \ni t \longmapsto (x_0(t), y_0(t)) \in \Omega$$

a smooth curve in an open set $\Omega \subseteq \mathbb{R}^2$. The harmonic deformation of the curve (x_0, y_0) consists then in finding functions $x(s,t)$ and $y(s,t)$ defined on some $J \times I$, J being an interval with $0 \in J$, such that

$$\forall\,(s,t) \in J \times I \qquad (x(s,t), y(s,t)) \in \Omega, \tag{1.1}$$

$$\forall\,t \in I \qquad x(0,t) = x_0(t), \quad y(0,t) = y_0(t), \tag{1.2}$$

$$\forall\,s \in J \qquad \int_I h(x(s,t), y(s,t))\,\mathrm{d}t = \int_I h(x_0(t), y_0(t))\,\mathrm{d}t \tag{1.3}$$

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for every harmonic function $h\colon \Omega \to \mathbb{R}$, for which $t \mapsto h(x_0(t), y_0(t))$ is integrable over I. The last equation can be considered as the invariance of the mean value of h over the curves $t \mapsto (x(s,t), y(s,t))$ (s being the parameter of deformation).

2. The d-potential and the equations of deformation

The principle of deformation is described in the following theorem.

Theorem 1. *Let Ω be a simply connected open subset of $\mathbb{R}^2$, $t \mapsto (x_0(t), y_0(t))$ a smooth curve in Ω defined on an interval I. Furthermore, let J be an open interval with $0 \in J$ and*

$$\begin{cases} J \times I & \longrightarrow & \Omega \\ (s,t) & \longmapsto & (x(s,t), y(s,t)) \end{cases}$$

a conformal mapping (that is, $\frac{\partial x}{\partial s} = \frac{\partial y}{\partial t}$ and $\frac{\partial x}{\partial t} = -\frac{\partial y}{\partial s}$) satisfying (1.2).

If $h\colon \Omega \to \mathbb{R}$ is a harmonic function for which there exists a bounded harmonic conjugate $\tilde{h}\colon \Omega \to \mathbb{R}$ that satisfies the condition

$$\forall\, s \in J \quad \lim_{t \to \inf I} \tilde{h}(x(s,t), y(s,t)) = \lim_{t \to \sup I} \tilde{h}(x(s,t), y(s,t)), \tag{2.1}$$

then (1.3) holds.

For a proof the reader is referred to [2] or [3] (the latter article addresses the more general case of weighted integrals). It should be noted that (2.1) is trivially satisfied when I is compact and all curves are closed.

For the conformal mapping in the theorem it follows that there exists a harmonic function $v(s,t)$ on $J \times I$ such that

$$x = \frac{\partial v}{\partial t}, \qquad y = \frac{\partial v}{\partial s}. \tag{2.2}$$

The harmonicity of v together with (1.2) leads to the series expansion

$$v(s,t) = v(0,t) + \sum_{k=1}^{\infty} \frac{(-1)^k x_0^{(2k-1)}(t)}{(2k)!} s^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k y_0^{(2k)}(t)}{(2k+1)!} s^{2k+1}, \tag{2.3}$$

(see details in [2] or [3]). This function is called a *d-potential*, because together with $\frac{\partial v(0,t)}{\partial t} = x_0(t)$ it determines the deformation by the equations (2.2). Obviously, a d-potential is determined up to a constant.

We now investigate, under which conditions and to what extent a given curve (x_0, y_0) is harmonically deformable. More precisely, we ask about the range of the parameter s of deformation.

Since a conformal mapping is complex analytic, (1.2) shows that (x_0, y_0) has to be a real analytic curve, otherwise it is not deformable. A sufficient condition for deformability is contained in the next proposition.

Proposition 2. *Let $I \ni t \mapsto (x_0(t), y_0(t))$ be a real analytic mapping, and let $\varrho_\tau \in (0, \infty]$ be the radius of convergence of its Taylor series at the point $t = \tau \in I$. Furthermore, let $\varrho := \inf_{\tau \in I} \varrho_\tau$ be (strictly) positive. Then, the curve (x_0, y_0) is harmonically deformable and the range of the parameter s of deformation comprises the interval $(-\varrho, \varrho)$.*

Proof: Assuming the condition, the series in (2.3) converges for $s \in (-\varrho, \varrho)$ and all $t \in I$, since the Taylor series of a function and that one of its derivative have the same radius of convergence. Furthermore, it is easily seen that the series in (2.3) can be differentiated term by term with respect to both $s \in (-\varrho, \varrho)$ and $t \in I$, so the deformation follows from (2.2). $\square$

Example I in the next section shows that the range of s may be precisely $(-\varrho, \varrho)$, whereas example II shows that it may be much larger. Besides, there are (real) analytic curves that are not deformable. In fact, let f be a holomorphic function on the unit disk, which is not holomorphically extendable along any boundary point (such functions exist). If $x_0(t) = \mathrm{Re}(f(\mathrm{i}t))$, $y_0(t) = \mathrm{Im}(f(\mathrm{i}t))$ for $t \in (-1, 1)$, then it is easily seen that there are no possible values for $s \neq 0$, because the deformation is given by $f(s + \mathrm{i}t)$.

Finally, it should be clear that the shape of the deformed curves heavily depends on the parametrization of the original curve and not only on its shape, as examples I and III in the next section show.

3. Applications

For the basic example of concentric circles from the introduction, now as an application of the d-potential, the reader is referred to [2], where also a generalization to confocal ellipses is given. Here we continue with further interesting examples.

I. The curve $(x_0(t), y_0(t)) = (1/(1+t^2), 0)$ for $t \in \mathbb{R}$ covers the left-open interval $(0, 1]$. Since the function $F(z) = 1/(1-z^2)$ is holomorphic on $\mathbb{C}\setminus\{-1, 1\}$, and $F(\mathrm{i}t) = 1/(1+t^2)$, it is this function that delivers the harmonic deformation, that is,

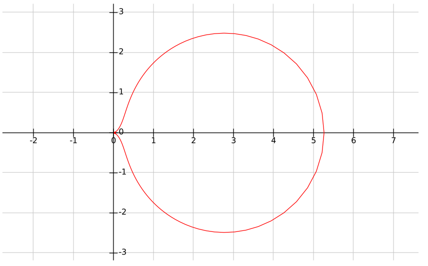
$$x(s,t) + \mathrm{i}y(s,t) = F(s + \mathrm{i}t) = \frac{1}{1 - (s + \mathrm{i}t)^2},$$

so

$$x(s,t) = \frac{1 - s^2 + t^2}{1 + (s^2 + t^2)^2 - 2s^2 + 2t^2}, \qquad y(s,t) = \frac{2st}{1 + (s^2 + t^2)^2 - 2s^2 + 2t^2}.$$

Obviously, the maximal open interval of the parameter of deformation s is $(-1, 1)$, and for the minimal radius of convergence (see the proposition above) we have $\varrho = 1$.

The following two figures show the curves for $s = 0.2$ (left) and $s = 0.9$ (right). (All figures in this article have been created by the software at http://fooplot.com .)

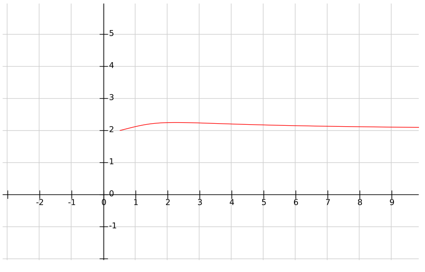


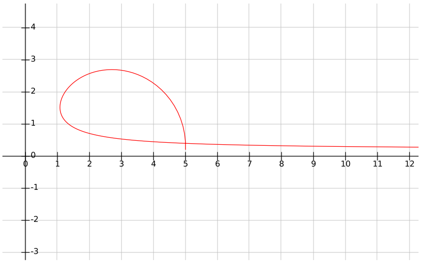
II. We consider the standard hyperbola $(x_0(t), y_0(t)) = (t, 1/t)$ for $t \in (0, \infty)$. Since the function $F(z) = -\mathrm{i}z - 1/z$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and $F(\mathrm{i}t) = t + \mathrm{i}/t$, it is this function that delivers the harmonic deformation, so

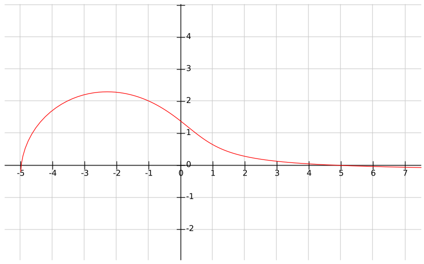
$$x(s,t) = \operatorname{Re} F(s + \mathrm{i}t) = t - \frac{s}{s^2 + t^2},$$
$$y(s,t) = \operatorname{Im} F(s + \mathrm{i}t) = -s + \frac{t}{s^2 + t^2}.$$

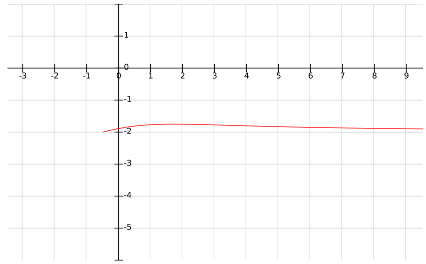
Obviously, there is no restriction for $s \in \mathbb{R}$, despite the fact that here $\varrho = 0$.

The following figures show the curves for the values $s = -2$, $s = -0.2$, $s = 0.2$, $s = 2$.









III. A different parametrization of the left-open interval $(0, 1]$ is given by $(x_0(t), y_0(t)) = (1/\cosh t, 0)$ for $t \in \mathbb{R}$. Since this is nothing else but $F(\mathrm{i}t)$ for the holomorphic function $F(z) = 1/\cos z$, the harmonic deformation is given by

$$
\begin{aligned}
x(s,t) &= \operatorname{Re} \frac{1}{\cos(s+\mathrm{i}t)} = \frac{\cos s \cosh t}{\cos^2 s + \sinh^2 t}, \\
y(s,t) &= \operatorname{Im} \frac{1}{\cos(s+\mathrm{i}t)} = -\frac{\sin s \sinh t}{\cos^2 s + \sinh^2 t}.
\end{aligned}
$$

The range of s is the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the deformed curves are the images of the confocal hyperbolas $x^2/\cos^2 s - y^2/\sin^2 s = 1$, $x > 0$ under the mapping $z \mapsto 1/z$. They satisfy the equations

$$
(x^2+y^2)^2 = \frac{x^2}{\cos^2 s} - \frac{y^2}{\sin^2 s},
$$

which define lemniscates of Booth. For $s \in \left\{-\frac{\pi}{4}, \frac{\pi}{4}\right\}$ in particular, the curves coincide with the right half of Bernoulli's lemniscate with equation $(x^2+y^2)^2 = 2(x^2-y^2)$. This gives an answer to an open problem in [2, Section 3, VI]. Concerning the whole lemniscate, say L, the following quadrature identity is mentioned in [1, Chapter 22, Section 4].

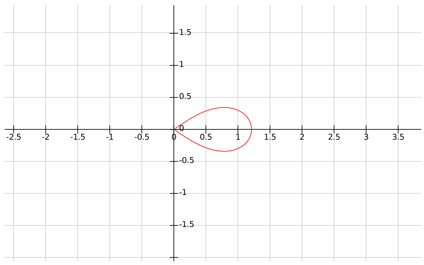
$$
\int_L h(z)\, \partial_n \log|z^2-1| \,\mathrm{d}s = h(-1) + h(1),
$$

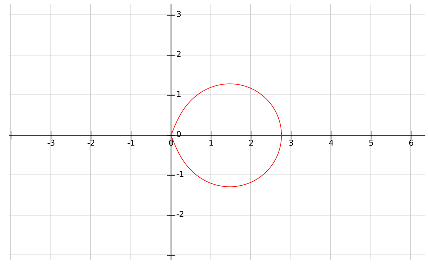
where h is a harmonic function on a neighborhood of L and its interior, ∂_n denotes the derivative in the outward normal direction, and $\mathrm{d}s$ stands for the arclength measure ($z = x + \mathrm{i}y$). In terms of our parametrization $t \mapsto \left(x\left(-\frac{\pi}{4}, t\right), y\left(-\frac{\pi}{4}, t\right)\right)$ one computes

$$
\partial_n \log|z^2-1| \,\mathrm{d}s = \frac{4}{1+2\sinh^2 t}\,\mathrm{d}t,
$$

whereas the measure throughout this article is just $\mathrm{d}t$, see (1.3). In fact, for our integral over the right half of the lemniscate to converge, it is necessary that the harmonic function vanishes sufficiently fast at zero.

The next figures show the curves for $s = 0.6$ and $s = 1.2$.





IV. The Archimedean spiral $(x_0(t), y_0(t)) = (a + bt)(\cos t, \sin t)$ for $t \in \mathbb{R}$ with the parameters $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$ is not contained in any proper simply connected open subset of $\mathbb{R}^2$, wherefore the right side of (1.3) does not seem to converge unless $h = 0$. Nevertheless, applying the principle of harmonic deformation to the spiral may be interesting in itself.

One computes

$$x_0^{(2k-1)}(t) = (-1)^{k-1}(2k-1)b\cos t + (-1)^k(a+bt)\sin t,$$
$$y_0^{(2k)}(t) = 2(-1)^{k-1}kb\cos t + (-1)^k(a+bt)\sin t,$$

which inserted in (2.3) render

$$v(s,t) = \mathrm{e}^s(a+bt)\sin t + b\mathrm{e}^s(1-s)\cos t$$

after the computations, if we take $v(0,t) = (a+bt)\sin t + b\cos t$, $s \in \mathbb{R}$.

By (2.2) we now obtain

$$\begin{pmatrix} x(s,t) \\ y(s,t) \end{pmatrix} = bs\,\mathrm{e}^s \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} + \mathrm{e}^s(a+bt)\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

The first vector on the right side can be expressed as $\left(\cos\left(t - \frac{\pi}{2}\right), \sin\left(t - \frac{\pi}{2}\right)\right)^T$, from which we infer that the deformed curve is a homothetical spiral with its base point moving on a circle of radius $bs\,\mathrm{e}^s$.

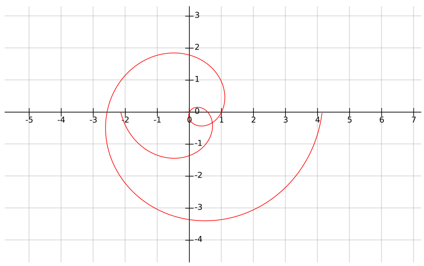
It is worth noting that the tangent vector with components

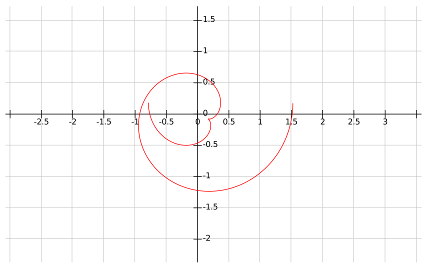
$$\frac{\partial x(s,t)}{\partial t} = \mathrm{e}^s[b(1+s)\cos t - (a+bt)\sin t]$$

and

$$\frac{\partial y(s,t)}{\partial t} = \mathrm{e}^s[b(1+s)\sin t + (a+bt)\cos t]$$

vanishes precisely in the case $s = -1$ and $t = -a/b$. This is visible at the cusp in the following figure on the right, $s = -1$, whereas the one on the left shows the original curve, $s = 0$. For both figures we have taken $a = 1$, $b = 0.5$, and $t \in [-2\pi, 2\pi]$.





V. The Kampyle of Eudoxus has the equation $x^4 = x^2 + y^2$ up to a homothety, with the restriction $x > 0$. We consider the parametrization $(x_0(t), y_0(t)) = \left(\cosh\frac{t}{2}, \frac{1}{2}\sinh t\right)$ for $t \in \mathbb{R}$. One computes

$$x_0^{(2k-1)}(t) = \frac{1}{2^{2k-1}}\sinh\frac{t}{2}, \qquad y_0^{(2k)}(t) = \frac{1}{2}\sinh t,$$

which inserted in (2.3) render

$$v(s,t) = 2\sinh\frac{t}{2}\cos\frac{s}{2} + \frac{1}{2}\sinh t\sin s$$

after the computations, if we take $v(0,t) = 2\sinh\frac{t}{2}$, $s \in \mathbb{R}$.

By (2.2) we obtain

$$x(s,t) = \cosh\frac{t}{2}\cos\frac{s}{2} + \frac{1}{2}\cosh t\sin s,$$
$$y(s,t) = \frac{1}{2}\sinh t\cos s - \sinh\frac{t}{2}\sin\frac{s}{2}.$$

The tangent vector with components

$$\frac{\partial x(s,t)}{\partial t} = \frac{1}{2}\cos\frac{s}{2}\sinh\frac{t}{2} + \frac{1}{2}\sin s\sinh t$$

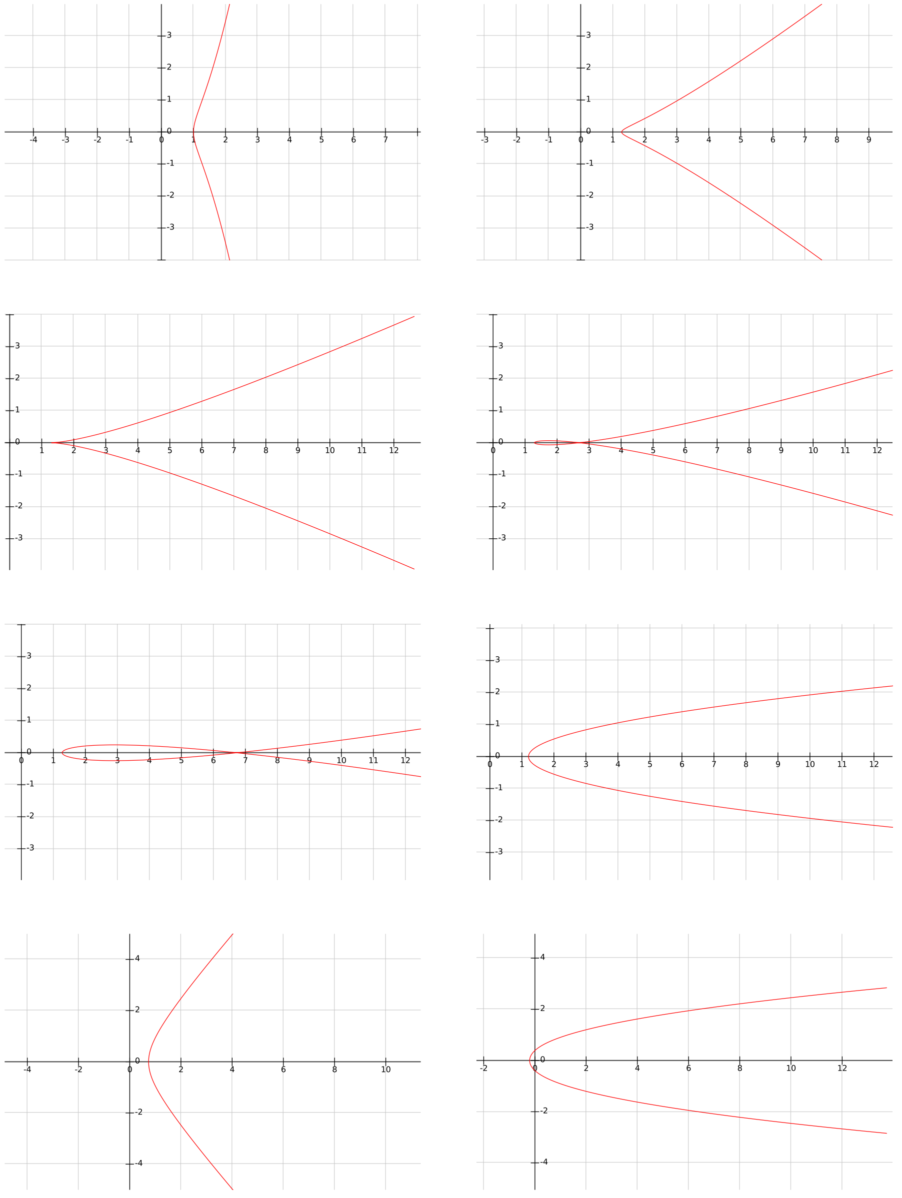
and

$$\frac{\partial y(s,t)}{\partial t} = \frac{1}{2}\cos s\cosh t - \frac{1}{2}\sin\frac{s}{2}\cosh\frac{t}{2}$$

vanishes if and only if $t = 0$ and $\left[s \in 3\pi + 4\pi\mathbb{Z} \text{ or } s \in \frac{\pi}{3} + 4\pi\mathbb{Z} \text{ or } s \in \frac{5\pi}{3} + 4\pi\mathbb{Z}\right]$.

Moreover, the mapping $s \mapsto 2\pi - s$ results in a reflection of the respective curve with respect to the y-axis, and the family is 4π-periodic in s. Therefore, it suffices to study the family for $s \in [0,\pi] \cup (2\pi, 3\pi]$.

Quaintly, for $s \in \{\pi, 3\pi\}$ the curve is just the y-axis. The next figures show the curves for $s \in \left\{0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{12}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{2}\right\}$.



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