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Conformal Killing graphs in foliated Riemannian spaces with density: rigidity and stability

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Abstract. In this paper we investigate the geometry of conformal Killing graphs in a Riemannian manifold $\overline{M}_f^{n+1}$ endowed with a weight function f and having a closed conformal Killing vector field V with conformal factor ψ_V, that is, graphs constructed through the flow generated by V and which are defined over an integral leaf of the foliation $V^\perp$ orthogonal to V. For such graphs, we establish some rigidity results under appropriate constraints on the f-mean curvature. Afterwards, we obtain some stability results for f-minimal conformal Killing graphs of $\overline{M}_f^{n+1}$ according to the behavior of ψ_V. Finally, related to conformal Killing graphs immersed in $\overline{M}_f^{n+1}$ with constant f-mean curvature, we study the strong stability.

Keywords: weighted Riemannian manifold; conformal Killing graph; f-mean curvature; Bakry–Émery–Ricci tensor; strong f-stability

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1. Introduction

Conformal Killing vector fields are important objects which have been widely used in order to understand the geometry of submanifolds and, more particularly, of hypersurfaces immersed in Riemannian spaces. In this setting, S. Montiel in [29] has studied the uniqueness of compact hypersurfaces with constant mean curvature in a complete Riemannian manifold endowed with a closed conformal Killing vector field, obtaining analogous results to the classical theorems of A. D. Alexandrov [2], [3] and J. J. Jellett-Liebmann [24], [27] concerning hypersurfaces in Euclidean space.

Later on, L. J. Alías, M. Dajczer and J. R. Ripoll in [4] extended the also classical Bernstein's theorem, see [9], to the context of complete minimal surfaces in Riemannian spaces of nonnegative Ricci curvature carrying a Killing vector field. This was done under the assumption that the sign of the angle function between

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a global Gauss mapping and the Killing vector field remains unchanged along the surface. In fact, their main result only requires the presence of a homothetic conformal Killing vector field. Next, M. Dajczer, P. Hinojosa and J. H. de Lira in [16] defined a notion of Killing graph in a class of Riemannian manifolds endowed with a Killing vector field and solved the corresponding Dirichlet problem for prescribed mean curvature under hypothesis involving domain data and the Ricci curvature of the ambient space.

In [10], A. Caminha established obstructions to the existence of closed conformal and nonparallel Killing vector fields on complete Riemannian manifolds with nonpositive Ricci curvature, generalizing a theorem due to T. K. Pan, see [31]. Moreover, he obtained general Bernstein-type theorems for certain complete hypersurfaces of Riemannian manifolds furnished with closed conformal Killing vector fields. Afterwards, M. Dajczer and J. H. de Lira in [15] extended the results of [16] by considering graphs which are constructed through the flow generated by a conformal Killing vector field V globally defined in a Riemannian manifold $\overline{M}^{n+1}$. According to the terminology established in [15], such graphs are called *conformal Killing graphs*. In this context, in [18], [17] the first and last of the authors together with H. F. de Lima studied the geometry of entire conformal Killing graphs; more specifically, under a suitable restriction on the norm of the gradient of the function z which determines such a graph $\Sigma(z)$, they have established sufficient conditions to ensure that $\Sigma(z)$ is totally umbilical and, in particular, an integral leaf of the distribution $V^{\perp}$ of all vector fields orthogonal to V. Afterwards, when the ambient space has constant sectional curvature, they get lower estimates for the index of minimum relative nullity of $\Sigma(z)$. More recently, J. A. Aledo and R. M. Rubio in [1] showed several stability results for minimal two-sided surfaces immersed in a wide class of 3-dimensional Riemannian warped products, which includes the class of Riemannian manifold equipped with a closed conformal Killing vector field, and, as a consequence, the authors were able to establish some Bernstein-type results.

On the other hand, in the branch of the geometric analysis many problems lead us to consider Riemannian manifolds endowed with a measure that has a smooth positive density with respect to the Riemannian one. The resulting spaces are the *weighted manifolds*, which are also called manifolds with density or smooth metric measure spaces in the current literature. More precisely, given a $(n+1)$-dimensional Riemannian manifold $\overline{M}^{n+1}$ with metric tensor g and a smooth function $f\colon \overline{M}^{n+1} \to \mathbb{R}$, the weighted manifold $\overline{M}_f^{n+1}$ is obtained by considering in $\overline{M}^{n+1}$ the weighted volume $\mathrm{d}\mu = \mathrm{e}^{-f}\,\mathrm{d}\overline{M}$, where $\mathrm{d}\overline{M}$ denotes the standard volume element of $\overline{M}^{n+1}$ induced by g. For simplicity, we denote such a weighted manifold by $\overline{M}_f^{n+1}$.

Appearing naturally in the study of self-shrinkers, Ricci solitons, harmonic heat flows and many others, weighted manifolds are important nontrivial generalizations of Riemannian manifolds and, nowadays, there are several geometric investigations concerning them. In particular, a theory of Ricci curvature for weighted manifolds goes back to A. Lichnerowicz, see [25], [26], and it was later developed by D. Bakry and M. Émery in their seminal work [6]. In this setting, as a crucial ingredient to understand the geometry of a weighted manifold $\overline{M}_f^{n+1}$, they introduced the so-called *Bakry–Émery–Ricci tensor*, which corresponds to an extension of the standard Ricci tensor (see definition (2.1)).

The study of variational questions associated to the area functional in weighted Riemannian manifolds has been a focus of attention in the last years. In this direction, C. Rosales, A. Cañete, V. Bayle and F. Morgan in [32] investigated the isoperimetric problem for Euclidean space endowed with a continuous density, showing that, for a radial log-convex density, balls about the origin are isoperimetric regions. Afterwards, A. Cañete and C. Rosales in [12] studied smooth Euclidean solid cones endowed with a smooth homogeneous weight function. In this context, they proved that the unique compact, orientable, second order minima of the weighted area under variations preserving the weighted volume and with free boundary in the boundary of the cone are intersections with the cone of round spheres centered at the vertex. In [23], D. Impera and M. Rimoldi established stability properties concerning *f-minimal* hypersurfaces (that is, with identically zero f-mean curvature) isometrically immersed in a weighted manifold with nonnegative Bakry–Émery–Ricci curvature under volume growth conditions. Meanwhile, K. Castro and C. Rosales in [13] obtained variational characterizations of critical points and second order minima of the weighted area with or without a volume constraint in weighted Riemannian manifolds with boundary.

In [22], D. Imprera, J. H. de Lira, S. Pigola and A. G. Setti aimed to obtain global height estimates for Killing graphs defined over a complete manifold with nonempty boundary. To this end, they first point out how the geometric analysis on a Killing graph is naturally related to a weighted manifold structure, where the weight is defined in terms of the length of the Killing vector field. According to this viewpoint, the authors introduce some potential theory on weighted manifolds with boundary and they proved a weighted volume estimate for intrinsic balls on the Killing graph. Finally, using these tools, they provided the desired estimate for the weighted height function in the assumption that the Killing graph has constant weighted mean curvature and the weighted geometry of the ambient space is suitably controlled.

Also in the branch of manifolds with density, M. Batista, M. P. Cavalcante and J. Pyo in [8] showed some general inequalities involving the weighted mean curvature of compact submanifolds immersed in $\overline{M}_f^{n+1}$. As application, they obtained an isoperimetric inequality for such submanifolds. Moreover, they also proved an extrinsic upper bound to the first nonzero eigenvalue of the f-Laplacian on closed submanifolds of $\overline{M}_f^{n+1}$. Concerning the weighted product space $\mathbb{G}^n \times \mathbb{R}$, where $\mathbb{G}^n$ stands for the so-called Gaussian space which is nothing but the Euclidian space $\mathbb{R}^n$ endowed with the Gaussian probability density $\mathrm{e}^{-f(x)} = (2\pi)^{-(n+1)/2}\mathrm{e}^{-|x|^2/2}$, $x \in \mathbb{R}^n$, D. T. Hieu and T. L. Nam in [21] extended the classical Bernstein's theorem showing that the only weighted minimal graphs $\Sigma(z)$ of smooth functions $z(x) = t$ over $\mathbb{G}^n$ are the affine hyperplanes $t =$ constant. Afterwards, M. McGonagle and J. Ross in [28] showed that the hyperplane is the only stable, smooth solution to the isoperimetric problem in the $\mathbb{G}^{n+1}$. Meanwhile, in the works [14], [19] it was applied suitable generalized maximum principles in order to obtain new Bernstein type results concerning complete hypersurfaces immersed in a class of weighted warped products.

Here, motivated by the works described above, our purpose is to investigate the geometry of conformal Killing graphs in a weighted Riemannian manifold $\overline{M}_f^{n+1}$ endowed with a complete conformal Killing vector field V, which are defined via the global flow associated to V over an integral leaf of the distribution $V^\perp$ (for more details see Section 2). Taking into account the Cheeger–Gromoll type splitting theorems due to G. Wei and W. Wylie, see [33], we assume that the weight function f does not depend on the parameter of the flow associated to unit vector field $\nu = -V/|V|$ (see Remark 3.6). In these circumstances, we calculate a formula for the f-Laplacian of the support function $g(N, V)$ (cf. Lemma 3.4), where N is the Gauss map of the conformal Killing graph $\Sigma(z)$. Afterwards, in Section 4, under a suitable restriction on the norm of the gradient of the function z which determines such a graph $\Sigma(z)$, we establish sufficient conditions to ensure that $\Sigma(z)$ is totally umbilical and, in particular, an integral leaf of $V^\perp$ (cf. Theorems 4.1, 4.2, 4.5 and 4.6 and Corollaries 4.3, 4.4, 4.7 and 4.8). Our approach is based on the use of the f-Laplacian of the supported function $g(N, V)$, the f-divergence of the tangent part of V on $\Sigma(z)$, jointly with a weighted version of Stoke's theorem to the context of complete weighted Riemannian manifolds (see Lemma 3.1).

Furthermore, in Section 5 we study the stability of f-minimal conformal Killing graphs of $\overline{M}_f^{n+1}$ according to the behavior of the derivative of the conformal factor ψ_V of V, obtaining sufficient conditions to guarantee that an f-minimal conformal Killing graphs be L_f-stable, where L_f stands for the weighted Jacobi operator (cf. Theorem 5.3 and Corollary 5.4). Finally, in Section 6 our goal is to investigate the strong f-stability of closed conformal Killing graphs in $\overline{M}_f^{n+1}$ with constant f-mean curvature. More specifically, we get sufficient conditions to a strong f-stable closed conformal Killing graphs be either f-minimal or isometric to a leaf of $V^\perp$ (cf. Theorem 6.2 and Corollary 6.3).

2. Preliminaries

Let $(\overline{M}^{n+1}, \overline{g}, \overline{\nabla}, \mathrm{d}\overline{\mu})$ be a *weighted oriented Riemannian manifold*, that is, an oriented Riemannian manifold $\overline{M}^{n+1}$ with metric tensor $\overline{g}$, Levi–Civita connection $\overline{\nabla}$, and endowed with a weighted volume form $\mathrm{d}\overline{\mu} = \mathrm{e}^{-f}\,\mathrm{d}\overline{M}$, where f is a real-valued smooth function on $\overline{M}^{n+1}$, which is called *weight function*, and $\mathrm{d}\overline{M}$ is the volume element induced by the metric $\overline{g}$. For simplicity of notation, we will denote $(\overline{M}^{n+1}, \overline{g}, \overline{\nabla}, \mathrm{d}\overline{\mu})$ by $\overline{M}_f^{n+1}$. We mean by $C^\infty(\overline{M})$ the ring of real functions of class C^∞ on $\overline{M}_f^{n+1}$ and by $\mathfrak{X}(\overline{M})$ the $C^\infty(\overline{M})$-module of vector fields of class C^∞ on $\overline{M}_f^{n+1}$.

In this setting, the *Bakry–Émery–Ricci tensor* $\overline{\mathrm{Ric}}_f$ of $\overline{M}_f^{n+1}$ is defined by

$$\overline{\mathrm{Ric}}_f = \overline{\mathrm{Ric}} + \overline{\mathrm{Hess}} f, \tag{2.1}$$

where $\overline{\mathrm{Ric}}$ and $\overline{\mathrm{Hess}}$ are the standard Ricci tensor and the Hessian in $\overline{M}_f^{n+1}$, respectively.

Along this work, we will consider hypersurfaces $x\colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$, namely, isometric immersions from a connected, n-dimensional oriented Riemannian manifold Σ^n into $\overline{M}_f^{n+1}$, and ∇ and $g = \overline{g}|_{\Sigma^n}$ will denote the Levi–Civita connection of Σ^n and the induced metric on Σ^n, respectively. Let N be the unit normal vector field, called the *Gauss map* of $x\colon \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$, globally defined on Σ^n.

In this setting, let A denote the *shape operator* of Σ^n with respect to N. So, at each $p \in \Sigma^n$, A restricts to a self-adjoint linear map $A_p\colon T_p\Sigma \to T_p\Sigma$ which is defined by $A_p(v) = -\overline{\nabla}_v N$ for all $v \in T_p\Sigma$. The *f-mean curvature* of Σ^n is the function H_f given by

$$nH_f = nH + \overline{g}(\overline{\nabla} f, N), \tag{2.2}$$

where $H = \frac{1}{n}\mathrm{tr}(A)$ denotes the classical mean curvature of Σ^n with respect to N. The *f-divergence* on Σ^n for any $X \in \mathfrak{X}(\Sigma)$ is defined by

$$\mathrm{div}_f X = \mathrm{div} X - g(\nabla f, X), \tag{2.3}$$

where $\operatorname{div}(X) = \operatorname{trace}\{Y \mapsto \nabla_Y X\}$ denotes the divergence relative to Σ^n. A direct calculation assures us that

$$\operatorname{div}_f(\varphi X) = \varphi \operatorname{div}_f X + g(\nabla\varphi, X) \tag{2.4}$$

for all $X \in \mathfrak{X}(\Sigma)$ and any $\varphi \in C^\infty(\Sigma)$. We define the *f-Laplacian* (or *drift Laplacian*) relative to Σ^n by

$$\Delta_f(\varphi) = \operatorname{div}_f(\nabla\varphi) = \Delta\varphi - g(\nabla f, \nabla\varphi) \tag{2.5}$$

for all $\varphi \in C^\infty(\Sigma)$, where Δ is the standard Laplacian relative to Σ^n. From (2.4) and (2.5) we can easily obtain the expression

$$\Delta_f(\varrho\,\varphi) = \varrho\Delta_f(\varphi) + \varphi\Delta_f(\varrho) + 2g(\nabla\varrho, \nabla\varphi), \tag{2.6}$$

which is valid for any pair of functions $\varrho, \varphi \in C^\infty(\Sigma)$.

In what follows, let us consider an $(n+1)$-dimensional weighted Riemannian manifold $\overline{M}_f^{n+1}$ endowed with a *conformal Killing vector field* V whose orthogonal distribution $\mathcal{D}$ is integrable. Thus, there exists a smooth function $\psi_V \in C^\infty(\overline{M})$ such that

$$\mathcal{L}_V\overline{g} = 2\psi_V\overline{g}, \tag{2.7}$$

where $\mathcal{L}$ stands for the Lie derivative of the metric $\overline{g}$ of $\overline{M}_f^{n+1}$; the function ψ_V is called the *conformal factor* of V.

In this setting, we denote by $\Phi\colon \mathbb{I} \times \mathbb{M}^n \to \overline{M}_f^{n+1}$ the flow generated by V, where $\mathbb{I} = (-\infty, a)$ is an interval with $a > 0$ and $\mathbb{M}^n$ is an arbitrarily fixed integral leaf of $\mathcal{D}$ labeled as $t = 0$ and which we will suppose to be connected and complete. It may happen that $a = \infty$, i.e., the vector field V is *complete*. Since $\Phi_t = \Phi(t, .)$ is a conformal map for any fixed $t \in \mathbb{R}$, there exists a positive function $\lambda \in C^\infty(\mathbb{I} \times \mathbb{M}^n)$ such that $\lambda(0, u) = 1$ and $\Phi_t^*\overline{g}(u) = \lambda^2(t, u)\overline{g}(u)$ for any $u \in \mathbb{M}^n$.

Throughout this paper, we restrict ourselves to the case where the function λ depends only on the variable t, that is, $\lambda \in C^\infty(\mathbb{I})$. Geometrically, as it was already observed in [15], this hypothesis allows us to relate the induced metrics in distinct leaves of the foliation orthogonal to V, which we will denote by $V^\perp$.

In the following, for $X, Y \in \mathfrak{X}(\overline{M})$ we write $\langle X, Y\rangle = \overline{g}(X, Y)$. From (2.7) we easily deduce the conformal Killing equation

$$\langle\overline{\nabla}_X V, Y\rangle + \langle X, \overline{\nabla}_Y V\rangle = 2\psi_V\langle X, Y\rangle$$

for any $X, Y \in \mathfrak{X}(\overline{M})$.

An interesting particular case of a conformal Killing vector field V is that in which

$$\overline{\nabla}_X V = \psi_V X \tag{2.8}$$

for all $X \in \mathfrak{X}(\overline{M})$; in this case we say that V is *closed*, an allusion to the fact that its dual 1-form is closed. Yet more particularly, a closed and conformal Killing vector field V is said to be *parallel* if its conformal factor ψ_V vanishes identically, and *homothetic* if ψ_V is constant.

Let $\mathbb{M}^n_t = \Phi_t(\mathbb{M}^n)$ be a leaf of $V^\perp$ furnished with the induced metric. From (2.8) we get

$$\overline{\nabla}\langle V, V\rangle = 2\psi_V V. \tag{2.9}$$

Consequently, $|V|^2$ is constant on the leaves of $V^\perp$. Moreover, computing covariant derivatives in (2.9), we have

$$(\overline{\operatorname{Hess}}\langle V, V\rangle)(X, Y) = 2X(\psi_V)\langle V, Y\rangle + 2\psi_V^2\langle X, Y\rangle.$$

Consequently, since both $\overline{\operatorname{Hess}}$ and the metric $\langle\,,\rangle$ are symmetric tensors, we get

$$X(\psi_V)\langle V, Y\rangle = Y(\psi_V)\langle V, X\rangle$$

for all $X, Y \in \mathfrak{X}(\overline{M})$. Now, taking $Y = V$ we arrive at

$$\overline{\nabla}\psi_V = \frac{V(\psi_V)}{|V|^2}V = \nu(\psi_V)\nu, \tag{2.10}$$

where $\nu = -V/|V|$ and, hence, ψ_V is also constant on the leaves of $V^\perp$.

Furthermore, with a straightforward computation, we verify that the shape operator A_t of a leaf $\mathbb{M}^n_t \in V^\perp$ with respect to ν is given by

$$A_t(X) = \overline{\nabla}_X\nu = \frac{\psi_V}{|V|}X$$

for any $X \in \mathfrak{X}(\mathbb{M}^n_t)$ and, hence, the leaves $\mathbb{M}^n_t$ are totally umbilical hypersurfaces with constant mean curvature $\mathcal{H} = \mathcal{H}(t)$ with respect to ν given by

$$\mathcal{H} = \frac{\psi_V}{|V|}. \tag{2.11}$$

Under the additional condition that the weight function f of $\overline{M}^{n+1}_f$ does not depend on the parameter of the flow associated to unit vector field ν, which means that $\langle\overline{\nabla} f, \nu\rangle = 0$ on $\overline{M}^{n+1}_f$, we obtain from (2.2) and (2.11) that the

f-mean curvature of a leaf $\mathbb{M}^n_t \in V^\top$ is given by

$$\mathcal{H}_f = \frac{\psi_V}{|V|}. \tag{2.12}$$

At the end of this first section, our purpose will be to give a description of our objects of study: conformal Killing graphs immersed in a weighted Riemannian manifold $\overline{M}^{n+1}_f$ endowed with closed conformal Killing vector field V. In this sense, following the ideas established in [15], given a domain Ω in $\mathbb{M}^n = \mathbb{M}^n_0$, we define the *conformal Killing graph* $\Sigma(z)$ of a smooth function z on $\overline{\Omega}$ as the hypersurface of $\overline{M}^{n+1}_f$ given by

$$\Sigma(z) = \{\Phi(z(u), u)\colon u \in \overline{\Omega}\},$$

where Φ is the flow generated by V. When $\Omega = \mathbb{M}^n$, $\Sigma(z)$ is said to be *entire.*

If we assign coordinates $x_0 = t, x_1, \dots, x_n$ to points in $\overline{M}^{n+1}_f$ of the form $\overline{u} = \Phi(t, u)$, where $x_1, \dots, x_n$ are local coordinates in $\mathbb{M}^n$, then the corresponding coordinate vector fields are

$$\partial_0|_{\overline{u}} = V(t) \qquad \text{and} \qquad \partial_i|_{\overline{u}} = \Phi_{t*}\partial_i|_u \qquad \text{for all } \ i \in \{1, \dots, n\}.$$

Thus, the conformal Killing graph $\Sigma(z)$ is parameterized in terms of local coordinates by $z(x_1, \dots, x_n), x_1, \dots, x_n$ and the tangent space to $\Sigma(z)$ is spanned by the vectors

$$\frac{\partial z}{\partial x_i}\partial_0|_{\Phi(z(u),u)} + \partial_i|_{\Phi(z(u),u)} \qquad \text{for all } \ i \in \{1, \dots, n\}. \tag{2.13}$$

Hence, from (2.13) we see that the metric induced on $\Sigma(z)$ is given by

$$\lambda^2(z(u))\Big(\frac{1}{\gamma}\,\mathrm{d}z^2 + \mathrm{d}\sigma^2\Big),$$

where $\gamma = 1/|V(0)|^2$ and $\mathrm{d}\sigma^2$ stands for the metric of the leaf $\mathbb{M}^n$.

Moreover, denoting by Dz the gradient of the function z with respect the metric $\mathrm{d}\sigma^2$, with a straightforward computation we verify that

$$N = \frac{1}{\lambda(z(u))\sqrt{\gamma + |Dz(u)|^2}}\big(\Phi_{z(u)*}Dz(u) - \gamma\partial_0|_{\Phi(z(u),u)}\big) \tag{2.14}$$

gives an orientation on $\Sigma(z)$ such that $\langle N, V\rangle < 0$.

3. Some auxiliary lemmas

This section is devoted to present the analytical machinery that will be used to establish our main results.

Let us denote by $\mathcal{L}^1_f(M)$ the set of integrable functions on the weighted Riemannian manifold M_f with respect to the weighted volume element $\mathrm{d}\mu = \mathrm{e}^{-f}\,\mathrm{d}M$, where $\mathrm{d}M$ stands for the volume element induced by the metric of M_f. Since from (2.3) we have that

$$\operatorname{div}_f X = \mathrm{e}^f \operatorname{div}(\mathrm{e}^{-f} X)$$

for all smooth vector field X on M_f, it is not difficult to see that from Proposition 2.1 of [10] we get the following extension of a result due to S. T. Yau in [34].

Lemma 3.1. *Let X be a smooth vector field on an oriented n-dimensional complete weighted Riemannian manifold M_f with weight function f such that $\operatorname{div}_f X$ does not change sign on M_f. If $|X| \in \mathcal{L}^1_f(M)$, then $\operatorname{div}_f X = 0$.*

The next lemma is due to G. Wei and W. Wylie, see [33], and it extends Theorem 7 of [34].

Lemma 3.2. *All complete noncompact Riemannian manifolds endowed with a bounded weighted function f and with nonnegative Bakry–Émery–Ricci tensor have at least linear f-volume growth.*

In the context of conformal Killing graphs immersed in a weighted Riemannian manifold, following the same ideas of Lemma 4.3 of [17], see also the proof of Theorem 4.2 of [18], we obtain the following

Lemma 3.3. *Let $\overline{M}^{n+1}_f$ be a weighted Riemannian manifold endowed with complete closed conformal Killing vector field V and let $\Sigma(z)$ be an entire conformal Killing graph in $\overline{M}^{n+1}_f$, defined on some leaf $\mathbb{M}^n$ of the foliation $V^\perp$. If $\Sigma(z)$ lies between two leaves of the foliation $V^\perp$ then $\Sigma(z)$ is complete. Moreover, if $|Dz| \in \mathcal{L}^1_f(\mathbb{M}^n)$, then the projection $V^\top$ of V onto $\Sigma(z)$ satisfies $|V^\top| \in \mathcal{L}^1_f(\Sigma(z))$.*

A particular class of Riemannian manifolds provided with a closed conformal Killing vector field is the so-called warped product of the type $\mathbb{I} \times_\phi \mathbb{F}^n$, where $\mathbb{I} \subset \mathbb{R}$ is an open interval with the metric $\mathrm{d}t^2$, $\mathbb{F}^n$ is an n-dimensional Riemannian manifold and $\phi\colon \mathbb{I} \to \mathbb{R}$ is positive and smooth. A warped product $\mathbb{I} \times_\phi \mathbb{F}^n$ endowed with a weight function f will be called a *weighted warped product* and it will be denoted by

$$(\mathbb{I} \times_\phi \mathbb{F}^n)_f. \tag{3.1}$$

For such a space, if π_I is the canonical projection onto $\mathbb{I}$, then the vector field $V = (\phi \circ \pi_I)\,\partial_t$ is conformal Killing and closed, with conformal factor $\psi_V = \phi' \circ \pi_I$, where the prime denotes differentiation with respect to $t \in \mathbb{I}$. Moreover, see [29], for $t_0 \in I$, the leaf $\mathbb{F}^n_{t_0} = \{t_0\} \times \mathbb{F}^n$ (also called *slice*) is totally umbilical, with constant mean curvature

$$\mathcal{H}(t_0) = \frac{\phi'(t_0)}{\phi(t_0)}$$

with respect to $-\partial_t$.

Conversely, let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold endowed with closed conformal Killing vector field V. If $p \in \overline{M}_f^{n+1}$ and $\mathbb{M}_p^n$ is the leaf of $V^\perp$ passing through p, then we can find a neighborhood $\mathcal{U}_p$ of p in $\mathbb{M}_p^n$ and an open interval $\mathbb{I} \subset \mathbb{R}$ containing 0 such that the flow Φ of V is defined on $\mathcal{U}_p$ for every $t \in \mathbb{I}$. Besides, if V is complete, following the ideas in Section 3 of [29], one can prove that

$$\begin{aligned} \mathbb{R} \times \mathbb{M}_p^n &\to \overline{M}_f^{n+1} \\ (t, u) &\mapsto \Phi(t, u) \end{aligned} \tag{3.2}$$

is a global parametrization on $\overline{M}_f^{n+1}$, so that $\overline{M}_f^{n+1}$ is isometric to the weighted warped product

$$\big(\mathbb{R} \times_\phi \mathbb{M}_p^n\big)_f, \tag{3.3}$$

where $\phi(t) = |V(\Phi(t, u))|$, $t \in \mathbb{R}$, and $u \in \mathbb{M}_p^n$ is an arbitrary point.

In what follows we assume that the weight function f of $\overline{M}_f^{n+1}$ does not depend on the parameter of the flow associated with the unit vector field $\nu = -V/|V|$, that is, $\langle \overline{\nabla} f, \nu \rangle = 0$. This condition has already been used in (2.12) for calculating the f-mean curvature of the leaves of $V^\perp$. In particular, when the ambient space is a warped product of the type $\mathbb{I} \times_\phi \mathbb{F}^n$, we will explicit this condition simply writing

$$\mathbb{I} \times_\phi \mathbb{F}_f^n, \tag{3.4}$$

and, in this case, from (2.12) we get that the f-mean curvature of the slice $\{t\} \times \mathbb{F}^n$ is given by

$$\mathcal{H}_f(t) = \frac{\phi'(t)}{\phi(t)} \tag{3.5}$$

with respect to the orientation given by $-\partial_t$.

In this scenario, we will consider the support function η_V on a conformal Killing graph $\Sigma(z)$ immersed in $\overline{M}_f^{n+1}$, which is defined by

$$\begin{aligned} \eta_V : \Sigma(z) &\to \mathbb{R} \\ p &\mapsto \eta_V(p) = \langle V(p), N(p) \rangle, \end{aligned} \tag{3.6}$$

where N is the Gauss map of $\Sigma(z)$ given in (2.14). We have that η_V is negative and

$$\nabla\eta_V = -A(V^\top), \tag{3.7}$$

where A is the shape operator of $\Sigma(z)$ with respect to N and $V^\top$ is the projection of vector field V on the tangent bundle of $\Sigma(z)$.

In our next lemma, we present a suitable formula for the drift Laplacian of η_V.

Lemma 3.4. *Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold endowed with closed conformal Killing vector field V having conformal factor ψ_V and such that the weight function f does not depend on the parameter of the flow associated to $\nu = -V/|V|$. If $\Sigma(z)$ is a conformal Killing graph in $\overline{M}_f^{n+1}$, with Gauss map N given in* (2.14)*, and η_V is the smooth function on $\Sigma(z)$ defined in* (3.6) *then*

$$\Delta_f(\eta_V) = -\{\overline{\mathrm{Ric}}_f(N,N) + |A|^2\}\eta_V - nV^\top(H_f) - n\{\psi_V H_f + N(\psi_V)\}, \tag{3.8}$$

where A and H_f are the shape operator and the f-mean curvature of $\Sigma(z)$ with respect to N, respectively, and $\overline{\mathrm{Ric}}_f$ denotes the Bakry–Émery–Ricci tensor of $\overline{M}_f^{n+1}$.

Proof: According to our previous digression, we have that (up to isometry) $\overline{M}_f^{n+1}$ can be regarded locally as a weighted warped product of the type (3.3). In this setting, we have that $V = \phi\,\partial_t$, $\psi_V = \phi'$, $\nu = -\partial_t$, $|V| = \phi$, and, consequently, $\langle\overline{\nabla} f, \partial_t\rangle = 0$.

Note that, from (2.2) we get

$$n\langle\partial_t, \nabla H\rangle = n\langle\partial_t^\top, \nabla H\rangle = n\langle\partial_t^\top, \nabla H_f\rangle - \partial_t^\top\langle\overline{\nabla} f, N\rangle, \tag{3.9}$$

where $\partial_t^\top = \partial_t - \langle N, \partial_t\rangle N$ is the projection of ∂_t on the tangent bundle of $\Sigma(z)$.

On the other hand,

$$\begin{aligned} \partial_t^\top\langle\overline{\nabla} f, N\rangle &= \langle\overline{\nabla}_{\partial_t^\top}\overline{\nabla} f, N\rangle + \langle\overline{\nabla} f, \overline{\nabla}_{\partial_t^\top} N\rangle \\ &= \langle\overline{\nabla}_{\partial_t - \langle N,\partial_t\rangle N}\overline{\nabla} f, N\rangle - \langle\overline{\nabla} f, A(\partial_t^\top)\rangle \\ &= \langle\overline{\nabla}_{\partial_t}\overline{\nabla} f, N\rangle - \langle N, \partial_t\rangle\overline{\mathrm{Hess}} f(N,N) - \langle\overline{\nabla} f, A(\partial_t^\top)\rangle. \end{aligned} \tag{3.10}$$

Now, taking into account that $\langle\overline{\nabla} f, \partial_t\rangle = 0$ and denoting by $\widetilde{\nabla}$ the Levi–Civita connection on $\mathbb{M}_p^n$, we have $\overline{\nabla} f = \phi^{-2}\widetilde{\nabla} f$. Then,

$$\langle\overline{\nabla}_{\partial_t}\overline{\nabla} f, N\rangle = \langle\overline{\nabla}_{\partial_t}(\phi^{-2}\widetilde{\nabla} f), N\rangle = \langle -2\phi^{-3}\phi'\widetilde{\nabla} f + \phi^{-2}\overline{\nabla}_{\partial_t}\widetilde{\nabla} f, N\rangle. \tag{3.11}$$

Hence, applying Proposition 7.35 of [30], from (3.11) we get

$$\begin{aligned}\langle\overline{\nabla}_{\partial_t}\overline{\nabla}f,N\rangle &= \langle -2\phi^{-3}\phi'\widetilde{\nabla}f+\phi^{-2}\phi^{-1}\phi'\widetilde{\nabla}f,N\rangle \\ &= -\phi'\phi^{-3}\langle\widetilde{\nabla}f,N\rangle = -\phi'\phi^{-1}\langle\overline{\nabla}f,N\rangle.\end{aligned}\tag{3.12}$$

Substituting (3.12) in equation (3.10) we get that

$$\partial_t^\top\langle\overline{\nabla}f,N\rangle = -\langle\overline{\nabla}f,N\rangle\phi^{-1}\phi' - \langle N,\partial_t\rangle\,\overline{\mathrm{Hess}}f(N,N) - \langle\overline{\nabla}f,A(\partial_t^\top)\rangle.\tag{3.13}$$

From equation (3.9) and (3.13) we conclude that

$$\begin{aligned}-n\phi\langle\partial_t,\nabla H\rangle =& -n\phi\langle\partial_t^\top,\nabla H_f\rangle - \phi'\langle\overline{\nabla}f,N\rangle \\ &- \phi\langle N,\partial_t\rangle\,\overline{\mathrm{Hess}}f(N,N) - \phi\langle\overline{\nabla}f,A(\partial_t^\top)\rangle.\end{aligned}\tag{3.14}$$

On the other hand, from Proposition 2.1 of [11] we have that

$$\begin{aligned}\Delta\langle N,\phi\,\partial_t\rangle =& -n\langle\phi\,\partial_t,\nabla H\rangle - n\{\phi' H + N(\phi')\} \\ &- \langle N,\phi\,\partial_t\rangle\{\overline{\mathrm{Ric}}(N,N)+|A|^2\}.\end{aligned}\tag{3.15}$$

So, substituting (3.14) in (3.15) and using (2.1) we obtain

$$\begin{aligned}\Delta\langle N,\phi\,\partial_t\rangle =& -n\langle\phi\,\partial_t,\nabla H_f\rangle - \langle N,\phi\,\partial_t\rangle\{\overline{\mathrm{Ric}}_f(N,N)+|A|^2\} \\ &- n\{\phi' H_f + N(\phi')\} - \langle\overline{\nabla}f,A(\phi\,\partial_t^\top)\rangle.\end{aligned}\tag{3.16}$$

Moreover, from (3.7) we verify that

$$\nabla\langle N,\phi\,\partial_t\rangle = -A(\phi\,\partial_t^\top).\tag{3.17}$$

We finish the proof using the equations (3.16) and (3.17) into (2.5). □

We conclude this section by providing an explicit expression for the f-divergence of the tangencial component $V^\top$ of V along a conformal Killing graph.

Lemma 3.5. *Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold endowed with closed conformal Killing vector field V having conformal factor ψ_V and such that the weight function f does not depend on the parameter of the flow associated to $\nu = -V/|V|$, and let $\Sigma(z)$ be a conformal Killing graph in $\overline{M}_f^{n+1}$. Then*

$$\mathrm{div}_f V^\top = n\psi_V + n\eta_V H_f,\tag{3.18}$$

where H_f is the f-mean curvature of $\Sigma(z)$ with respect to N and η_V is the smooth function on $\Sigma(z)$ defined in (3.6).

Proof: Since $\langle\overline{\nabla}f,V\rangle = 0$, then, writing $V = V^\top + \eta_V N$, we get

$$\langle\nabla f,V^\top\rangle = -\eta_V\langle\overline{\nabla}f,N\rangle.\tag{3.19}$$

On the other hand, from equation (8.4) of [5] we have

$$\mathrm{div} V^{\top} = n\psi_V + n\eta_V H, \tag{3.20}$$

where H is the standard mean curvature of $\Sigma(z)$. Hence, from (2.3), (3.20) and (3.19) we obtain (3.18). □

Remark 3.6. We observe that the following result is a consequence of a Cheeger–Gromoll type splitting theorem due to G. Wei and W. Wylie (cf. Theorem 6.1 of [33], see also Theorem 1.1 of [20]):

> "Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold that contains a line. If the Bakry–Émery–Ricci tensor of $\overline{M}_f^{n+1}$ is nonnegative and the weight function f is bounded then f must be constant along the line."

Consequently, in any weighted Riemannian manifold $\overline{M}_f^{n+1}$ endowed with complete closed conformal Killing vector field V, having nonnegative Bakry–Émery–Ricci tensor and with bounded weight function f, we have that f does not depend on the parameter of the flow associated with the unit vector field $\nu = -V/|V|$, that is, $\langle \overline{\nabla} f, \nu\rangle = 0$. In this case, we can see that the hypotheses adopted in Lemmas 3.4 and 3.5 on the weight function f are naturally verified.

4. Rigidity results for conformal killing graphs in $\overline{M}_f^{n+1}$

Taking into account (2.12), we establish our first rigidity result:

Theorem 4.1. *Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold endowed with complete closed conformal Killing vector field V and such that the weight function f does not depend on the parameter of the flow associated to $\nu = -V/|V|$, and let $\Sigma(z)$ be an entire conformal Killing graph in $\overline{M}_f^{n+1}$, defined on some leaf $\mathbb{M}^n$ of the foliation $V^{\perp}$, which lies between two leaves of $V^{\perp}$. Suppose that the f-mean curvature H_f (not necessarily constant) of $\Sigma(z)$ satisfies the following inequality*

$$0 < H_f \leq \mathcal{H}_f, \tag{4.1}$$

where $\mathcal{H}_f$ is the f-mean curvature of $\mathbb{M}^n$ given in (2.12). If $|Dz| \in \mathcal{L}^1_f(\mathbb{M}^n)$, then $\Sigma(z)$ is isometric to a leaf of $V^{\perp}$.

Proof: Let θ be the angle between ν and N. From (3.18) and (4.1), we get

$$\mathrm{div}_f V^{\top} = n|V|\{\mathcal{H}_f - H_f \cos\theta\} \geq n(1-\cos\theta)H_f|V| \geq 0. \tag{4.2}$$

On the other hand, from Lemma 3.3 we obtain that $\Sigma(z)$ is complete and $|V^{\top}| \in \mathcal{L}^1_f(\Sigma(z))$. Consequently, we can apply Lemma 3.1 to guarantee that $\mathrm{div}_f V^{\top}$ vanishes identically on $\Sigma(z)$. Therefore, returning to (4.2) we conclude that $\cos\theta = 1$ on $\Sigma(z)$, that is, the unit vector fields N and ν determine the same direction on $\Sigma(z)$ and, hence, $\Sigma(z)$ must be isometric to a leaf of the foliation $V^{\perp}$. □

A conformal Killing graph is said *f-minimal* when its f-mean curvature vanishes identically on it. Thus, from the analysis of the sign of $\mathrm{div}_f(V^{\top})$ in the proof of Theorem 4.1, we obtain the following

Theorem 4.2. *Let $\overline{M}^{n+1}_f$ be a weighted Riemannian manifold endowed with complete closed conformal Killing vector field V and such that the weight function f does not depend on the parameter of the flow associated to $\nu = -V/|V|$, and let $\Sigma(z)$ be an entire conformal Killing graph in $\overline{M}^{n+1}_f$, defined on some leaf $\mathbb{M}^n$ of the foliation $V^{\perp}$, which lies between two leaves of $V^{\perp}$. Suppose that the f-mean curvature H_f of $\Sigma(z)$ is constant and satisfies*

$$0 \leq H_f \leq \mathcal{H}_f,$$

where $\mathcal{H}_f$ is the f-mean curvature of $\mathbb{M}^n$ given in (2.12). If $|Dz| \in \mathcal{L}^1_f(\mathbb{M}^n)$, then $\Sigma(z)$ is either f-minimal or isometric to a leaf of $V^{\perp}$.

We recall that a *slab* of a warped product $\mathbb{I} \times_\phi \mathbb{F}^n$ is a region of the type

$$[t_1, t_2] \times_\phi \mathbb{F}^n = \{(t, q) \in \mathbb{I} \times_\phi \mathbb{F}^n :\ t_1 \leq t \leq t_2\}.$$

Then, in the case where the ambient space in Theorems 4.1 and 4.2 is a weighted warped product of the type (3.4), noting that $\mathcal{H}_f$ admits the expression (3.5), we get the following results.

Corollary 4.3. *Let $\Sigma(z)$ be an entire conformal Killing graph in a weighted warped product $\mathbb{R} \times_\phi \mathbb{F}^n_f$, defined on a slice $\mathbb{F}^n_{t_0} = \{t_0\} \times \mathbb{F}^n$, $t_0 \in \mathbb{R}$, which lies in a slab of $\mathbb{R} \times_\phi \mathbb{F}^n_f$. Suppose that the f-mean curvature H_f (not necessarily constant) of $\Sigma(z)$ satisfies the following inequality*

$$0 < \phi H_f \leq \phi'.$$

If $|Dz| \in \mathcal{L}^1_f(\mathbb{F}^n_{t_0})$, then $\Sigma(z)$ is isometric to slice $\{t\} \times \mathbb{F}^n$ for some $t \in \mathbb{R}$.

Corollary 4.4. *Let $\Sigma(z)$ be an entire conformal Killing graph in a weighted warped product $\mathbb{R} \times_\phi \mathbb{F}^n_f$, defined on a slice $\mathbb{F}^n_{t_0} = \{t_0\} \times \mathbb{F}^n$, $t_0 \in \mathbb{R}$, which lies in a slab of $\mathbb{R} \times_\phi \mathbb{F}^n_f$. Suppose that the f-mean curvature H_f of $\Sigma(z)$ is constant and satisfies*

$$0 \leq \phi H_f \leq \phi'.$$

If $|Dz| \in \mathcal{L}^1_f(\mathbb{F}^n_{t_0})$, then $\Sigma(z)$ is either f-minimal or isometric to slice $\{t\} \times \mathbb{F}^n$ for some $t \in \mathbb{R}$.

Continuing with our study, if the f-mean curvature of the conformal Killing graph and the conformal factor of the conformal Killing vector field have opposite signs, we have established the following result.

Theorem 4.5. *Let $\overline{M}^{n+1}_f$ be a weighted Riemannian manifold with nonnegative Bakry–Émery–Ricci tensor $\overline{\mathrm{Ric}}_f$, endowed with complete closed conformal Killing vector field V having conformal factor ψ_V and such that the weight function f is bounded. Let $\Sigma(z)$ be an entire conformal Killing graph in $\overline{M}^{n+1}_f$, defined on some leaf $\mathbb{M}^n$ of the foliation $V^\perp$, which lies between two leaves of $V^\perp$, and with Gauss map N given in* (2.14)*. Suppose that ψ_V and the f-mean curvature H_f of $\Sigma(z)$ verify one of the following conditions:*

(a) *$H_f \geq 0$ and $\psi_V \leq 0$ on $\Sigma(z)$;*
(b) *$H_f \leq 0$ and $\psi_V \geq 0$ on $\Sigma(z)$.*

If the norm of the second fundamental form $|A|$ of $\Sigma(z)$ is bounded and $|Dz| \in \mathcal{L}^1_f(\mathbb{M}^n)$, then $\Sigma(z)$ is totally geodesic and $\overline{\mathrm{Ric}}_f$ in the direction of N vanishes identically. In addition, if $\Sigma(z)$ is noncompact and the Bakry–Émery–Ricci tensor of $\Sigma(z)$ is also nonnegative, then $\Sigma(z)$ is isometric to a totally geodesic leaf of $V^\perp$.

Proof: First of all, we note that f does not depend on the parameter of the flow associated with ν, see Remark 3.6.

Since the support function η_V defined in (3.6) is negative, from either item (a) or (b) jointly with equation (3.18) we obtain that $\mathrm{div}_f(V^\top)$ does not change sign on $\Sigma(z)$. Since $\Sigma(z)$ lies between two leaves of the foliation $V^\perp$ and $|Dz| \in \mathcal{L}^1_f(\mathbb{M}^n)$, from Lemma 3.3 we obtain that $\Sigma(z)$ is complete and $|V^\top| \in \mathcal{L}^1_f(\Sigma(z))$. So, Lemma 3.1 gives $\mathrm{div}_f(V^\top) = 0$ on $\Sigma(z)$. Therefore, $\psi_V = 0$ and $H_f = 0$ on $\Sigma(z)$.

Now, considering (3.8), we obtain

$$\Delta_f(\eta_V) = -\{\overline{\mathrm{Ric}}_f(N,N) + |A|^2\}\eta_V \geq 0$$

on $\Sigma(z)$. Moreover, we note that the boundedness of $|A|$ on $\Sigma(z)$ gives

$$|\nabla \eta_V| \leq |A||V^\top| \in \mathcal{L}^1_f(\Sigma(z)).$$

Applying again Lemma 3.1, we get $\Delta_f(\eta_V) = 0$ on $\Sigma(z)$ and, consequently,

$$\overline{\mathrm{Ric}}_f(N,N) + |A|^2 = 0$$

on $\Sigma(z)$. Since $\overline{\mathrm{Ric}}_f(N,N) \geq 0$, we get $\overline{\mathrm{Ric}}_f(N,N) = 0$ and $A = 0$ on $\Sigma(z)$, that is, $\Sigma(z)$ is totally geodesic.

Proceeding, in view of (3.7), we obtain that $\nabla\eta_V = 0$ on $\Sigma(z)$ and, hence, $\eta_V = \langle V, N\rangle$ is constant and nonzero on $\Sigma(z)$. On the order hand, since V is parallel on $\Sigma(z)$, from (2.9) we have that $\langle V, V\rangle$ is constant on $\overline{M}_f^{n+1}$. Thus,

$$|V^\top|^2 = |V - \langle V, N\rangle N|^2 = \langle V, V\rangle - \langle V, N\rangle^2 \tag{4.3}$$

is also constant on $\Sigma(z)$. Therefore,

$$\infty > \int_{\Sigma(z)} |V^\top|\,\mathrm{d}\mu = |V^\top|\mathrm{vol}_f(\Sigma(z)), \tag{4.4}$$

where $\mathrm{vol}_f(\Sigma(z))$ is the weighted volume of $\Sigma(z)$. If, in addition, we assume $\Sigma(z)$ is noncompact and that the Bakry–Émery–Ricci tensor of $\Sigma(z)$ is also nonnegative, Lemma 3.2 gives $\mathrm{vol}_f(\Sigma(z)) = \infty$ and, consequently, the only possibility that we have for validity of (4.4) is that $|V^\top| = 0$ on $\Sigma(z)$. Thus, from (4.3) we get

$$|\langle V, N\rangle| = |V|.$$

Therefore, Cauchy–Schwarz inequality gives that V is parallel to N and, hence, $\Sigma(z)$ must be isometric to a totally geodesic leaf of $V^\perp$. □

When the f-mean curvature of a conformal Killing graph and the conformal factor of the conformal Killing vector field have the same sign, we have the following

Theorem 4.6. *Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold with nonnegative Bakry–Émery–Ricci tensor $\overline{\mathrm{Ric}}_f$, endowed with complete closed conformal Killing vector field V having conformal factor ψ_V and such that the weight function f is bounded. Let $\Sigma(z)$ be an entire conformal Killing graph in $\overline{M}_f^{n+1}$, defined on some leaf $\mathbb{M}^n$ of the foliation $V^\perp$, which lies between two leaves of $V^\perp$, with Gauss map N given in (2.14), and with norm of the second fundamental form $|A|$ and f-mean curvature H_f both bounded. Suppose that $|Dz| \in \mathcal{L}_f^1(\mathbb{M}^n)$, H_f has the same sign as ψ_V and*

$$\frac{1}{|V|}\frac{\partial\psi_V}{\partial t} \leq -n(H_f)^2, \tag{4.5}$$

where $t \in \mathbb{R}$ is the parameter of the flow associated with the unit vector field $\nu = -V/|V|$. Then $\Sigma(z)$ is totally geodesic and $\overline{\mathrm{Ric}}_f$ in the direction of N vanishes identically. In addition, if $\Sigma(z)$ is noncompact, $\langle V, V\rangle$ is constant on $\Sigma(z)$ and the Bakry–Émery–Ricci tensor of $\Sigma(z)$ is also nonnegative, then $\Sigma(z)$ is isometric to a totally geodesic leaf of $V^\perp$.

Proof: We have that f does not depend on the parameter of the flow associated with ν, see Remark 3.6. From (2.10) we observe that

$$N(\psi_V) = \langle N, \overline{\nabla}\psi_V\rangle = -\frac{\nu(\psi_V)}{|V|}\eta_V = -\frac{1}{|V|}\frac{\partial \psi_V}{\partial t}\,\eta_V, \tag{4.6}$$

where η_V is the negative support function defined in (3.6). Thus, in (3.8) we have

$$\Delta_f(\eta_V) = -n\langle \nabla H_f, V\rangle - \{\overline{\mathrm{Ric}}_f(N,N) + |A|^2\}\eta_V - n\psi_V H_f + \frac{n}{|V|}\frac{\partial \psi_V}{\partial t}\eta_V.$$

From hypothesis (4.5), we get

$$\Delta_f(\eta_V) \geq -n\langle \nabla H_f, V\rangle - \{\overline{\mathrm{Ric}}_f(N,N) + |A|^2\}\eta_V - n\psi_V H_f - n^2(H_f)^2\eta_V. \tag{4.7}$$

Now, let us consider on $\Sigma(x)$ the smooth vector field

$$X = \nabla \eta_V + nH_f V^\top.$$

Since $\Sigma(z)$ lies between two leaves of the foliation $V^\perp$ and $|Dz| \in \mathcal{L}^1_f(\mathbb{M}^n)$, from Lemma 3.3 we obtain that $\Sigma(z)$ is complete and $|V^\top| \in \mathcal{L}^1_f(\Sigma(z))$. Then, from (3.7) we obtain

$$|X| \leq \{|A| + n|H_f|\}|V^\top| \in \mathcal{L}^1_f(\Sigma(z)),$$

since H_f and $|A|$ are bounded on $\Sigma(z)$.

Moreover, from (2.3), (2.4), (3.18) and (4.7) we have

$$\begin{aligned}
\mathrm{div}_f X &= \Delta_f(\eta_V) + n\langle \nabla H_f, V\rangle + nH_f\,\mathrm{div}_f(V^\top)\\
&\geq -n\langle \nabla H_f, V\rangle - \{\overline{\mathrm{Ric}}_f(N,N) + |A|^2\}\eta_V\\
&\quad - n\psi_V H_f - n^2(H_f)^2\eta_V + n\langle \nabla H_f, V\rangle\\
&\quad + n^2\psi_V H_f + n^2(H_f)^2\eta_V\\
&\quad - \{\overline{\mathrm{Ric}}_f(N,N) + |A|^2\}\eta_V + n(n-1)\psi_V H_f \geq 0,
\end{aligned} \tag{4.8}$$

where in the last inequality we used that η_V is negative, $\overline{\mathrm{Ric}}_f$ is nonnegative and the assumption that H_f and ψ_V have the same sign on Σ^n. Thus, Lemma 3.1 gives $\mathrm{div}_f X = 0$ on $\Sigma(z)$. Therefore, by returning to (4.8) we obtain that $\overline{\mathrm{Ric}}_f(N,N) = 0$ and $\Sigma(z)$ is totally geodesic.

Finally, if $\Sigma(z)$ is noncompact, $\langle V, V\rangle$ is constant on $\Sigma(z)$ and the Bakry–Émery–Ricci tensor of $\Sigma(z)$ is also nonnegative, then (4.3) holds and we can reason as in the last part of the proof of Theorem 4.5 to conclude that $\Sigma(z)$ is isometric to a totally geodesic leaf of $V^\top$. □

If the ambient space $\overline{M}^{n+1}_f$ in Theorems 4.5 and 4.6 is a weighted warped product $\mathbb{R}\times_\phi \mathbb{F}^n_f$, described in (3.4), we observe that the hypotheses about the

Bakry–Émery–Ricci tensor of $\overline{M}_f^{n+1}$ and the weight function f can be disregarded, because in this case we already have that the weighted function f does not depend on the parameter of the flow associated with the unit vector field $-\partial_t$. Hence, when $\overline{M}_f^{n+1} = \mathbb{R} \times_\phi \mathbb{F}_f^n$ we have that Theorems 4.5 and 4.6 can be rescripted, respectively, in the following way.

Corollary 4.7. *Let $\mathbb{R} \times_\phi \mathbb{F}_f^n$ be a weighted warped product with bounded weight function f and let $\Sigma(z)$ be an entire conformal Killing graph in $\mathbb{R} \times_\phi \mathbb{F}_f^n$, defined on a slice $\mathbb{F}_{t_0}^n = \{t_0\} \times F^n$, $t_0 \in \mathbb{R}$, which lies in a slab of $\mathbb{R} \times_\phi \mathbb{F}_f^n$, and with Gauss map N given in (2.14). Suppose that the warped function ϕ and the f-mean curvature H_f of $\Sigma(z)$ verify one of the following conditions:*

(a) *$H_f \geq 0$ and $\phi' \leq 0$ on $\Sigma(z)$;*
(b) *$H_f \leq 0$ and $\phi' \geq 0$ on $\Sigma(z)$.*

If the norm of the second fundamental form $|A|$ of $\Sigma(z)$ is bounded and $|Dz| \in \mathcal{L}_f^1(\mathbb{F}_{t_0}^n)$, then $\Sigma(z)$ is totally geodesic and the Bakry–Émery–Ricci tensor of $\mathbb{R} \times_\phi \mathbb{F}_f^n$ in the direction of N vanishes identically. In addition, if $\Sigma(z)$ is noncompact and the Bakry–Émery–Ricci tensor of $\Sigma(z)$ is nonnegative, then $\Sigma(z)$ is isometric to a totally geodesic slice $\{t\} \times \mathbb{F}^n$ for same $t \in \mathbb{R}$.

Corollary 4.8. *Let $\mathbb{R} \times_\phi \mathbb{F}_f^n$ be a weighted warped product with bounded weight function f and let $\Sigma(z)$ be an entire conformal Killing graph in $\mathbb{R} \times_\phi \mathbb{F}_f^n$, defined on a slice $\mathbb{F}_{t_0}^n = \{t_0\} \times \mathbb{F}^n$, $t_0 \in \mathbb{R}$, which lies in a slab of $\mathbb{R} \times_\phi \mathbb{F}_f^n$, with Gauss map N given in (2.14), and with norm of the second fundamental form $|A|$ and f-mean curvature H_f both bounded. Suppose that $|Dz| \in \mathcal{L}_f^1(\mathbb{F}_{t_0}^n)$, H_f has the same sign as the derivative of the warped function ϕ and*

$$\phi'' \leq -n\phi\,(H_f)^2.$$

Then $\Sigma(z)$ is totally geodesic and the Bakry–Émery–Ricci tensor of $\mathbb{R} \times_\phi \mathbb{F}_f^n$ in the direction of N vanishes identically. In addition, if $\Sigma(z)$ is noncompact, $\langle V, V\rangle$ is constant on $\Sigma(z)$ and the Bakry–Émery–Ricci tensor of $\Sigma(z)$ is nonnegative, then $\Sigma(z)$ is isometric to a totally geodesic slice $\{t\} \times \mathbb{F}^n$, $t \in \mathbb{R}$.

5. Stability of f-minimal conformal killing graphs

Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold with weight function f and endowed with closed conformal Killing vector field V, and let $x\colon \Sigma(z) \hookrightarrow \overline{M}_f^{n+1}$ be a conformal Killing graph with Gauss map N defined in (2.14). In this setting, we denote by $\mathrm{d}\Sigma(z)$ the volume element with respect to the metric induced by $x\colon \Sigma(z) \hookrightarrow \overline{M}_f^{n+1}$ and we mean by $C_0^\infty(\Sigma(z))$ the set of all functions of class C^∞ on $\Sigma(z)$ supported compactly.

It is well known that, given a function $\varphi \in C_0^\infty(\Sigma(z))$ there exists a *normal variation with compact support a fixed boundary*

$$x_s \colon \Sigma(z) \to \overline{M}_f^{n+1} \qquad \text{for} \;\; s \in (-\varepsilon, \varepsilon), \tag{5.1}$$

of $x \colon \Sigma(z) \hookrightarrow \overline{M}_f^{n+1}$, that is,

(i) $x_s = \mathrm{Id}$ outside a compact subset of $\Sigma(z)$;

(ii) for $s \in (-\varepsilon, \varepsilon)$, the map $x_s \colon \Sigma(z) \to \overline{M}_f^{n+1}$ is an immersion such that $x_0(p) = x(p)$ for all $p \in \Sigma(z)$;

(iii) $x_s(p) = p$ for all $p \in \partial\Sigma(z)$.

Moreover, associated with $x_s \colon \Sigma(z) \to \overline{M}_f^{n+1}$ we have that *the variational normal field* is φN and the first variation of the *weighted area functional*

$$\begin{aligned} &\mathcal{A}_f \colon (-\varepsilon, \varepsilon) \to \mathbb{R} \\ &\qquad s \mapsto \mathcal{A}_f(s) = \mathrm{Area}_f\big(x_s(\Sigma(z))\big) = \int_{\Sigma(z)} \mathrm{d}\mu_s \ , \end{aligned} \tag{5.2}$$

where $\mathrm{d}\mu_s = \mathrm{e}^{-f}\, \mathrm{d}\Sigma(z)_s$ and $\mathrm{d}\Sigma(z)_s$ denotes the volume element of $\Sigma(z)$ with respect to the metric induced by $x_s \colon \Sigma(z) \to \overline{M}_f^{n+1}$, is given by (see, for instance, [13], Lemma 3.2)

$$\delta_\varphi(\mathcal{A}_f) = \frac{\mathrm{d}\mathcal{A}}{\mathrm{d}s}(0) = n \int_{\Sigma(z)} \varphi H_f \, \mathrm{d}\mu. \tag{5.3}$$

As a consequence, $x \colon \Sigma(z) \hookrightarrow \overline{M}_f^{n+1}$ is a f-minimal if and only if $\delta_\varphi(\mathcal{A}_f) = 0$ for every smooth function $\varphi \in C_0^\infty(\Sigma(z))$. In other words, f-minimal conformal killing graphs in $\overline{M}_f^{n+1}$ are characterized as critical points of $\mathcal{A}_f$.

The stability operator of this variational problem is given by the second variation formula for the f-area, which in our case is written as follows, see Proposition 3.5 of [13] for $H_f = 0$,

$$\delta^2_\varphi(\mathcal{A}_f) = \frac{\mathrm{d}^2\mathcal{A}}{\mathrm{d}s^2}(0) = -\int_{\Sigma(z)} \varphi L_f(\varphi) \, \mathrm{d}\mu \tag{5.4}$$

with

$$L_f = \Delta_f + |A|^2 + \overline{\mathrm{Ric}}_f(N, N),$$

where Δ_f is the drift Laplacian operator on $\Sigma(z)$, N is the Gauss map of $\Sigma(z)$, $|A|$ denotes the length of the shape operator A of $\Sigma(z)$ and $\overline{\mathrm{Ric}}_f$ is the Bakry–Émery–Ricci tensor of $\overline{M}_f^{n+1}$.

For f-minimal conformal Killing graphs in $\overline{M}_f^{n+1}$, the above discussion motivates the following notion of stability.

Definition 5.1. Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold with weight function f and endowed with closed conformal Killing vector field V, and let $x\colon \Sigma(z) \hookrightarrow \overline{M}_f^{n+1}$ be a f-minimal conformal Killing graph. We say that $x\colon \Sigma(z) \hookrightarrow \overline{M}_f^{n+1}$ is *L_f-stable* if $\delta^2_\varphi(\mathcal{A}_f) \geq 0$ for every $\varphi \in C_0^\infty(\Sigma(z))$.

In order to prove our main theorem in this section, we will need to use the following auxiliary result, which gives a sufficient condition for a f-minimal hypersurfaces be L_f-stable.

Lemma 5.2. *Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold with weight function f and endowed with closed conformal Killing vector field V, and let $x\colon \Sigma(z) \hookrightarrow \overline{M}_f^{n+1}$ be a f-minimal conformal Killing graph. If there exists a positive smooth function $u \in C^\infty(\Sigma(z))$ such that $L_f(u) \leq 0$, then $x\colon \Sigma(z) \hookrightarrow \overline{M}_f^{n+1}$ is L_f-stable.*

Proof: Let us assume that there exists such a function u and take $\varphi \in C_0^\infty(\Sigma(z))$. Then, we can choose $\varrho \in C_0^\infty(\Sigma(z))$ satisfying $\varphi = \varrho u$. Hence, from (2.6) and (5.4) we have

$$\begin{aligned}
\delta^2_\varphi(\mathcal{A}_f) &= -\int_{\Sigma(z)} \varphi L_f(\varphi)\,\mathrm{d}\mu = -\int_{\Sigma(z)} \varrho\, u L_f(\varrho\, u)\,\mathrm{d}\mu \\
&= -\int_{\Sigma(z)} (\varrho^2 u L_f(u) + \varrho\, u^2 \Delta_f(\varrho) + 2\varrho\, u \langle \nabla\varrho, \nabla u\rangle)\,\mathrm{d}\mu \\
&\geq -\int_{\Sigma(z)} (\varrho\, u^2 \Delta(\varrho) + 2\varrho\, u \langle \nabla\varrho, \nabla u\rangle - \varrho\, u^2 \langle \nabla\varrho, \nabla f\rangle)\,\mathrm{d}\mu \\
&= -\int_{\Sigma(z)} \Big(\varrho\, u^2 \Delta(\varrho) + \frac{1}{2}\langle \nabla\varrho^2, \nabla u^2\rangle - \varrho\, u^2 \langle \nabla\varrho, \nabla f\rangle\Big)\,\mathrm{d}\mu.
\end{aligned} \tag{5.5}$$

On the other hand, we can see that

$$\operatorname{div}(u^2 \nabla\varrho^2) = \langle \nabla u^2, \nabla\varrho^2\rangle + u^2\Delta(\varrho^2) = \langle \nabla u^2, \nabla\varrho^2\rangle + 2\varrho\, u^2 \Delta(\varrho) + 2u^2|\nabla\varrho|^2.$$

Therefore, from the weighted version of divergence theorem, see Lemma 2.2 of [12], we get from last equation together with (5.5) that

$$\begin{aligned}
\delta^2_\varphi(\mathcal{A}_f) &\geq -\int_{\Sigma(z)} \Big(\frac{1}{2}\operatorname{div}(u^2\nabla\varrho^2) - u^2|\nabla\varrho|^2 - \varrho\, u^2\langle \nabla\varrho, \nabla f\rangle\Big)\,\mathrm{d}\mu \\
&= -\int_{\Sigma(z)} \Big(\frac{1}{2}\operatorname{div}_f(u^2\nabla\varrho^2) - u^2|\nabla\varrho|^2\Big)\,\mathrm{d}\mu = \int_{\Sigma(z)} u^2|\nabla\varrho|^2\,\mathrm{d}\mu \geq 0
\end{aligned}$$

and, therefore, $x\colon \Sigma(z) \hookrightarrow \overline{M}_f^{n+1}$ is L_f-stable. □

Now, analyzing the behavior of the conformal factor ψ_V along a conformal Killing graph, we will state and prove our main result concerning L_f-stability. In what follows, $t \in \mathbb{R}$ denotes the parameter of the flow associated with the unit vector field $\nu = -V/|V|$.

Theorem 5.3. *Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold with nonnegative Bakry–Émery–Ricci tensor, endowed with complete closed conformal Killing vector field V having conformal factor ψ_V and whose weight function f is bounded, and let $x\colon \Sigma(z) \hookrightarrow \overline{M}_f^{n+1}$ be a f-minimal conformal Killing graph.*

(a) *If $\partial\psi_V/\partial t \leq 0$ on $\Sigma(z)$ then $x\colon \Sigma(z) \hookrightarrow \overline{M}_f^{n+1}$ is L_f-stable.*
(b) *If $\Sigma(z)$ is compact and $\partial\psi_V/\partial t \geq 0$ on $\Sigma(z)$ then $x\colon \Sigma(z) \hookrightarrow \overline{M}_f^{n+1}$ is L_f-stable if and only if ψ_V is constant on $\Sigma(z)$.*
(c) *If $\Sigma(z)$ is compact and $\partial\psi_V/\partial t > 0$ on $\Sigma(z)$ then $x\colon \Sigma(z) \hookrightarrow \overline{M}_f^{n+1}$ cannot be L_f-stable.*

Proof: We have that f does not depend on the parameter of the flow associated with ν, see Remark 3.6. On $\Sigma(z)$, we consider the smooth positive function $u = -\eta_V$, where η_V is defined in (3.6). Then, from (3.8) and (4.6) we obtain

$$L_f(u) = \frac{n}{|V|}\frac{\partial\psi_V}{\partial t}\, u, \tag{5.6}$$

and, with a direct application of Lemma 5.2, the result of item (a) is obtained immediately.

Now, let us consider (b). Note that in this case $C_0^\infty(\Sigma(z)) = C^\infty(\Sigma(z))$. So, if $x\colon \Sigma(z) \hookrightarrow \overline{M}_f^{n+1}$ is L_f-stable, from Definition 5.1 and equation (5.6) we get

$$0 \leq \delta_u^2(\mathcal{A}_f) = -\int_{\Sigma(z)} uL_f(u)\,\mathrm{d}\mu = -n\int_{\Sigma(z)} \frac{u^2}{|V|}\frac{\partial\psi_V}{\partial t}\,\mathrm{d}\mu \leq 0, \tag{5.7}$$

which guarantees us $\partial\psi_V/\partial t = 0$ on $\Sigma(z)$. The converse follows from item (a).

Finally, we prove (c). Assuming the opposite, if we would have $x\colon \Sigma(z) \hookrightarrow \overline{M}_f^{n+1}$ L_f-stable then, from the analysis of signals studied in (5.7), we obtain

$$0 \leq -n\int_{\Sigma(z)} \frac{u^2}{|V|}\frac{\partial\psi_V}{\partial t}\,\mathrm{d}\mu < 0,$$

which is absurd. □

When the ambient space is a weighted warped product of the type (3.4), we can apply Theorem 5.3 to obtain the following result.

Corollary 5.4. *Let $x\colon \Sigma(z) \hookrightarrow \mathbb{R}\times_\phi \mathbb{F}_f^n$ be a f-minimal conformal Killing graph.*

(a) *If the warping function ϕ satisfies $\phi'' \leq 0$ on $\Sigma(z)$ then $x\colon \Sigma(z) \hookrightarrow \mathbb{R}\times_\phi\mathbb{F}_f^n$ is L_f-stable.*

(b) *If $\Sigma(z)$ is compact and the warping function ϕ satisfies $\phi'' \geq 0$ on $\Sigma(z)$ then $x\colon \Sigma(z) \hookrightarrow \mathbb{R} \times_\phi \mathbb{F}^n_f$ is L_f-stable if and only if $\phi = at + b$ on $\Sigma(z)$ for some $a, b \in \mathbb{R}$.*

(c) *If $\Sigma(z)$ is compact and the warping function ϕ satisfies $\phi'' > 0$ on $\Sigma(z)$ then $x\colon \Sigma(z) \hookrightarrow \overline{M}^{n+1}_f$ cannot be L_f-stable.*

6. Stability of constant f-mean curvature conformal killing graphs

Let $\overline{M}^{n+1}_f$ be a weighted Riemannian manifold with weight function f and endowed with closed conformal Killing vector field V, and let $x\colon \Sigma(z) \hookrightarrow \overline{M}^{n+1}_f$ be a *closed* (that is, compact and without boundary) conformal Killing graph with Gauss map N defined in (2.14).

In what follows we consider the set

$$\mathcal{G} = \left\{ \varphi \in C^\infty(\Sigma(z))\colon \int_{\Sigma(z)} \varphi \,\mathrm{d}\mu = 0 \right\},$$

formed by all the smooth functions on $\Sigma(z)$ with weighted integral mean equal to zero, where $\mathrm{d}\mu = \mathrm{e}^{-f}\,\mathrm{d}\Sigma(z)$ and $\mathrm{d}\Sigma(z)$ is the volume element with respect to the metric induced by $x\colon \Sigma(z) \hookrightarrow \overline{M}^{n+1}_f$.

According to the ideas established in Lemmas 2.1 and 2.2 of [7], see also Lemma 3.2 of [13], every smooth function $\varphi \in \mathcal{G}$ induces a normal variation (namely, a smooth function of form (5.1) checking only item (ii)) of $x\colon \Sigma(z) \hookrightarrow \overline{M}^{n+1}_f$ with variational normal field φN and with first variation $\delta_\varphi(\mathcal{A}_f)$ of the weighted area functional $\mathcal{A}_f\colon (-\varepsilon, \varepsilon) \to \mathbb{R}$, defined in (5.2), given by the expression (5.3). As a consequence of (5.3), any closed conformal Killing graph $x\colon \Sigma(z) \hookrightarrow \overline{M}^{n+1}_f$ with constant f-mean curvature H_f is a critical point of $\mathcal{A}_f$ restricted to all functions φ belonging to $\mathcal{G}$. Geometrically, this condition means that the variations under consideration preserve a certain weighted volume function (for more details, see Section 3 of [13]). For these critical points, Proposition 3.5 of [13], see also Proposition 2.5 of [7], asserts that the stability of the corresponding variational problem is given by the second variation

$$\delta^2_\varphi(\mathcal{A}_f) = -\int_{\Sigma(z)} \{\Delta_f(\varphi) + (|A|^2 + \overline{\mathrm{Ric}}_f(N,N))(\varphi)\}\varphi \,\mathrm{d}\mu \tag{6.1}$$

where Δ_f is the drift Laplacian operator on $\Sigma(z)$, N is the Gauss map of $\Sigma(z)$, $|A|$ denotes the length of the shape operator A of $\Sigma(z)$ and $\overline{\mathrm{Ric}}_f$ is the Bakry–Émery–Ricci tensor of $\overline{M}^{n+1}_f$.

From (6.1), let us now note that $\delta^2_\varphi(\mathcal{A}_f)$ depends only on $\varphi \in C^\infty(\Sigma(z))$. The following notion of stability now makes sense.

Definition 6.1. Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold with weight function f and endowed with closed conformal Killing vector field V, and let $x\colon \Sigma(z) \hookrightarrow \overline{M}_f^{n+1}$ be a closed conformal Killing graph with constant f-mean curvature H_f. We say that $x\colon \Sigma(z) \hookrightarrow \overline{M}_f^{n+1}$ is strongly f-stable when $\delta_u^2(\mathcal{A}_f) \geq 0$ for every $\varphi \in C^\infty(\Sigma(z))$.

We are now in position to state and prove the following rigidity result for strongly f-stable conformal Killing graphs.

Theorem 6.2. *Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold with nonnegative Bakry–Émery–Ricci tensor, endowed with complete closed conformal Killing vector field V having conformal factor ψ_V and whose weight function f is bounded. Let $x\colon \Sigma(z) \hookrightarrow \overline{M}_f^{n+1}$ be a strongly f-stable closed conformal Killing graph. Suppose that*

$$\frac{\partial \psi_V}{\partial t} \geq \max\{\psi_V H_f, 0\}, \tag{6.2}$$

where $t \in \mathbb{R}$ is the parameter of the flow associated with the unit vector field $\nu = -V/|V|$. If the set where $\psi_V = 0$ has empty interior in $\Sigma(z)$, then $\Sigma(z)$ is either f-minimal or isometric to a leaf of the foliation $V^\perp$.

Proof: As seen in Remark 3.6, we have that f does not depend on $t \in \mathbb{R}$. Let us consider in $\overline{M}_f^{n+1}$ the global parametrization (3.2). Since $x\colon \Sigma(z) \hookrightarrow \overline{M}_f^{n+1}$ is strongly f-stable, it follows from Definition 6.1 and (6.1) that

$$-\int_{\Sigma(z)} \{\Delta_f(\varphi) + \{\overline{\mathrm{Ric}}_f(N,N) + |A|^2\}\varphi\}\varphi \, \mathrm{d}\mu \geq 0 \tag{6.3}$$

for all $\varphi \in C^\infty(\Sigma(z))$. In particular, since H_f is constant on $\Sigma(z)$, taking the negative function η_V defined in (3.6) we get from (3.8) that

$$\Delta_f(\eta_V) + \{\overline{\mathrm{Ric}}_f(N,N) + |A|^2\}\eta_V = -n\{\psi_V H_f + N(\psi_V)\}.$$

Thus, from (6.3) we have that

$$\int_{\Sigma(z)} \{\psi_V H_f + N(\psi_V)\}\eta_V \, \mathrm{d}\mu \geq 0. \tag{6.4}$$

On the other hand, it follows from (2.10) that

$$N(\psi_V) = \langle N, \overline{\nabla}\psi_V\rangle = \nu(\psi_V)\langle N, \nu\rangle = -\frac{\partial \psi_V}{\partial t}\cos\theta,$$

where θ is the angle between N and $-\nu$. Substituting the above into (6.4), we finally arrive at

$$\int_{\Sigma(z)} \Big(\psi_V H_f - \frac{\partial \psi_V}{\partial t}\cos\theta\Big)|V|\cos\theta \, \mathrm{d}\mu \geq 0.$$

Now, from (6.2) we obtain

$$\begin{aligned} 0 &\leq \int_{\Sigma(z)} \Big\{\psi_V H_f - \frac{\partial \psi_V}{\partial t}\cos\theta\Big\}|V|\cos\theta \, \mathrm{d}\mu \\ &\leq \int_{\Sigma(z)} (1-\cos\theta)\frac{\partial \psi_V}{\partial t}|V|\cos\theta \, \mathrm{d}\mu \leq 0. \end{aligned}$$

Hence,

$$(1-\cos\theta)\frac{\partial \psi_V}{\partial t} = 0 \qquad \text{and} \qquad \frac{\partial \psi_V}{\partial t} = -\psi_V H_f$$

on $\Sigma(z)$. But, since H_f is constant on $\Sigma(z)$, $\Sigma(z)$ is either f-minimal or $H_f \neq 0$ on $\Sigma(z)$. If this last case occurs, the condition on the zero set of ψ_V on $\Sigma(z)$ together with the above give $\partial\psi_V/\partial t \neq 0$ on a dense subset of $\Sigma(z)$ and, hence, $\cos\theta = 1$ on this set. By continuity, $\cos\theta = 1$ on $\Sigma(z)$. Therefore, in this case, $\Sigma(z)$ must be a leaf of the foliation $V^\perp$. □

We close our paper observing that, when the ambient space is a weighted warped product of the type (3.4), we can apply Theorem 6.2 to obtain the following result.

Corollary 6.3. *Let $x\colon \Sigma(z) \hookrightarrow \mathbb{R} \times_\phi \mathbb{F}^n_f$ be a strongly f-stable closed conformal Killing graph. Suppose that the warped function ϕ satisfies*

$$\phi'' \geq \max\{\phi' H_f, 0\}.$$

If the set where $\phi' = 0$ has empty interior in $\Sigma(z)$, then $\Sigma(z)$ is either f-minimal or isometric to the slice $\{t_0\} \times \mathbb{F}^n$ for some $t_0 \in \mathbb{R}$.

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