Chromatic number of the product of graphs, graph homomorphisms, antichains and cofinal subsets of posets without AC

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Abstract. In set theory without the axiom of choice (AC), we observe new relations of the following statements with weak choice principles.

∘ If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable.

∘ If in a partially ordered set, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$ for any regular $\aleph_\alpha$.

∘ Every partially ordered set without a maximal element has two disjoint cofinal sub sets – CS.

∘ Every partially ordered set has a cofinal well-founded subset – CWF.

∘ Dilworth's decomposition theorem for infinite partially ordered sets of finite width – DT.

We also study a graph homomorphism problem and a problem due to A. Hajnal without AC. Further, we study a few statements restricted to linearly-ordered structures without AC.

Keywords: chromatic number of product of graphs; ultrafilter lemma; permutation model; Dilworth's theorem; chain; antichain; Loeb's theorem; application of Loeb's theorem

Classification: 05C15, 03E25, 03E35

1. Introduction

Firstly, the first author observes the following in ZFA (Zermelo–Fraenkel set theory with atoms).

(1) In Problem 15, Chapter 11 of [13], applying Zorn's lemma, P. Komjáth and V. Totik proved the statement 'Every partially ordered set without a maximal element has two disjoint cofinal subsets' (CS). In Theorem 3.26 of [9], P. Howard, D. I. Saveliev and E. Tachtsis proved that CS does not imply 'there are no amorphous sets' in ZFA. We observe that CS $\not\rightarrow$ $\mathrm{AC}^{\omega}_{\mathrm{fin}}$ (the axiom of choice for countably infinite familes of nonempty finite sets), CS $\not\rightarrow$ AC^{-}_{n} (Every infinite family of n-element sets has a partial choice

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function)[1] for every $2 \leq n < \omega$ and CS $\not\to$ LOKW_4^- (every infinite linearly orderable family $\mathcal{A}$ of 4-element sets has a partial Kinna–Wagner selection function)[2] in ZFA.

(2) In Problem 14, Chapter 11 of [13], applying the well-ordering theorem, P. Komjáth and V. Totik proved the statement 'Every partially ordered set has a cofinal well-founded subset' (CWF). In Theorem 10 (ii) of [18], E. Tachtsis proved that CWF holds in the basic Fraenkel model. Moreover, in Lemma 5 of [18], E. Tachtsis proved that CWF is equivalent to AC in ZF. We observe that CWF $\not\to$ $\mathrm{AC}^{\omega}_{\mathrm{fin}}$, CWF $\not\to$ AC_n^- for every $2 \leq n < \omega$ and CWF $\not\to$ LOKW_4^- in ZFA.

(3) In Problem 7, Chapter 11 of [13], applying Zorn's lemma, P. Komjáth and V. Totik proved that if in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable. We observe that the statement *'If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable'* neither implies AC_n^- nor implies 'There are no amorphous sets' in ZFA for any $n \geq 2$. Moreover, we prove that *'For any regular $\aleph_\alpha$, if in a partially ordered set, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$'* neither implies AC_n^- nor implies 'There are no amorphous sets' in ZFA for any $n \geq 2$.

(4) R. P. Dilworth in [3] proved the following statement in ZFC: *'If $\mathbb{P}$ is an arbitrary partially ordered set (p.o. set), and k is a natural number such that $\mathbb{P}$ has no antichains of size $k+1$ while at least one k-element subset of $\mathbb{P}$ is an antichain, then $\mathbb{P}$ can be partitioned into k chains'*, we abbreviate by DT, see Problem 4, Chapter 11 of [13] also. E. Tachtsis in [19] investigated the possible placement of DT in the hierarchy of weak choice principles. He proved that DT does not imply $\mathrm{AC}^{\omega}_{\mathrm{fin}}$ as well as AC_2 (every family of pairs has a choice function). We observe that DT does not imply AC_n^- for any $2 \leq n < \omega$ in ZFA. In particular, we observe that DT holds in the permutation model of Theorem 8 of [5], due to L. Halbeisen and E. Tachtsis. We also observe that a weaker form of Łoś's lemma, Form 253 of [8], fails in the permutation model of Theorem 8 of [5].

(5) P. Komjáth sketched the following generalization of the n-coloring theorem (for every graph $G = (V, E)$ such that if every finite subgraph of G is n-colorable then G is n-colorable) applying the Boolean prime ideal theorem (BPI): *'For an infinite graph $G = (V_G, E_G)$ and a finite graph $H = (V_H, E_H)$, if every finite subgraph of G has a homomorphism*

[1] It is easy to see that AC_n^- follows from AC_n (axiom of choice for n-element sets).

[2] We denote the principle 'Every infinite linearly orderable family $\mathcal{A}$ of n-element sets has a partial Kinna–Wagner selection function' by LOKW_n^-, see [5].

into H, then so has G' we abbreviate by $\mathcal{P}_{G,H}$. We observe that if $X \in \{\mathrm{AC}_3, \mathrm{AC}^{\omega}_{\mathrm{fin}}\}$, then $\mathcal{P}_{G,H}$ restricted to finite graph H with 2 vertices does not imply X in ZFA.

Secondly, we study a weaker formulation of a problem due to A. Hajnal in ZFA.

(1) In Theorem 2 of [4], A. Hajnal proved that if the chromatic number of a graph G_1 is finite (say $k < \omega$), and the chromatic number of another graph G_2 is infinite, then the chromatic number of $G_1 \times G_2$ is k using the Gödel's compactness theorem. In the solution of Problem 12, Chapter 23 of [13], P. Komjáth provided another argument using the Ultrafilter lemma. For a natural number $k < \omega$, we denote by $\mathcal{P}_k$ the following statement.

$$\text{`}\chi(E_{G_1}) = k < \omega \text{ and } \chi(E_{G_2}) \geq \omega \text{ implies } \chi(E_{G_1 \times G_2}) = k.\text{'}$$

We observe that if $X \in \{\mathrm{AC}_3, \mathrm{AC}^{\omega}_{\mathrm{fin}}\}$, then $\mathcal{P}_k \not\rightarrow X$ in ZFA when $k = 3$.

Lastly, we study a few algebraic and graph-theoretic statements restricted to linearly-ordered structures without AC. We abbreviate the statement 'The union of a well-orderable family of finite sets is well-orderable' by UT(WO, fin,WO). In Theorem 3.1 (i) of [19], E. Tachtsis proved DT for well-ordered infinite p.o. sets with finite width in ZF applying the following theorem.

Theorem 1.1 (Theorem 1 of [14]). *Let $\{X_i\}_{i\in I}$ be a family of compact spaces which is indexed by a set I on which there is a well-ordering $\leq$. If I is an infinite set and there is a choice function F on the collection $\{C\colon C$ is closed, $C \neq \emptyset$, $C \subset X_i$ for some $i \in I\}$, then the product space $\prod_{i\in I} X_i$ is compact in the product topology.*

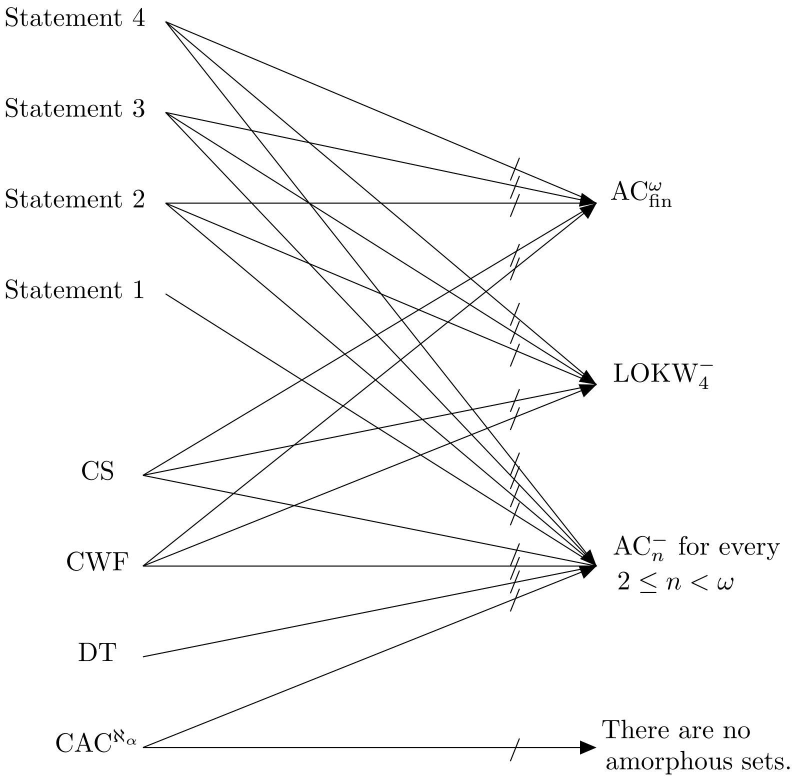
Using the same technique from Theorem 3.1 of [19], we prove a few algebraic and graph-theoretic statements restricted to well-ordered sets, either in ZF or in ZF + UT(WO, fin,WO). Consequently, those statements restricted to linearly ordered sets are true, in permutation models where LW (every linearly ordered set can be well-ordered) holds. In particular, we observe the following.

(1) In Theorem 18 of [10], P. E. Howard and E. Tachtsis obtained that for every finite field $\mathcal{F} = \langle F, \dots\rangle$ and for every nontrivial vector space V over $\mathcal{F}$, there exists a nonzero linear functional $f\colon V \to F$ applying BPI. Fix an arbitrary $2 \leq n < \omega$. We observe that '*For every finite field $\mathcal{F} = \langle F, \dots\rangle$ and for every nontrivial linearly-ordered vector space V over $\mathcal{F}$, there exists a nonzero linear functional $f\colon V \to F$*' $\not\rightarrow \mathrm{AC}^{\omega}_{\mathrm{fin}}$, $\not\rightarrow \mathrm{LOKW}^{-}_{4}$, and $\not\rightarrow \mathrm{AC}^{-}_{n}$ in ZFA.

(2) Fix an arbitrary $2 \leq n < \omega$. We observe that '*For an infinite graph $G = (V_G, E_G)$ on a linearly-ordered set of vertices V_G and a finite graph*

$H = (V_H, E_H)$, if every finite subgraph of G has a homomorphism into H, then so has G' $\not\rightarrow \mathrm{AC}^{\omega}_{\mathrm{fin}}$, $\not\rightarrow \mathrm{LOKW}^{-}_{4}$, and $\not\rightarrow \mathrm{AC}^{-}_{n}$ in ZFA.

(3) Fix an arbitrary $2 \leq n < \omega$. We prove that for every $3 \leq k < \omega$, the statement '$\mathcal{P}_k$ if the graph G_1 is on some linearly-orderable set of vertices' $\not\rightarrow \mathrm{AC}^{\omega}_{\mathrm{fin}}$, $\not\rightarrow \mathrm{LOKW}^{-}_{4}$, and $\not\rightarrow \mathrm{AC}^{-}_{n}$ in ZFA.



$\mathcal{P}_{G,H}$ restricted to finite graph H with 2 vertices, $\mathcal{P}_3 \not\longrightarrow \mathrm{AC}_3, \mathrm{AC}^{\omega}_{\mathrm{fin}}$

FIGURE 1.

(4) M. Hall in [6] proved that if S is a set and $\{S_i\}_{i\in I}$ is an indexed family of *finite* subsets of S, and if the following property

(P) for every finite $F \subseteq I$, there is an injective choice function for $\{S_i\}_{i\in F}$,

holds then there is an injective choice function for $\{S_i\}_{i\in I}$. We abbreviate the above assertion by MHT. We recall that BPI implies MHT and MHT implies the axiom of choice for finite sets ($\mathrm{AC}_{\mathrm{fin}}$) in ZF, c.f. [8]. Fix an arbitrary $2 \leq n < \omega$. We prove that MHT restricted to a linearly-ordered collection of finite subsets of a set does not imply AC_n^- in ZFA.

In Figure 1 we sketch the results of this note in ZFA. For each regular $\aleph_\alpha$, we denote by $\mathrm{CAC}^{\aleph_\alpha}$ the statement 'if in a partially ordered set, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$'. We use *Statement 4* to denote '$\mathcal{P}_k$ for the graph G_1 on some linearly-orderable set of vertices' for a natural number $k \geq 3$. We use *Statement 3* to denote 'For every finite field $\mathcal{F} = \langle F, \dots \rangle$ and for every nontrivial linearly-ordered vector space V over $\mathcal{F}$, there exists a nonzero linear functional $f \colon V \to F$'. We use *Statement 2* to denote 'For an infinite graph $G = (V_G, E_G)$ on a linearly-ordered set of vertices V_G and a finite graph $H = (V_H, E_H)$, if every finite subgraph of G has a homomorphism into H, then so has G'. We use *Statement 1* to denote Marshall Hall's theorem for linearly-ordered collection of finite subsets of a set.

2. A list of forms and definitions

(1) The *axiom of choice*, AC (Form 1 in [8]): Every family of nonempty sets has a choice function.

(2) The *axiom of choice for finite sets*, $\mathrm{AC}_{\mathrm{fin}}$ (Form 62 in [8]): Every family of nonempty finite sets has a choice function.

(3) AC_2 (Form 88 in [8]): Every family of pairs has a choice function.

(4) AC_n for each $n \in \omega$, $n \geq 2$ (Form 61 in [8]): Every family of n element sets has a choice function. We denote by AC_n^- the statement '*Every infinite family of n-element sets has a partial choice function*' (Form 342 (n) in [8], denoted by C_n^- in Definition 1 (2) of [5]). We denote by LOKW_n^- the statement '*Every infinite linearly orderable family $\mathcal{A}$ of n-element sets has a partial Kinna–Wagner selection function*' (c.f. Definition 1 (2) of [5]).

(5) $\mathrm{AC}_{\mathrm{fin}}^\omega$ (Form 10 in [8]): Every countably infinite family of nonempty finite sets has a choice function. We denote by $\mathrm{PAC}_{\mathrm{fin}}^\omega$ the statement *'Every countably infinite family of nonempty finite sets has a partial choice function'*.

(6) The *principle of dependent choice*, DC (Form 43 in [8]): If S is a relation on a nonempty set A and for all $x \in A$ there exists $y \in A(xSy)$, then there is a sequence $a_0, a_1, \dots$ of elements of A such that for all $n \in \omega$, $(a_n S a_{n+1})$.

(7) LW (Form 90 in [8]): Every linearly-ordered set can be well-ordered.

(8) UT(WO, WO, WO) (Form 231 in [8]): The union of a well-ordered collection of well-orderable sets is well-orderable.

(9) UT(WO, fin, WO) (Form 122A in [8]): The union of a well-orderable family of finite sets is well-orderable.

(10) For all α, UT$(\aleph_\alpha, \aleph_\alpha, \aleph_\alpha)$ (Form 23 in [8]): For every ordinal α, if A and every member of A has cardinality $\aleph_\alpha$, then $\left|\bigcup A\right| = \aleph_\alpha$.

(11) UT($\aleph_0$, fin, $\aleph_0$) (Form 10A in [8]): The union of a denumerable collection of finite sets is countable.

(12) The *Boolean prime ideal theorem*, BPI (Form 14 in [8]): Every Boolean algebra has a prime ideal. We recall the following equivalent formulations of BPI.
   - (Form 14AW in [8]): The Compactness theorem for propositional logic.
   - The *Ultrafilter lemma*, UL (Form 14A in [8]): Every proper filter over a set S in $\mathcal{P}(S)$ can be extended to an ultrafilter.
   - The *n-coloring theorem for* $n \geq 3$, (Form 14G(n) $n \in \omega$, $n \geq 3$ in [8]): For every graph $G = (V, E)$ if every finite subgraph of G is n-colorable then G is n-colorable. This is De Bruijn–Erdős theorem for $n \geq 3$ colorings.

(13) The *Principle of consistent choice*, PCC (Form 14AH in [8]): Let $\mathcal{A} = \{A_i\}_{i\in I}$ be a family of finite sets and $\mathcal{R}$ is a symmetric binary relation on $\bigcup_{i\in I} A_i$. Suppose that for every finite $W \subset I$, there is an $\mathcal{R}$-consistent choice function for $\{A_i\}_{i\in W}$, then there is an $\mathcal{R}$-consistent choice function for $\{A_i\}_{i\in I}$.

We note that Form 14AH in [8] is different than the above formulation. J. Łoś and C. Ryll–Nardzewski in [15] introduced both the formulations where it was noted that they are equivalent. Let $n \in \omega\backslash\{0, 1\}$. We recall the notation F_n introduced by R. H. Cowen in [2], which is PCC restricted to families $\mathcal{A} = \{A_i : i \in I\}$, where $|A_i| \leq n$ for all $i \in I$.

(14) *Marshall Hall's theorem*, MHT (Form 107 in [8]): If S is a set and $\{S_i\}_{i\in I}$ is an indexed family of *finite* subsets of S, and if the following property

(P) for every finite $F \subseteq I$, there is an injective choice function for $\{S_i\}_{i\in F}$,

holds then there is an injective choice function for $\{S_i\}_{i\in I}$.

Philip Hall's theorem states that the property (P) is equivalent to Hall's condition which states that 'for all $F \in [I]^{<\omega}$, $\left|\bigcup_{i\in F} S_i\right| \geq |F|$'. We recall that Philip Hall's theorem or finite Hall's theorem can be proved in ZF without using any choice principles.

(15) *A weaker form of Łoś's lemma*, LT (Form 253 in [8]): If $\mathcal{A} = \langle A, \mathcal{R}^{\mathcal{A}}\rangle$ is a nontrivial relational $\mathcal{L}$-structure over some language $\mathcal{L}$, and $\mathcal{U}$ be an ultrafilter on a nonempty set I, then the ultrapower $\mathcal{A}^I/\mathcal{U}$ and $\mathcal{A}$ are elementarily equivalent.

(16) MCC (c.f. Definition 5 and Definition 6 of [18]): Every topological space with the minimal cover property is compact.

(17) Bounded and unbounded amorphous sets: An infinite set X is called *amorphous* if X cannot be written as a disjoint union of two infinite subsets. There are two types of amorphous sets, namely bounded amorphous sets and unbounded amorphous sets. Let $\mathcal{U}$ be a finitary partition of an amorphous set X. Then all but finitely many elements of $\mathcal{U}$ have the same cardinality, say $n(\mathcal{U})$. Let $\Pi(X)$ be the set of all finitary partitions of X and $n(X) = \sup\{n(\mathcal{U})\colon \mathcal{U} \in \Pi(X)\}$. If $n(X)$ is finite, then X is called *bounded amorphous* and if $n(X)$ is infinite, then X is called *unbounded amorphous*. We recall Theorem 6 of [18] which states that MCC implies 'there are no bounded amorphous sets'.

(18) (Form 64 in [8]): There are no amorphous sets.

(19) *Martin's axiom* (c.f. [16]): If κ is a well-ordered cardinal, we denote by $MA(\kappa)$ the principle 'If $(P, <)$ is a nonempty, countable chain condition (c.c.c.) quasi order and $\mathcal{D}$ is a family of $\leq \kappa$ dense sets in P, then there is a filter $\mathcal{F}$ of P such that $\mathcal{F} \cap D \neq \emptyset$ for all $D \in \mathcal{D}$'. We recall from Remark 2.7 of [16] that $\mathrm{AC}^{\omega}_{\mathrm{fin}} + MA(\aleph_0)$ implies 'for every infinite set X, 2^X is Baire' and 'for every infinite set X, 2^X is Baire' implies 'there are no amorphous sets'.

(20) *Dilworth's decomposition theorem* for infinite p.o. sets of finite width, DT, c.f. [19]: If $\mathbb{P}$ is an arbitrary p.o. set, and k is a natural number such that $\mathbb{P}$ has no antichains of size $k + 1$ while at least one k-element subset of $\mathbb{P}$ is an antichain, then $\mathbb{P}$ can be partitioned into k chains. We abbreviate the above formulation as DT. We recall Theorem 3.1 (i) of [19], which states that DT for well-ordered infinite p.o. sets with finite width is provable in ZF.

(21) The *Chain/Antichain principle*, CAC (Form 217 in [8]): Every infinite p.o. set has an infinite chain or an infinite antichain. We recall that CAC implies $\mathrm{AC}^{\omega}_{\mathrm{fin}}$ from Lemma 4.4 of [20].

(22) CS, c.f. [9]: Every partially ordered set without a maximal element has two disjoint cofinal subsets.

(23) CWF, c.f. Definition 6 (11) of [18]: Every partially ordered set has a cofinal well-founded subset.

(24) *Chromatic number of the product of graphs*: We recall a few basic terminology of graphs. An *independent set* is a set of vertices in a graph, no two of which are connected by an edge. A *good coloring* of a graph $G = (V_G, E_G)$ with a color set C is a mapping $f\colon V_G \to C$ such that for every $\{x, y\} \in E_G$, $f(x) \neq f(y)$. The *chromatic number* $\chi(E_G)$ of a graph $G =$

(V_G, E_G) is the smallest cardinal κ such that the graph G can be colored by κ colors. We define the cartesian product of two graphs $G_1 = (V_{G_1}, E_{G_1})$ and $G_2 = (V_{G_2}, E_{G_2})$ as the graph $G_1 \times G_2 = (V_{G_1\times G_2}, E_{G_1\times G_2}) = (V_{G_1} \times V_{G_2}, \{\{(x_0, x_1), (y_0, y_1)\} \colon \{x_0, y_0\} \in E_{G_1},\ \{x_1, y_1\} \in E_{G_2}\})$ where $V_{G_1} \times V_{G_2}$ is the cartesian product of the vertex sets V_{G_1} and V_{G_2}. It can be seen that $\chi(E_{G_1\times G_2}) \le \min(\chi(E_{G_1}), \chi(E_{G_2}))$. In particular, if $\chi(E_{G_1}) = k < \omega$ then $\chi(E_{G_1\times G_2}) = k$, since if $f \colon V_{G_1} \to \{1, \dots, k\}$ is a good k-coloring of G_1, then $F(\langle x, y\rangle) = f(x)$ is a good k-coloring of $G_1 \times G_2$. In Theorem 2 of [4], A. Hajnal proved that if $\chi(E_{G_1})$ is finite (say $k < \omega$), and $\chi(E_{G_2})$ is infinite, then $\chi(E_{G_1\times G_2})$ is k. For a natural number $k < \omega$, we denote by $\mathcal{P}_k$ the following statement.

$$\text{‘}\chi(E_{G_1}) = k < \omega \ \textit{ and } \ \chi(E_{G_2}) \ge \omega \ \textit{ implies } \ \chi(E_{G_1\times G_2}) = k.\text{’}$$

2.1 Permutation models. Let M be a model of ZFA + AC where A is a set of atoms or ur-elements. Each permutation $\pi \colon A \to A$ extends uniquely to a permutation of $\pi' \colon M \to M$ by ε-induction. Let $\mathcal{G}$ be a group of permutations of A and $\mathcal{F}$ be a normal filter of subgroups of $\mathcal{G}$. For $x \in M$, we denote the symmetric group with respect to $\mathcal{G}$ by $\mathrm{sym}_{\mathcal{G}}(x) = \{g \in \mathcal{G} \colon g(x) = x\}$. We say x is *$\mathcal{F}$-symmetric* if $\mathrm{sym}_{\mathcal{G}}(x) \in \mathcal{F}$ and x is *hereditarily $\mathcal{F}$-symmetric* if x is $\mathcal{F}$-symmetric and each element of transitive closure of x is symmetric. We define the permutation model $\mathcal{N}$ with respect to $\mathcal{G}$ and $\mathcal{F}$ to be the class of all hereditarily $\mathcal{F}$-symmetric sets. It is well-known that $\mathcal{N}$ is a model of ZFA, see Theorem 4.1 of [11]. If $\mathcal{I} \subseteq \mathcal{P}(A)$ is a normal ideal, then the set $\{\mathrm{fix}_{\mathcal{G}}E \colon E \in \mathcal{I}\}$ generates a normal filter over $\mathcal{G}$. Let $\mathcal{I}$ be a normal ideal generating a normal filter $\mathcal{F}_{\mathcal{I}}$ over $\mathcal{G}$. Let $\mathcal{N}$ be the permutation model determined by M, $\mathcal{G}$, and $\mathcal{F}_{\mathcal{I}}$. We say $E \in \mathcal{I}$ supports a set $\sigma \in \mathcal{N}$ if $\mathrm{fix}_{\mathcal{G}}E \subseteq \mathrm{sym}_{\mathcal{G}}(\sigma)$.

3. Well-ordered structures in ZF

3.1 Applications of Loeb's theorem. We recall the following fact from [12].

Lemma 3.1 (ZF)**.** *If X is well-orderable, then 2^X is compact.*

Remark. We can also prove Lemma 3.1 applying Theorem 1 of [14].

Observation 3.2. UT(WO, fin, WO) *implies Marshall Hall's theorem for any well-ordered collection of finite subsets of a set.*

Proof: Let S be a set and $\{S_i\}_{i\in I}$ be a well-ordered indexed family of *finite* subsets of S such that the following property holds,

(P) for every finite $F \subseteq I$, there is an injective choice function for $\{S_i\}_{i\in F}$.

We work with the propositional language $\mathcal{L}$ with the following sentence symbols

$$A'_{i,j} \qquad \text{where } j \in S_i \text{ and } i \in I.$$

Let $\mathcal{F}$ be the set of all formulae of $\mathcal{L}$ and $\Sigma \subset \mathcal{F}$ be the collection of the following formulae

(1) $\neg(A'_{i,m} \wedge A'_{j,m})$ for $i \neq j$, $m \in S_i \cap S_j$,
(2) $\neg(A'_{i,j} \wedge A'_{i,l})$ for any $l \neq j \in S_i$ where $i \in I$,
(3) $A'_{i,y_1} \vee A'_{i,y_2} \vee \cdots \vee A'_{i,y_k}$ for each $i \in I$ where $S_i = \{y_1, \ldots, y_k\}$.

We enumerate $\mathrm{Var} = \{A'_{i,j} : i \in I, j \in S_i\}$ since each S_i is finite, I is well-orderable and UT(WO, fin, WO) *is assumed*. For every $W \in [I]^{<\omega}\backslash\{\emptyset\}$, we let Σ_W be the subset of $\mathcal{F}$, which is defined as Σ except that the subscripts in the formulae are from the set $W \cup \bigcup_{i\in W} S_i$. Endow the discrete 2-element space $\{0,1\}$ with the discrete topology and consider the product space 2^{Var} with the product topology. Let $F_W = \{f \in 2^{\mathrm{Var}} \colon \text{ for all } \varphi \in \Sigma_W(f'(\varphi) = 1)\}$ where for $f \in 2^{\mathrm{Var}}$, the element f' of $2^{\mathcal{F}}$ denotes the valuation mapping determined by f. By Philip Hall's theorem which is provable in ZF without using any choice principles, each F_W is nonempty and the family $\mathcal{X} = \{F_W : W \in [I]^{<\omega}\backslash\{\emptyset\}\}$ has the finite intersection property. Also for each $W \in [I]^{<\omega}\backslash\{\emptyset\}$, F_W is closed in the topological space 2^{Var}. By Lemma 3.1 since 2^{Var} is compact in ZF, $\bigcap \mathcal{X}$ is nonempty. Pick an $f \in \bigcap \mathcal{X}$ and let $f' \in 2^{\mathcal{F}}$ be the unique valuation mapping that extends f. Clearly, $f'(\varphi) = 1$ for all $\varphi \in \Sigma$. Consequently, we obtain an injective choice function for $\{S_i\}_{i\in I}$ by the following claim.

Claim 3.3. *If v is a truth assignment which satisfies Σ, then we can define a system of distinct representatives by*

$$y \in S_i \qquad \textit{if and only if } \; v(A'_{i,y}) = T.$$

PROOF: By (2) and (3) for each $i \in I$, each collection S_i gets assigned a unique representative. By (1), distinct sets S_i and S_j get assigned distinct representatives. □

The proof of Observation 3.2 is completed. □

B. Banaschewski in [1] proved the uniqueness of the algebraic closure of an arbitrary field applying BPI.[3]

Observation 3.4. UT(WO, fin, WO) *implies 'If a field $\mathcal{K}$ has an algebraic closure, and the ring of polynomials $\mathcal{K}[x]$ is well-orderable, then the algebraic closure is unique'.*

[3] C.f. the last paragraph of page 384 and page 385 of [1].

Proof: Let $\mathcal{K}$ be a field, and suppose $\mathcal{E}$ and $\mathcal{F}$ be two algebraic closures of $\mathcal{K}$. We prove that there is an isomorphism from $\mathcal{E}$ onto $\mathcal{F}$ which fix $\mathcal{K}$ pointwise. Let $\mathcal{E}_u$ and $\mathcal{F}_u$ be the splitting fields of $u \in \mathcal{K}[x]$ inside $\mathcal{E}$ and $\mathcal{F}$, respectively. Let $\mathcal{H}_u$ be the set of all isomorphisms from $\mathcal{E}_u$ onto $\mathcal{F}_u$ which fix $\mathcal{K}$. Clearly, $\mathcal{H}_u$ is a nonempty finite set. Also, we can see that $\bigcup_u \mathcal{E}_u = \mathcal{E}$ and $\bigcup_u \mathcal{F}_u = \mathcal{F}$. Let $\mathcal{H} = \prod_{u\in\mathcal{K}[x]} \mathcal{H}_u$, and if $v|w$ define $H_{v,w} = \{(h_u) \in \mathcal{H}\colon h_v = h_w \restriction E_v\}$. Clearly, $H_{v,w}$ has finite intersection property and they are closed in the product topology of $\mathcal{H}$, where each $\mathcal{H}_u$ is discrete. Since $\mathcal{K}[x]$ *is well-orderable* as assumed and for each $u \in \mathcal{K}[x]$, $\mathcal{H}_u$ is finite, we have that $\bigcup_{u\in\mathcal{K}[x]} \mathcal{H}_u$ is well-orderable by UT(WO, fin, WO). By Theorem 1 of [14], $\mathcal{H}$ is compact. Consequently, $\bigcap_{v|w} \mathcal{H}_{v,w} \neq \emptyset$ and each (h_u) in this intersection determines a unique embedding $h\colon \bigcup_u \mathcal{E}_u \to \bigcup_u \mathcal{F}_u$ which is onto and fixes $\mathcal{K}$. □

Observation 3.5. *The statement 'For an infinite graph $G = (V_G, E_G)$ on a well-ordered set of vertices V_G and a finite graph $H = (V_H, E_H)$, if every finite subgraph of G has a homomorphism into H, then so has G' is provable in* ZF.

Proof: Fix a finite graph $H = (V_H, E_H)$ and a graph $G = (V_G, E_G)$ on a well-ordered set of vertices V_G. We consider $V_H = \{v_1, \ldots, v_k\}$ for some $k < \omega$. We work with the propositional language $\mathcal{L}$ with the following sentence symbols

$$A'_{x_i,v_j} \qquad \text{where } v_j \in V_H \text{ and } x_i \in V_G.$$

Let $\mathcal{F}$ be the set of all formulae of $\mathcal{L}$ and $\Sigma \subset \mathcal{F}$ be the collection of the following formulae

(1) $A'_{x_i,v_m} \wedge A'_{x_j,v_l}$ if and only if $\{x_i, x_j\} \in E_G$ implies $\{v_m, v_l\} \in E_H$,
(2) $\neg(A'_{x_i,v_j} \wedge A'_{x_i,v_l})$ for any $v_l, v_j \in V_H$ such that $v_l \neq v_j$ and each $x_i \in V_G$,
(3) $A'_{x_i,v_1} \vee A'_{x_i,v_2} \vee \cdots \vee A'_{x_i,v_k}$ for each $x_i \in V_G$.

By our assumption V_G is well-orderable and V_H is finite. So V_H is well-orderable. Consequently, $V_G \times V_H$ is well-orderable in ZF. We enumerate Var $= \{A'_{x_i,v_j}\colon x_i \in V_G,\ v_j \in V_H\}$. By assumption, for every $s \in [V_G]^{<\omega}$ there is a homomorphism $f_s\colon G \restriction s \to H$ of $G \restriction s$ into H. Following the methods used in the proof of Observation 3.2, we may obtain an $f' \in 2^{\mathcal{F}}$ such that $f'(\varphi) = 1$ for all $\varphi \in \Sigma$. Consequently, we can obtain a homomorphism h from G to H. □

Observation 3.6. *The statement 'For every finite field $\mathcal{F} = \langle F, \ldots\rangle$ and for every nontrivial well-ordered vector space V over $\mathcal{F}$, there exists a nonzero linear functional $f\colon V \to F$' is provable in* ZF.

Proof: We follow the proof of Theorem 3.14 of [10] and modify it in the context of well-orderable vector space. Fix a finite field $\mathcal{F} = \langle F, \ldots\rangle$ where $F = \{v_1, \ldots, v_k\}$ and a nontrivial well-ordered vector space V over $\mathcal{F}$. We work with the propositional language $\mathcal{L}$ with the following sentence symbols

$$A'_{x_i,v_j} \qquad \text{where } v_j \in F \text{ and } x_i \in V.$$

Let $\mathcal{F}'$ be the set of all formulae of $\mathcal{L}$ and $\Sigma \subset \mathcal{F}'$ be the collection of the following formulae

(1) $A'_{a,1}$,
(2) $A'_{x_i,v_j} \to A'_{v_k x_i, v_k v_j}$ for $v_k, v_j \in F$ and $x_i \in V$,
(3) $A'_{x_i,v_j} \wedge A'_{x_{i'},v_{j'}} \to A'_{x_i+x_{i'},v_j+v_{j'}}$ for $x_i, x_{i'} \in V$ and $v_j, v_{j'} \in F$,
(4) $\neg(A'_{x_i,v_j} \wedge A'_{x_i,v_l})$ for any $v_l, v_j \in F$ such that $v_l \neq v_j$ and each $x_i \in V$,
(5) $A'_{x_i,v_1} \vee A'_{x_i,v_2} \vee \dots \vee A'_{x_i,v_k}$ for each $x_i \in V$.

By our assumption V is well-orderable and F is finite. So F is well-orderable. Consequently, $V \times F$ is well-orderable. We enumerate $\mathrm{Var} = \{A'_{x_i,v_j} : x_i \in V, v_j \in F\}$. Fix $V' \in [V]^{<\omega}$. Let W be the subspace of V generated by the finite set $V' \cup \{a\}$. We can see that W is finite since F is finite. Consequently, a linear functional $f\colon W \to F$ with $f(a) = 1$ can be constructed in ZF. Following the methods used in the proof of Observation 3.2, we can obtain a nonzero linear functional $f\colon V \to F$. □

Observation 3.7. *For every $3 \leq k < \omega$, the statement $\mathcal{P}_k$ for the graph G_1 on some well-orderable set of vertices is provable under* ZF.

Proof: Fix $3 \leq k < \omega$. Suppose $\chi(E_{G_1}) = k$, $\chi(E_{G_2}) \geq \omega$, and G_1 is a graph on some well-orderable set of vertices. First we observe that if $g\colon V_{G_1} \to \{1,\dots,k\}$ is a good k-coloring of G_1, then $G(\langle x,y\rangle) = g(x)$ is a good k-coloring of $G_1 \times G_2$. So, $\chi(E_{G_1\times G_2}) \leq k$. For the sake of contradiction assume that $F\colon V_{G_1} \times V_{G_2} \to \{1,\dots,k-1\}$ is a good coloring of $G_1 \times G_2$. For each color $c \in \{1,\dots,k-1\}$ and each vertex $x \in V_{G_1}$ we let $A_{x,c} = \{y \in V_{G_2} : F(x,y) = c\}$.

Claim 3.8 (ZF)**.** *For all finite $F \subset V_{G_1}$, there exists a mapping $i_F\colon F \to \{1,\dots,k-1\}$ such that for any $x, x' \in F$, $A_{x,i_F(x)} \cap A_{x',i_F(x')}$ is not independent.*

Proof: Since any superset of nonindependent set is nonindependent, it is enough to show that for all finite $F \subset V_{G_1}$, there exists an $i_F\colon F \to \{1,\dots,k-1\}$ such that $\bigcap_{x\in F} A_{x,i_F(x)}$ is not independent. For the sake of contradiction assume that there exists a finite $F \subset V_{G_1}$ such that for all $i_F\colon F \to \{1,\dots,k-1\}$, $\bigcap_{x\in F} A_{x,i_F(x)}$ is independent. Now, $V_{G_2} = \bigcup_{i_F\colon F\to\{1,\dots,k-1\}} \bigcap_{x\in F} A_{x,i_F(x)}$. Consequently V_{G_2} can be written as a finite union of independent sets which contradicts the fact that $\chi(E_{G_2})$ is infinite. Thus for all finite $F \subset V_{G_1}$, we can obtain a mapping $i_F\colon F \to \{1,\dots,k-1\}$ such that $\bigcap_{x\in F} A_{x,i_F(x)}$ is not independent. □

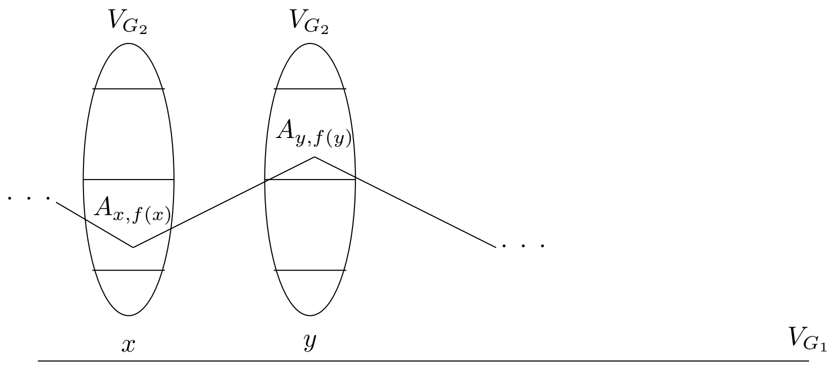


FIGURE 2. A map $f\colon V_{G_1} \to \{1,\dots,k-1\}$ such that intersection of any two elements in $\{A_{x,f(x)}\colon x \in V_{G_1}\}$ is not independent.

Endow $\{1,2,\dots,k-1\}$ with the discrete topology. Since V_{G_1} is well-orderable, $\{1,2,\dots,k-1\} \times V_{G_1}$ is well-orderable under ZF. Applying Theorem 1 of [14], $\{1,2,\dots,k-1\}^{V_{G_1}}$ is compact. For $s \in [V_{G_1}]^{<\omega}$, let $F_s = \{f \in \{1,2,\dots,k-1\}^{V_{G_1}}\colon x,y \in s,\ x \neq y \to A_{x,f(x)} \cap A_{y,f(y)}$ is not independent$\}$. By Claim 3.8 for each $s \in [V_{G_1}]^{<\omega}$ we have that F_s is nonempty. We can see that $\{F_s\colon s \in [V_{G_1}]^{<\omega}\}$ has finite intersection property as $F_{s_0\cup\dots\cup s_k} \subseteq F_{s_0} \cap \dots \cap F_{s_k}$. Thus by compactness of $\{1,2,\dots,k-1\}^{V_{G_1}}$, there is a $f \in \bigcap\{F_s\colon s \in [V_{G_1}]^{<\omega}\}$. Clearly, for any $x,x' \in V_{G_1}$, $A_{x,f(x)} \cap A_{x',f(x')}$ is not independent, see Figure 2. Since $x \to f(x)$ is not a good coloring in G_1 as $\chi(E_{G_1}) = k$, there are $x,x' \in V_{G_1}$ with $f(x) = f(x') = j$ and $\{x,x'\} \in E_{G_1}$. Consequently, $A' = A_{x,f(x)} \cap A_{x',f(x')}$ is not independent. Pick $y,y' \in A'$ joined by an edge in E_{G_2}. Then (x,y) and (x',y') are joined in $E_{G_1} \times E_{G_2}$ and get the same color j which is a contradiction to the fact that F is a good coloring of $G_1 \times G_2$. $\square$

3.2 On partially ordered sets based on a well-ordered set of elements. The first author modifies the arguments from Claim 5 of [17] and observes the following.

Observation 3.9. *The following holds.*

(1) $\mathrm{UT}(\aleph_0,\aleph_0,\aleph_0)$ *implies 'If in a partially ordered set based on a well-ordered set of elements, all chains are finite and all antichains are countable, then the set is countable'.*
(2) $\mathrm{UT}(\aleph_\alpha,\aleph_\alpha,\aleph_\alpha)$ *implies 'If in a partially ordered set based on a well-ordered set of elements, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$' for any regular $\aleph_\alpha$.*

Proof: We prove Observation 3.9 (1). For the sake of contradiction, assume that all antichains of $(P, \leq)$ are countable, all chains of $(P, \leq)$ are finite, but the set P is uncountable and well-ordered. We construct an infinite chain in $(P, \leq)$ using $\mathrm{UT}(\aleph_0, \aleph_0, \aleph_0)$ and obtain the desired contradiction.

Claim 3.10. *We have that $\leq$ is a well-founded relation on P, i.e., every nonempty subset of P has a $\leq$-minimal element.*

Proof: Let P be well-orderable, say by $\preceq$. We claim that $\leq$ is a well-founded relation on P. Otherwise, there is a nonempty subset $P_1 \subseteq P$ with no minimal elements. Consequently, using the fact that $\preceq$ is a well-ordering in P, we can obtain a strictly $\leq$-decreasing sequence of elements of P_1. This contradicts the assumption that P has no infinite chains. □

Without loss of generality we may assume $P = \bigcup\{P_\alpha \colon \alpha < \kappa\}$ where κ is a well-ordered cardinal, P_0 is the set of minimal elements of P and for each $\alpha < \kappa$, P_α is the set of minimal elements of $P \setminus \bigcup\{P_\beta \colon \beta < \alpha\}$. For each $\alpha < \kappa$, P_α is countable since P_α is an antichain.

- We note that $P = \bigcup\{P_p \colon p \in P_0\}$ where $P_p = \{q \in P \colon p \leq q\}$. Since P is uncountable and P_0 is countable, P_p is uncountable for some $p \in P_0$. Otherwise for all $p \in P_0$, P_p is either countable or finite and $\mathrm{UT}(\aleph_0, \aleph_0, \aleph_0) + \mathrm{UT}(\aleph_0, \mathrm{fin}, \aleph_0)$ implies P is countable which is a contradiction. Now $\mathrm{UT}(\aleph_0, \aleph_0, \aleph_0)$ implies $\mathrm{UT}(\aleph_0, \mathrm{fin}, \aleph_0)$ in ZF, thus $\mathrm{UT}(\aleph_0, \aleph_0, \aleph_0)$ suffices. Since $\{q \in P_0 \colon P_q \text{ is uncountable}\}$ is a nonempty subset of P, we can find a least $p_0 \in P_0$ with respect to $\preceq$ such that P_{p_0} is uncountable.
- Let us consider $P' = P_{p_0} \setminus \{p_0\}$. So, P' is uncountable. Again if P'_1 is the set of minimal elements of P', we can write $P' = \bigcup\{P_p \colon p \in P'_1\}$ where $P_p = \{q \in P \colon p \leq q\}$. Since P' is uncountable and P'_1 is countable (since all antichains of $(P, \leq)$ are countable by assumption), once again applying $\mathrm{UT}(\aleph_0, \aleph_0, \aleph_0)$ as in the previous paragraph, P_p is uncountable for some $p \in P'_1$. Since $\{q \in P'_1 \colon P_q \text{ is uncountable}\}$ is a nonempty subset of P, we can find a least $p_1 \in P'_1$ with respect to $\preceq$ such that P_{p_1} is uncountable. We can see that $p_0 < p_1$.

Continuing this process step by step we obtain a sequence $\langle p_n \colon n \in \omega\rangle$ of elements of P such that $p_n < p_{n+1}$ for each $n \in \omega$. Consequently, we obtain an infinite chain. □

Remark. Similarly for any regular $\aleph_\alpha$, assuming $\mathrm{UT}(\aleph_\alpha, \aleph_\alpha, \aleph_\alpha)$ we can prove the following since alephs are well-ordered. '*If in a partially ordered set based on well-ordered set of elements, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$.*' Consequently, we can prove Observation 3.9 (2).

4. Consistency results

Theorem 4.1. *For every natural number $n \geq 2$, there is a permutation model $\mathcal{N}$ of* ZFA *where* CAC *holds and* AC_n^- *fails. Moreover, we can observe the following in the model.*

(1) CS, *as well as* CWF, *holds.*
(2) DT *holds,* $MA(\aleph_0)$ *fails,* MCC *fails,* LT *fails.*
(3) *If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable. Moreover, if in a partially ordered set, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$ for any regular $\aleph_\alpha$.*
(4) *The following statements hold.*
   (a) *Marshall Hall's theorem for linearly-ordered collection of finite subsets of a set.*
   (b) *For every $3 \leq k < \omega$, $\mathcal{P}_k$ holds for any graph G_1 on some linearly-orderable set of vertices.*
   (c) *For an infinite graph $G = (V_G, E_G)$ on a linearly-ordered set of vertices V_G and a finite graph $H = (V_H, E_H)$, if every finite subgraph of G has a homomorphism into H, then so has G.*
   (d) *For every finite field $\mathcal{F} = \langle F, \dots \rangle$ and for every nontrivial linearly-orderable vector space V over $\mathcal{F}$, there exists a nonzero linear functional $f \colon V \to F$.*

Proof: In Theorem 8 of [5], L. Halbeisen and E. Tachtsis constructed a permutation model $\mathcal{N}$ where for arbitrary $n \geq 2$, AC_n^- fails but CAC holds. We fix an arbitrary integer $n \geq 2$ and recall the model constructed in the proof of Theorem 8 of [5] as follows.

- Defining the ground model M. We start with a ground model M of ZFA + AC where A is a countably infinite set of atoms written as a disjoint union $\bigcup\{A_i \colon i \in \omega\}$ where for each $i \in \omega$, $A_i = \{a_{i_1}, a_{i_2}, \dots, a_{i_n}\}$.
- Defining the group $\mathcal{G}$ of permutations and the filter $\mathcal{F}$ of subgroups of $\mathcal{G}$.
  - Defining $\mathcal{G}$. The group $\mathcal{G}$ is defined in [5] in a way so that *if $\eta \in \mathcal{G}$, then η only moves finitely many atoms* and for all $i \in \omega$, $\eta(A_i) = A_k$ for some $k \in \omega$. We recall the details from [5] as follows. For all $i \in \omega$, let τ_i be the n-cycle $a_{i_1} \mapsto a_{i_2} \mapsto \cdots \mapsto a_{i_n} \mapsto a_{i_1}$. For every permutation ψ of ω, which moves only finitely many natural numbers, let φ_ψ be the permutation of A defined by $\varphi_\psi(a_{i_j}) = a_{\psi(i)_j}$ for all $i \in \omega$ and $j = 1, 2, \dots, n$. Let $\eta \in \mathcal{G}$ if and only if $\eta = \varrho\varphi_\psi$ where ψ is a permutation of ω which moves only finitely many natural numbers and ϱ is a permutation of A for which there

is a finite $F \subseteq \omega$ such that for every $k \in F$, $\varrho \restriction A_k = \tau_k^j$ for some $j < n$, and ϱ fixes A_m pointwise for every $m \in \omega \backslash F$.

– Defining $\mathcal{F}$. Let $\mathcal{F}$ be the filter of subgroups of $\mathcal{G}$ generated by $\{\mathrm{fix}_{\mathcal{G}}(E)\colon E \in [A]^{<\omega}\}$.

∘ Defining the permutation model. Consider the permutation model $\mathcal{N}$ determined by M, $\mathcal{G}$ and $\mathcal{F}$.

Following point 1 in the proof of Theorem 8 of [5], both A and $\mathcal{A} = \{A_i\}_{i\in\omega}$ are amorphous in $\mathcal{N}$ and no infinite subfamily $\mathcal{B}$ of $\mathcal{A}$ has a Kinna–Wagner selection function. Consequently, AC_n^- fails. The first author observes the following.

Lemma 4.2. *In $\mathcal{N}$,* DT, CS *as well as* CWF *holds. Moreover the following holds in $\mathcal{N}$.*

- ∘ *If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable.*
- ∘ *'If in a partially ordered set, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$' for any regular $\aleph_\alpha$.*

Proof: We follow the steps below.

(1) Let $(P, \leq)$ be a p.o. set in $\mathcal{N}$ and $E \in [A]^{<\omega}$ be a support of $(P, \leq)$. We can write P as a disjoint union of $\mathrm{fix}_{\mathcal{G}}(E)$-orbits, i.e., $P = \bigcup\{\mathrm{Orb}_E(p)\colon p \in P\}$, where $\mathrm{Orb}_E(p) = \{\varphi(p)\colon \varphi \in \mathrm{fix}_{\mathcal{G}}(E)\}$ for all $p \in P$. The family $\{\mathrm{Orb}_E(p)\colon p \in P\}$ is well-orderable in $\mathcal{N}$ since $\mathrm{fix}_{\mathcal{G}}(E) \subseteq \mathrm{Sym}_{\mathcal{G}}(\mathrm{Orb}_E(p))$ for all $p \in P$ (cf. Claim 3.6 of [19]).

(2) Since *if $\eta \in \mathcal{G}$, then η only moves finitely many atoms*, $\mathrm{Orb}_E(p)$ is an antichain in P for each $p \in P$. Otherwise there is a $p \in P$, such that $\mathrm{Orb}_E(p)$ is not an antichain in $(P, \leq)$. Thus, for some $\varphi, \psi \in \mathrm{fix}_{\mathcal{G}}(E)$, $\varphi(p)$ and $\psi(p)$ are comparable. Without loss of generality we may assume $\varphi(p) < \psi(p)$. Since *if $\eta \in \mathcal{G}$, then η only moves finitely many atoms*, there exists some $k < \omega$ such that $\varphi^k = 1_A$. Let $\pi = \psi^{-1}\varphi$. Consequently, $\pi(p) < p$ and $\pi^k = 1_A$ for some $k \in \omega$. Thus, $p = \pi^k(p) < \pi^{k-1}(p) < \cdots < \pi(p) < p$. By transitivity of $<$, $p < p$, which is a contradiction (cf. Claim 3.5 of [19]).

(3) We prove that in $\mathcal{N}$, DT holds. Let $E \subset A$ be a finite support of an infinite p.o. set $\mathbb{P} = (P, <)$ with finite width. Then $P = \bigcup\{\mathrm{Orb}_E(p)\colon p \in P\}$. Following (2), $\mathrm{Orb}_E(p)$ is an antichain in $\mathbb{P}$. Consequently, $\mathrm{Orb}_E(p)$ is finite for each $p \in P$ since the width of $\mathbb{P}$ is finite. Following (1), $\{\mathrm{Orb}_E(p)\colon p \in P\}$ is well-orderable in $\mathcal{N}$. Following point 4 in the proof of Theorem 8 of [5] and Lemma 3 of [17], UT(WO,WO,WO) holds in $\mathcal{N}$, and so P is well-orderable in $\mathcal{N}$. Applying Theorem 3.1 (i) of [19], DT holds in $\mathcal{N}$.

(4) To see that CS as well as CWF holds in $\mathcal{N}$ we follow Theorem 3.26 of [9] and Theorem 10 (ii) of [18], respectively. We sketch the important steps below.

(a) We follow Theorem 3.26 of [9] to see that CS holds in $\mathcal{N}$ as follows. Let $(P, \leq)$ be a poset without maximal elements supported by E. Following (1), $\mathcal{O} = \{\mathrm{Orb}_E(p) \colon p \in P\}$ is a well-ordered partition of P. Define $\preceq$ on $\mathcal{O}$, as

$$X \preceq Y \leftrightarrow \exists x \in X, \exists y \in Y \text{ such that } x \leq y.$$

Since $(P, \leq)$ has no maximal element, $(\mathcal{O}, \preceq)$ has no maximal element following (2). Since $\mathcal{O}$ is well-ordered there exists a partition $\mathcal{U}_{\mathcal{O}} = \{\mathcal{Q}, \mathcal{R}\}$ of $\mathcal{O}$ in 2 cofinal subsets. Consequently, $\mathcal{U}_P = \left\{\bigcup \mathcal{Q}, \bigcup \mathcal{R}\right\}$ is a partition of P in 2 cofinal subsets.

(b) We follow Theorem 10 (ii) of [18] to see that CWF holds in $\mathcal{N}$ as follows. Let $(P, \leq)$ be a poset supported by $\mathcal{N}$. Since $\mathcal{O} = \{\mathrm{Orb}_E(p) \colon p \in P\}$ is well-orderable, it has a cofinal well-founded subset $\mathcal{W} = \{W_\alpha \colon \alpha < \gamma\}$ such that for $\beta < \alpha$, $W_\alpha \not\preceq W_\beta$ for all $\beta, \alpha < \gamma$. Consequently, $C = \bigcup \mathcal{W}$ is a cofinal well-founded subset of P.

(5) We show the following in $\mathcal{N}$.

'If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable.'

It is known that in every FM-model UT(WO,WO,WO) implies for all α UT$(\aleph_\alpha, \aleph_\alpha, \aleph_\alpha)$, c.f. page 176 of [8]. Consequently, UT$(\aleph_0, \aleph_0, \aleph_0)$ holds in $\mathcal{N}$. Let $(P, <)$ be an uncountable p.o. set in $\mathcal{N}$ where all antichains are countable and $E \in [A]^{<\omega}$ be a support of $(P, <)$. Following (1), $\mathcal{O} = \{\mathrm{Orb}_E(p) \colon p \in P\}$ is a well-ordered partition of P since for all $p \in P$, E is a support of $\mathrm{Orb}_E(p)$. Following (2), $\mathrm{Orb}_E(p)$ is an antichain and hence countable. Consequently, $\mathrm{Orb}_E(p)$ is well-orderable. Since UT(WO,WO,WO) holds in $\mathcal{N}$, P is well-orderable. By Observation 3.9 (1), since UT$(\aleph_0, \aleph_0, \aleph_0)$ holds in $\mathcal{N}$, there is an infinite chain in $\mathcal{N}$.

(6) Following (5) and Observation 3.9 (2), we can prove the following in $\mathcal{N}$.

'If in a partially ordered set $(P, <)$, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$'.

□

Remark. The referee pointed out that the statements 'If in a partially ordered set based on a well-ordered set of elements all chains are finite and all antichains are countable then the set is countable' and 'If in a partially ordered set based on a well-ordered set of elements all chains are finite and all antichains have size $\aleph_\alpha$ then the set has size $\aleph_\alpha$' are true in all Fraenkel–Mostowski permutation models. So Observation 3.9 (1) and Observation 3.9 (2) are not needed in the proofs of parts (5) and (6) of the proof of Lemma 4.2.

Lemma 4.3. *In $\mathcal{N}$, $MA(\aleph_0)$ fails.*

Proof: Since A is amorphous, the statement 'for all infinite X, 2^X is Baire' is false following Remark 2.7 of [16]. Since CAC holds in $\mathcal{N}$, $\mathrm{AC}^{\omega}_{\mathrm{fin}}$ holds as well, c.f. Lemma 4.4 of [20]. Consequently, $MA(\aleph_0)$ fails following Remark 2.7 of [16]. $\square$

Lemma 4.4. *In $\mathcal{N}$,* MCC *fails.*

Proof: Modifying the proof of Theorem 8 (ii) of [18], we can see that $n(A) = n$. Thus there is a bounded amorphous set A. Consequently, MCC fails by Theorem 6 of [18]. $\square$

Lemma 4.5. *In $\mathcal{N}$, LT fails.*

Proof: Since $\mathcal{A}$ is an amorphous set of nonempty sets which has no choice function in $\mathcal{N}$, following Lemma 4.1 (i) of [20], LT fails in $\mathcal{N}$. $\square$

Lemma 4.6. *In $\mathcal{N}$, the following statements hold for linearly-ordered structures.*

(1) *Marshall Hall's theorem for linearly-ordered collection of finite subsets of a set.*
(2) *For every $3 \leq k < \omega$, $\mathcal{P}_k$ holds for any graph G_1 on some linearly-orderable set of vertices.*
(3) *For an infinite graph $G = (V_G, E_G)$ on a linearly-ordered set of vertices V_G and a finite graph $H = (V_H, E_H)$, if every finite subgraph of G has a homomorphism into H, then so has G.*
(4) *For every finite field $\mathcal{F} = \langle F, \dots \rangle$ and for every nontrivial linearly-orderable vector space V over $\mathcal{F}$, there exists a nonzero linear functional $f : V \to F$.*

Proof: Since UT(WO,WO,WO) and LW holds in $\mathcal{N}$, c.f. pt 4 and pt 3 in the proof of Theorem 8 in [5], (1), (2), (3) and (4) hold in $\mathcal{N}$ following the observations in Section 3. $\square$

The proof of Theorem 4.1 is completed. $\square$

Remark 4.7. In Theorem 7 of [20], E. Tachtsis generalized the above construction and proved that $\mathrm{AC}^{\mathrm{LO}}$+LW $\not\to$ LT by constructing a permutation model $\mathcal{N}$. Since $\mathrm{AC}^{\mathrm{WO}}$ holds in $\mathcal{N}$, DC holds in $\mathcal{N}$ as well, c.f. Theorem 8.2 of [11]. We observe another standard argument to see that DC holds in $\mathcal{N}$. Since $\mathcal{I}$ is closed under countable unions in the model, we can see that DC holds in $\mathcal{N}$. Let $\mathcal{R}$ be a relation in $\mathcal{N}$ such that if $x \in \mathrm{dom}(\mathcal{R})$, there exists a y such that xRy. Consequently, there is a sequence $\langle x_n : n \in \omega \rangle$ in the ground model M such that for each $n \in \omega$, $x_n R x_{n+1}$. If x_n is supported by E_n for every $n \in \omega$, then $\langle x_n : n \in \omega \rangle$ is supported by $\bigcup_{n\in\omega} E_n$. Since $\mathcal{I}$ is closed under countable unions, the sequence $\langle x_n : n \in \omega\rangle$ is in $\mathcal{N}$.

A class of models $M_{\aleph_\alpha}$ for any regular cardinal $\aleph_\alpha$ (similar to the model $M_{\aleph_1}$ constructed in Theorem 7 of [20]) can be defined where $\mathrm{AC}^{\mathrm{LO}}$ and LW hold but LT fails, by replacing $\aleph_1$ by $\aleph_\alpha$. Moreover in $M_{\aleph_\alpha}$, $\mathrm{DC}_{<\aleph_\alpha}$ holds since $\mathcal{I}$ is closed under less than $\aleph_\alpha$ unions.

Remark 4.8. In the permutation model $\mathcal{N}$ of [17], CS, as well as CWF, holds following the work in this section. Moreover, the following statement holds in $\mathcal{N}$, following the work in this section.

'If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable.'

Theorem 4.9. *There is a permutation model $\mathcal{N}$ of* ZFA*, where there is an amorphous set. Moreover, the following holds in $\mathcal{N}$.*

(1) *If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable.*
(2) *If in a partially ordered set, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$ for any regular $\aleph_\alpha$.*

Proof: We consider the basic Fraenkel model (labeled as Model $\mathcal{N}_1$ in [8]) where 'there are no amorphous sets' is false and UT(WO,WO,WO) holds, c.f. [8]. Let $(P, \leq)$ be a p.o. set in $\mathcal{N}_1$, and E be a finite support of $(P, \leq)$. By (1) in the proof of Lemma 4.2, $\mathcal{O} = \{\mathrm{Orb}_E(p) : p \in P\}$ is a well-ordered partition of P. Now for each $p \in P$, $\mathrm{Orb}_E(p)$ is an antichain, c.f. the proof of Lemma 9.3 in [11]. Thus, by methods of Lemma 4.2, (1) and (2) hold in $\mathcal{N}_1$. □

Theorem 4.10. *There is a permutation model of* ZFA *where* CS*, as well as* CWF*, holds, but $\mathrm{AC}^{\omega}_{\mathrm{fin}}$ fails. Moreover, the following statements hold in the model.*

(1) *For every $3 \leq k < \omega$, $\mathcal{P}_k$ holds for any graph G_1 on some linearly-orderable set of vertices.*
(2) *For an infinite graph $G = (V_G, E_G)$ on a linearly-ordered set of vertices V_G and a finite graph $H = (V_H, E_H)$, if every finite subgraph of G has a homomorphism into H, then so has G.*
(3) *For every finite field $\mathcal{F} = \langle F, \dots\rangle$ and for every nontrivial linearly-orderable vector space V over $\mathcal{F}$, there exists a nonzero linear functional $f : V \to F$.*

Proof: We recall Lévy's permutation model (labeled as Model $\mathcal{N}_6$ in [8]).

∘ Defining the ground model M. We start with a ground model M of ZFA + AC where A is a countably infinite set of atoms written as a disjoint union $\bigcup\{P_n : n \in \omega\}$, where $P_n = \{a_1^n, \dots, a_{p_n}^n\}$ such that p_n is the nth-prime number.

- ∘ Defining the group $\mathcal{G}$ of permutations and the filter $\mathcal{F}$ of subgroups of $\mathcal{G}$.
  - Defining $\mathcal{G}$. Let $\mathcal{G}$ be the group generated by the following permutations π_n of A.

$$\pi_n : a_1^n \mapsto a_2^n \mapsto \cdots \mapsto a_{p_n}^n \mapsto a_1^n \quad \text{and} \quad \pi_n(x) = x \qquad \text{for all } x \in A \backslash P_n.$$

  - Defining $\mathcal{F}$. Let $\mathcal{F}$ be the filter of subgroups of $\mathcal{G}$ generated by $\{\text{fix}_{\mathcal{G}}(E) : E \in [A]^{<\omega}\}$.

∘ Defining the permutation model. Consider the permutation model $\mathcal{N}_6$ determined by M, $\mathcal{G}$ and $\mathcal{F}$.

It is well-known that in $\mathcal{N}_6$, $\text{AC}_{\text{fin}}^{\omega}$ fails since $\{P_i : i \in \omega\}$ has no (partial) choice function, c.f. [11]. Consequently, following Lemma 4.4 of [20], CAC fails in $\mathcal{N}_6$. Since *every permutation* $\varphi \in \mathcal{G}$ *moves only finitely many atoms*, following the arguments in Lemma 4.2, we can observe that CS, as well as CWF, holds in $\mathcal{N}_6$.

Lemma 4.11. *In* $\mathcal{N}_6$, LW *holds.*

Proof: Let $(X, \leq)$ be a linearly ordered set in $\mathcal{N}_6$ supported by E. We show $\text{fix}_{\mathcal{G}}E \subseteq \text{fix}_{\mathcal{G}}X$ which implies that X is well-orderable in $\mathcal{N}_6$. For the sake of contrary assume $\text{fix}_{\mathcal{G}}E \not\subseteq \text{fix}_{\mathcal{G}}X$. So there is an element $y \in X$ which is not supported by E and there is a $\varphi \in \text{fix}_{\mathcal{G}}E$ such that $\varphi(y) \neq y$. Since $\varphi(y) \neq y$ and $\leq$ is a linear order on X, we obtain either $\varphi(y) < y$ or $y < \varphi(y)$. Let $\varphi(y) < y$. Since *every permutation* $\varphi \in \mathcal{G}$ *moves only finitely many atoms* there exists some $k < \omega$ such that $\varphi^k = 1_A$. Thus, $p = \varphi^k(p) < \varphi^{k-1}(p) < \cdots < \varphi(p) < p$ which is a contradiction. Similarly we can arrive at a contradiction if we assume $y < \varphi(y)$. □

Since LW holds in $\mathcal{N}_6$, we can observe (1), (2) and (3) in $\mathcal{N}_6$ by observations in Section 3. □

Theorem 4.12. *There is a permutation model of* ZFA *where* CS, *as well as* CWF, *holds, but* LOKW_4^- *fails. Moreover, the following statements hold in the model.*

1. *For every* $3 \leq k < \omega$, $\mathcal{P}_k$ *holds for any graph* G_1 *on some linearly-orderable set of vertices.*
2. *For an infinite graph* $G = (V_G, E_G)$ *on a linearly-ordered set of vertices* V_G *and a finite graph* $H = (V_H, E_H)$, *if every finite subgraph of* G *has a homomorphism into* H, *then so has* G.
3. *For every finite field* $\mathcal{F} = \langle F, \dots \rangle$ *and for every nontrivial linearly-orderable vector space* V *over* $\mathcal{F}$, *there exists a nonzero linear functional* $f : V \to F$.

Proof: We recall the permutation model $\mathcal{M}$ from the second assertion of Theorem 10 (ii) of [5].

$\circ$ Defining the ground model M. Let κ be any infinite well-ordered cardinal number. We start with a ground model M of ZFA + AC where A is a κ-sized set of atoms written as a disjoint union $\bigcup\{A_\alpha : \alpha < \kappa\}$, where $A_\alpha = \{a_{\alpha,1}, a_{\alpha,2}, a_{\alpha,3}, a_{\alpha,4}\}$ such that $|A_\alpha| = 4$ for all $\alpha < \kappa$.

$\circ$ Defining the group $\mathcal{G}$ of permutations and the filter $\mathcal{F}$ of subgroups of $\mathcal{G}$.

- Defining $\mathcal{G}$. Let $\mathcal{G}$ be the weak direct product of $\mathcal{G}_\alpha$'s where $\mathcal{G}_\alpha$ is the alternating group on $\mathcal{A}_\alpha$ for each $\alpha < \kappa$.
- Defining $\mathcal{F}$. Let $\mathcal{F}$ be the normal filter of subgroups of $\mathcal{G}$ generated by $\{\mathrm{fix}_{\mathcal{G}}(E) : E \in [A]^{<\omega}\}$.

$\circ$ Defining the permutation model. Consider the permutation model $\mathcal{M}$ determined by M, $\mathcal{G}$ and $\mathcal{F}$.

In $\mathcal{M}$, LOKW_4^- fails, c.f. Theorem 10 (ii) of [5]. Since *every permutation $\varphi \in \mathcal{G}$ moves only finitely many atoms*, following the arguments in Lemma 4.2, we can observe that CS, as well as CWF, holds in $\mathcal{M}$. Since LW holds in $\mathcal{M}$, c.f. Theorem 10 (ii) of [5], we can observe (1), (2) and (3) in $\mathcal{M}$ by observations in Section 3. $\square$

5. Observations in Howard's model

Theorem 5.1. *For any $3 \leq k < \omega$, $\mathcal{P}_k$ follows from F_{k-1} in* ZF. *Moreover, if $X \in \{\mathrm{AC}_3, \mathrm{AC}^\omega_{\mathrm{fin}}\}$, then the statement $\mathcal{P}_k$ does not imply X in* ZFA *when $k = 3$.*

Proof: Fix $3 \leq k < \omega$. Suppose $\chi(E_{G_1}) = k$, $\chi(E_{G_2}) \geq \omega$ and G_1 is a graph on some well-orderable set of vertices. First we observe that if $g : V_{G_1} \to \{1, \ldots, k\}$ is a good k-coloring of G_1, then $G(\langle x, y\rangle) = g(x)$ is a good k-coloring of $G_1 \times G_2$. So, $\chi(E_{G_1 \times G_2}) \leq k$. For the sake of contradiction assume that $F : V_{G_1} \times V_{G_2} \to \{1, \ldots, k-1\}$ is a good coloring of $G_1 \times G_2$. For each color $c \in \{1, \ldots, k-1\}$ and each vertex $x \in V_{G_1}$ we let $A_{x,c} = \{y \in V_{G_2} : F(x, y) = c\}$. Define a relation R on $\{1, \ldots, k-1\}$ as $(v_1, i)R(v_2, j)$ if and only if '*$v_1 \neq v_2$ implies $A_{v_1,i} \cap A_{v_2,j}$ is not independent*' for $v_1, v_2 \in V_{G_1}$. By F_{k-1} and Claim 3.8 there exists a choice function f such that for any $x, x' \in V_{G_1}$, $A_{x,f(x)} \cap A_{x',f(x')}$ is not independent. Since $x \to f(x)$ is not a good coloring in G_1 as $\chi(E_{G_1}) = k$, there are $x, x' \in V_{G_1}$ with $f(x) = f(x') = j$ and $\{x, x'\} \in E_{G_1}$. Consequently, $A' = A_{x,f(x)} \cap A_{x',f(x')}$ is not independent. Pick $y, y' \in A'$ joined by an edge in E_{G_2}. Then (x, y) and (x', y') are joined in $E_{G_1} \times E_{G_2}$ and get the same color j which is a contradiction to the fact that F is a good coloring of $G_1 \times G_2$.

For the second assertion, we consider the permutation model $\mathcal{N}$ from Section 3 of [7] where AC_3 fails, and F_2 holds. Consequently, $\mathcal{P}_3$ holds in $\mathcal{N}$. In $\mathcal{N}$, there is a countable family $\mathcal{A} = \{A_i \colon i \in \omega\}$ which has no partial choice function. Consequently, $\mathrm{PAC}^{\omega}_{\mathrm{fin}}$ fails. Since $\mathrm{PAC}^{\omega}_{\mathrm{fin}}$ is equivalent to $\mathrm{AC}^{\omega}_{\mathrm{fin}}$, see the proof of Lemma 4.4 of [20], $\mathrm{AC}^{\omega}_{\mathrm{fin}}$ fails in $\mathcal{N}$. □

Question 5.2. If $k > 3$, does UL follow from $\mathcal{P}_k$? Otherwise is there any model of ZF or ZFA, where $\mathcal{P}_k$ holds for $k > 3$, but UL fails?

Theorem 5.3. *For any $2 \leq k < \omega$, $\mathcal{P}_{G,H}$ restricted to finite graph H with k vertices follows from F_k in* ZF. *Moreover, if $X \in \{\mathrm{AC}_3, \mathrm{AC}^{\omega}_{\mathrm{fin}}\}$, then $\mathcal{P}_{G,H}$ restricted to finite graph H with 2 vertices does not imply X in* ZFA.

Proof: Fix $2 \leq k < \omega$. Let $V_H = \{v_1, \ldots, v_k\}$. For each $x \in V_G$, let $A_x = \{(x, v_1), \ldots, (x, v_k)\}$. Define a relation R on $\bigcup_{x \in V_G} A_x$ by $(x, v_i) R (x', v_j)$ if and only if '$\{x, x'\} \in E_G$ *implies* $\{v_i, v_j\} \in E_H$' for $(x, v_i) \in A_x, (x', v_j) \in A_{x'}$. By assumption, for all finite $F \subset V_G$, there exists a homomorphism $h_F \colon G \restriction F \to H$. For any finite $F \subset V_G$, and a homomorphism h_F of F, let ${h_F}^*(j) = (j, h_F(j))$ for $j \in F$. Clearly, ${h_F}^*$ is an R-consistent choice function for $\{A_x\}_{x \in F}$. By F_k, there is an R-consistent choice function ${h_F}^*$ for $\{A_x\}_{x \in V_G}$. Define h_{V_G} on V_G by ${h_{V_G}}^*(j) = (j, h_{V_G}(j))$ for $j \in V_G$. Let $(j, j') \in E_G$ such that $j, j' \in V_G$. Since ${i_I}^*$ is R-consistent, $(j, h_{V_G}(j)) R (j', h_{V_G}(j'))$. By the definition of R, $(h_{V_G}(j), h_{V_G}(j')) \in E_H$.

For the second assertion, we once more consider the permutation model $\mathcal{N}$ from Section 3 of [7] where AC_3 and $\mathrm{AC}^{\omega}_{\mathrm{fin}}$ fail, and F_2 holds. Consequently, '$\mathcal{P}_{G,H}$ for a finite graph H with 2 vertices' holds in $\mathcal{N}$. □

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