Some interpretations of the (k, p)-Fibonacci numbers

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Abstract. In this paper we consider two parameters generalization of the Fibonacci numbers and Pell numbers, named as the (k, p)-Fibonacci numbers. We give some new interpretations of these numbers. Moreover using these interpretations we prove some identities for the (k, p)-Fibonacci numbers.

Keywords: Fibonacci number; Pell number; tiling

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1. Introduction

In general we use the standard notation, see [6], [8]. The nth Fibonacci number F_n is defined recursively as follows $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$, with $F_0 = F_1 = 1$. By numbers of the Fibonacci type we mean numbers defined recursively by the rth order linear recurrence relation of the form

$$a_n = b_1 a_{n-1} + b_2 a_{n-2} + \cdots + b_r a_{n-r} \qquad \text{for } n \geqslant r, \tag{1}$$

where $r \geqslant 2$ and $b_i \geqslant 0$, $i = 1, 2, \cdots, r$, are integers.

For special values of r and b_i, $i = 1, 2, \cdots, r$, the equality (1) defines other well-known numbers of the Fibonacci type. We list some of them:

(1) Lucas numbers: $L_n = L_{n-1} + L_{n-2}$ for $n \geqslant 2$, with $L_0 = 2$, $L_1 = 1$.
(2) Pell numbers: $P_n = 2P_{n-1} + P_{n-2}$ for $n \geqslant 2$, with $P_0 = 0$, $P_1 = 1$.
(3) Pell–Lucas numbers: $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geqslant 2$, with $Q_0 = 1$, $Q_1 = 3$.
(4) Jacobsthal numbers: $J_n = J_{n-1} + 2J_{n-2}$ for $n \geqslant 2$, with $J_0 = 0$, $J_1 = 1$.
(5) Padovan numbers: $Pv(n) = Pv(n-2) + Pv(n-3)$ for $n \geqslant 3$, with $Pv(0) = Pv(1) = Pv(2) = 1$.
(6) Tribonacci numbers of the first kind: $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geqslant 3$, with $T_0 = T_1 = T_2 = 1$.

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There are many generalizations of the classical Fibonacci numbers and numbers of the Fibonacci type. We list some of these generalized numbers. Let k, n, p be integers.

(1) k-generalized Fibonacci numbers, see E. P. Miles, Jr., [14]: $F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}$ for $k \geqslant 2$ and $n \geqslant k$, with $F_0^{(k)} = F_1^{(k)} = \cdots = F_{k-2}^{(k)} = 0$, $F_{k-1}^{(k)} = 1$.
(2) Fibonacci p-numbers, see A. P. Stakhov, [15]: $F_p(n) = F_p(n-1) + F_p(n-p-1)$ for $p \geqslant 1$ and $n > p+1$, with $F_p(0) = \cdots = F_p(p+1) = 1$.
(3) Generalized Fibonacci numbers, see M. Kwaśnik, I. Włoch, [12]: $F(k,n) = F(k,n-1) + F(k,n-k)$ for $k \geqslant 1$ and $n \geqslant k+1$, with $F(k,n) = n+1$ for $0 \leqslant n \leqslant k$.
(4) k-Fibonacci numbers, see S. Falcón, Á. Plaza, [9]: $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ for $k \geqslant 1$, $n \geqslant 2$, with $F_{k,0} = 0$, $F_{k,1} = 1$.
(5) Generalized Pell numbers, see I. Włoch, [17]: $P(k,n) = P(k,n-1) + P(k,n-k+1) + P(k,n-k)$ for $k \geqslant 2$, $n \geqslant k+1$, with $P(2,0) = 0$, $P(k,0) = 1$ for $k \geqslant 3$ and $P(k,1) = 1$, $P(k,n) = 2n-2$ for $2 \leqslant n \leqslant k$.
(6) Generalized Pell (p,i)-numbers, see E. Kiliç, [10]: $P_p^{(i)}(n) = 2P_p^{(i)}(n-1) + P_p^{(i)}(n-p-1)$ for $p \geqslant 1$, $0 \leqslant i \leqslant p$, $n > p+1$, with $P_p^{(i)}(1) = \cdots = P_p^{(i)}(i) = 0$ and $P_p^{(i)}(i+1) = \cdots = P_p^{(i)}(p+1) = 1$.
(7) k-Pell numbers, see P. Catarino, [7]: $P_{k,n} = 2P_{k,n-1} + kP_{k,n-2}$ for $k \geqslant 1$, $n \geqslant 2$, with $P_{k,0} = 0$, $P_{k,1} = 1$.
(8) (k,c)-generalized Jacobsthal numbers, see D. Marques, P. Trojovský, [13]: $J_n^{(k,c)} = J_{n-1}^{(k,c)} + J_{n-2}^{(k,c)} + \cdots + J_{n-k}^{(k,c)}$ for $k \geqslant 2$ and $n \geqslant k$, with $J_0^{(k,c)} = J_1^{(k,c)} = \cdots = J_{k-2}^{(k,c)} = 0$, $J_{k-1}^{(k,c)} = 1$.

For other generalizations of numbers of the Fibonacci type see for example [5].

In [1] a new two-parameters generalization, named as the (k,p)-Fibonacci numbers, was introduced and studied. We recall this definition.

Let $k \geqslant 2, n \geqslant 0$ be integers and let $p \geqslant 1$ be a rational number. The (k,p)-Fibonacci numbers denoted by $F_{k,p}(n)$ are defined recursively in the following way

$$F_{k,p}(n) = pF_{k,p}(n-1) + (p-1)F_{k,p}(n-k+1) + F_{k,p}(n-k) \qquad \text{for } n \geqslant k \tag{2}$$

with initial conditions

$$F_{k,p}(0) = 0 \quad \text{and} \quad F_{k,p}(n) = p^{n-1} \qquad \text{for } 1 \leqslant n \leqslant k-1. \tag{3}$$

For special values k, n, p the equality (2) gives well-known number of the Fibonacci type. We list these special cases.

(1) If $k=2$, $p=1$, $n \geqslant 0$ then $F_{2,1}(n+1)=F_n$.
(2) If $k \geqslant 2$, $p=1$, $n \geqslant k$ then $F_{k,1}(n)=F(k,n-k)$.
(3) If $k \geqslant 2$, $p=1$, $n \geqslant 1$ then $F_{k,1}(n)=F_{k-1}(n)$.
(4) If $k=2$, $p=3/2$, $n \geqslant 0$ then $F_{2,3/2}(n)=P_n$.
(5) If $k=2$, $p=t/2$, $t \in \mathbb{N}, t \geqslant 2$ and $n \geqslant 0$ then $F_{2,p}(n)=F_{2p-1,n}$.

The properties of these numbers were studied in [1].

Theorem 1.1 ([1])**.** *Let $k \geqslant 2$ be an integer and let $p \geqslant 1$ be a rational number. The generating function of the sequence $F_{k,p}(n)$ has the following form*

$$f_{k,p}(x)=\frac{x}{1-px-(p-1)x^{k-1}-x^k}.$$

The generating function for the (k,p)-Fibonacci numbers generalized other well-known generating functions for Fibonacci numbers, Pell numbers and k-Fibonacci numbers.

2. Main results

The Fibonacci numbers and numbers of the Fibonacci type have many interesting interpretations also in graphs, see for example [10], [11], [12], [17]. The graph interpretation of the Fibonacci numbers was initiated by H. Prodinger and R. F. Tichy in [16]. In [5] a total graph interpretation for numbers of the Fibonacci type was given. In this paper we shall show that this interpretation works also for the (k,p)-Fibonacci numbers. We recall some of necessary definitions and notations.

Let G be an undirected, simple graph with the vertex set $V(G)$ and the edge set $E(G)$. By $P(m), T(m), S(m)$ and $C(m)$ we denote a path, a tree, a star and a cycle of size m, respectively. Let $\mathcal{I}=\{1,2,\cdots,k\}$, $k \geqslant 2$, and let $\mathcal{I}_i=\{1,\cdots,b_i\}$, $b_i \geqslant 1$. Let $\mathcal{C}=\bigcup_{i\in\mathcal{I}}\mathcal{C}_i$ be a nonempty family of colors, where $\mathcal{C}_i=\{iA_j : j \in \mathcal{I}_i\}$ for $i=1,2,\cdots,k$. The set $\mathcal{C}_i$ will be called as the set of b_i shades of the colour i. Consequently, for all $i \neq p$, $1 \leqslant i$, $p \leqslant k$, it holds $iA_j \neq pA_j$ and this implies that the family $\mathcal{C}$ has exactly $\sum_{i=1}^{k}|\mathcal{C}_i|=\sum_{i=1}^{k}b_i$ colours.

A graph G is $(iA_j : i \in \mathcal{I},\ j \in \mathcal{I}_i)$*-edge coloured by monochromatic path* if for every maximal iA_j-monochromatic subgraph H of G, where $iA_j \in \mathcal{C}_i$, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant b_i$, there exists a partition of H into edge disjoint paths of the length i. If $b_1 \neq 0$ then $(iA_j : i \in \mathcal{I},\ j \in \mathcal{I}_i)$-edge colouring by monochromatic paths always exists.

Now we define special graph parameter associated with this edge colouring of the graph. Let G be a graph which can be $(iA_j : i \in \mathcal{I},\ j \in \mathcal{I}_i)$-edge coloured by monochromatic paths. Let $\mathcal{F}$ be a family of distinct $(iA_j : i \in \mathcal{I},$

$j \in \mathcal{I}_i$)-edge coloured graphs obtained by colouring of the graph G. Let $\mathcal{F} = \{G^{(1)}, G^{(2)}, \cdots, G^{(l)}\}$, $l \geqslant 1$, where $G^{(p)}$, $1 \leqslant p \leqslant l$, denotes a graph obtained by $(iA_j\colon i \in \mathcal{I},\ j \in \mathcal{I}_i)$-edge colouring by monochromatic paths of a graph G.

For $(iA_j\colon i \in \mathcal{I},\ j \in \mathcal{I}_i)$-edge coloured graph $G^{(p)}$, $1 \leqslant p \leqslant l$, by $\theta(G^{(p)})$ we denote the number of all partitions of iA_j-monochromatic subgraphs of $G^{(p)}$ into edge disjoint paths of the length i. If $G^{(p)}$ is $1A_s$-monochromatic, $1 \leqslant s \leqslant p$, then we put $\theta(G^{(p)}) = 1$.

The number of all $(iA_j\colon i \in \mathcal{I},\ j \in \mathcal{I}_i)$-edge colourings is defined as the graph parameter as follows

$$\sigma_{(iA_j\colon i \in \mathcal{I},\ j \in \mathcal{I}_i)}(G) = \sum_{p=1}^{l} \theta(G^{(p)}).$$

The parameter $\sigma_{(A_1, 2A_1)}(G)$ was studied for different classes of graphs i.e. paths, trees and unicyclic graphs, see [2], [3], [4], [5].

Theorem 2.1 ([5]). *Let m be an integer. Then*

$$\sigma_{(A_1,2A_1)}(\mathbb{P}(m)) = F_m \qquad \textit{for } \ m \geqslant 1,$$
$$\sigma_{(A_1,2A_1)}(\mathbb{C}(m)) = L_m \qquad \textit{for } \ m \geqslant 2.$$

Theorem 2.2 ([5]). *Let $T(m)$ be a tree of size m, $m \geqslant 1$. Then*

$$F_m \leqslant \sigma_{(A_1,2A_1)}(T(m)) \leqslant 1 + \sum_{j \geq 1} \binom{m}{2j} \prod_{p=0}^{j-1} [2j - (2p+1)].$$

Moreover

$$\sigma_{(A_1,2A_1)}(\mathbb{P}(m)) = F_m \quad \textit{and} \quad \sigma_{(A_1,2A_1)}(S(m)) = 1 + \sum_{j \geq 1} \binom{m}{2j} \prod_{p=0}^{j-1} [2j - (2p+1)].$$

Theorem 2.3 ([2]). *Let G be a unicyclic graph of the size m, $m \geqslant 3$. Then $\sigma_{(A_1,2A_1)}(G) \geqslant L_m$. The equality holds if $G \cong C(m)$.*

For future investigation we use following notation. Let $e \in E(G)$ be a fixed edge. If e is coloured by iA_j then we write $c(e) = iA_j$ and $\sigma_{iA_j(e)}(G)$ be the number of all $(iA_j\colon i \in \mathcal{I},\ j \in \mathcal{I}_i)$-edge colouring of the graph G with $c(e) = iA_j$, $i \in \mathcal{I}$, $j \in \mathcal{I}$.

For convenience in the next part of this section instead of $(A_1, \cdots, A_p, kB, (k-1)C_1, \cdots, (k-1)C_{p-1})$-edge colouring of the graph G we will write α-edge colouring of the graph G. Consequently instead of

$$\sigma_{(A_1,\cdots,A_p,kB,(k-1)C_1,\cdots,(k-1)C_{p-1})}(G)$$

we put $\sigma_\alpha(G)$.

Theorem 2.4. *Let $k \geqslant 2$, $m \geqslant 1$, $p \geqslant 1$ be integers. Then for fixed k, p*

$$\sigma_\alpha(P(m)) = F_{k,p}(m+1). \tag{4}$$

Proof: We use induction on m. Let $P(m)$ be the path of size m with $E(P(m)) = \{e_1, e_2, \cdots, e_m\}$ and the numbering of edges in the natural fashion. We will prove that for fixed k, p

$$\sigma_\alpha(P(m)) = F_{k,p}(m+1).$$

By the definition of α-edge colouring it follows that edges of the path $P(m)$ can be coloured by colours $A_1, \cdots, A_p, kB, (k-1)C_1, \cdots, (k-1)C_{p-1}$.

Let $m = 1$. If $k = 2$ then it is obvious that the unique edge $e_1 \in E(P(1))$ can be coloured using one of colours $A_1, \cdots, A_p, C_1, \cdots, C_{p-1}$ so $\sigma_\alpha(P(1)) = 2p-1 = F_{2,p}(2)$. If $k \geqslant 3$ then the unique edge $e_1 \in E(P(1))$ can be coloured by colours $A_1, \cdots, A_p$. Since the colour can be chosen into p ways so $\sigma_\alpha(P(1)) = p = F_{k,p}(2)$.

Let $m \geqslant 2$ and for $t < m$ we have $\sigma_\alpha(P(t)) = F_{k,p}(t+1)$. We shall show that

$$\sigma_\alpha(P(m)) = F_{k,p}(m+1).$$

Let us consider an arbitrary α-edge colourings of $P(m)$ and let $e_m \in E(P(m))$. We have the following possibilities

(1) Let $c(e_m) = A_i$, $i = 1, \cdots, p$. Since the colour of e_m can be chosen into p ways so by the induction's hypothesis we have

$$\sum_{i=1}^{p} \sigma_{A_i(e_m)}(P(m)) = p \cdot \sigma_\alpha(P(m-1)) = pF_{k,p}(m).$$

(2) Let $c(e_m) = kB$. Then there exists a kB-monochromatic path $e_{m-k+1} - \cdots - e_m$ in the graph $P(m)$. This path has the length k and using the induction's hypothesis we obtain that

$$\sigma_{kB(e)}(P(m)) = \sigma_\alpha(P(m-k)) = F_{k,p}(m-k+1).$$

(3) Let $c(e) = (k-1)C_j$, $j = 1, \cdots, p-1$. Then there exists a $(k-1)C_j$-monochromatic path $e_{m-k+2} - \cdots - e_m$ in the graph $P(m)$. This path has the length $k-1$. Because we have exactly $p-1$ possibilities of colouring of the path $e_{m-k+2} - \cdots - e_m$, so from the induction's hypothesis we have

$$\sum_{j=1}^{p-1} \sigma_{(k-1)C_j(e)}(P(m)) = (p-1)\sigma_\alpha(P(m-k+1)) = (p-1)F_{k,p}(m-k+2).$$

From above possibilities and (2) we obtained that

$$\sigma_\alpha(P(m)) = pF_{k,p}(m) + F_{k,p}(m-k+1) + (p-1)F_{k,p}(m-k+2) = F_{k,p}(m+1),$$

which ends the proof. $\square$

We can use the above interpretation as the proving tool for some identities.

Theorem 2.5. *Let $k \geqslant 2$, $n \geqslant k-2$, $m \geqslant k$, $p \geqslant 1$ be integers. Then*

$$\begin{aligned} F_{k,p}(m+n) = {} & pF_{k,p}(m-1)F_{k,p}(n+1) \\ & + (p-1)\sum_{i=1}^{k-1} F_{k,p}(m-k+i)F_{k,p}(n-i+2) \\ & + \sum_{j=0}^{k-1} F_{k,p}(m-k+j)F_{k,p}(n-j+1). \end{aligned} \tag{5}$$

Proof: Let $P(m-1+n)$ be the path of size $m-1+n$ with $E(P(m)) = \{e_1, \cdots, e_{m-1}, e_m, \cdots, e_{m-1+n}\}$ and the numbering of edges in the natural fashion. From Theorem 2.4 we have

$$\sigma_\alpha(P(m-1+n)) = F_{k,p}(m+n).$$

We shall show that

$$\begin{aligned} \sigma_\alpha(P(m-1+n)) = {} & pF_{k,p}(m-1)F_{k,p}(n+1) \\ & + (p-1)\sum_{i=1}^{k-1} F_{k,p}(m-k+i)F_{k,p}(n-i+2) \\ & + \sum_{j=0}^{k-1} F_{k,p}(m-k+j)F_{k,p}(n-j+1). \end{aligned}$$

Consider the following cases

(1) Let $c(e_{m-1}) = A_i$, $i = 1, 2, \cdots, p$. Then from Theorem 2.4

$$\begin{aligned} \sum_{i=1}^{p} \sigma_{A_i(e_{m-1})}(P(m-1+n)) &= \sigma_\alpha(P(m-2)) \cdot p \cdot \sigma_\alpha(P(n)) \\ &= pF_{k,p}(m-1)F_{k,p}(n+1). \end{aligned}$$

(2) Let $c(e_{m-1}) = (k-1)C_j$, $j = 1, 2, \cdots, p-1$. Then there exists a $(k-1)C_j$-monochromatic path $P = e_i - \cdots - e_{m-1} - \cdots - e_{i+k-2}$ of the length $k-1$. Of course this path P can be coloured in $p-1$ ways. For future investigations let us denote from now by $P(m-1)$ the path of size $m-1$ such that $E(P(m-1)) = \{e_1, e_2, \cdots, e_{m-1}\}$ and by $P(n)$ the path of size n with $E(P(n)) = \{e_m, e_{m+1}, \cdots, e_{m-1+n}\}$. Let us consider the following cases

i) Let $e_{m-1} = e_{i+k-2}$. Then $P \subseteq P(m-1)$. Because paths $P(m-k)$, $P(n)$ can be α-edge coloured in $F_{k,p}(m-k+1)$, $F_{k,p}(n+1)$ ways, respectively, so

$$\sum_{j=1}^{p-1} \sigma_{(k-1)C_j(e_{m-1})}(P(m-1+n)) = (p-1)F_{k,p}(m-k+1)F_{k,p}(n+1).$$

ii) Let $e_{m-1} = e_{i+k-3}$. Then $P \setminus \{e_{i+k-2}\} \subseteq P(m-1)$ and $e_{i+k-2} - e_m \in E(P(n))$. Because $\sigma_\alpha(P(m-k+1)) = F_{k,p}(m-k+2)$ and $\sigma_\alpha(P(n-1)) = F_{k,p}(n)$, so

$$\sum_{j=1}^{p-1} \sigma_{(k-1)C_j(e)}(P(m-1+n)) = (p-1)F_{k,p}(m-k+2)F_{k,p}(n).$$

$$\vdots$$

iii) Let $e_{m-1} = e_i$. Consequently $P \setminus \{e_{m-1}\} \subseteq P(n)$ and of course $e_{m-1} \subseteq E(P(m-1))$. Because paths $P(m-2)$ and $P(n-k+2)$ can be α-edge coloured in $F_{k,p}(m-1)$ and $F_{k,p}(n-k+3)$ ways, respectively, we obtain that

$$\sum_{j=1}^{p-1} \sigma_{(k-1)C_j(e)}(P(m-1+n)) = (p-1)F_{k,p}(m-1)F_{k,p}(n-k+3).$$

From all above cases we have that

$$\begin{aligned}\sum_{j=1}^{p-1} \sigma_{(k-1)C_j(e)}(P(m-1+n)) &= (p-1)F_{k,p}(m-k+1)F_{k,p}(n+1)\\ &\quad + (p-1)F_{k,p}(m-k+2)F_{k,p}(n)\\ &\quad + \cdots + (p-1)F_{k,p}(m-1)F_{k,p}(n-k+3)\\ &= (p-1)\sum_{i=1}^{k-1} F_{k,p}(m-k+i)F_{k,p}(n-i+2).\end{aligned}$$

(3) Let $c(e) = kB$. Then there exists a kB-monochromatic path $e_j - \cdots - e_{m-1} - \cdots - e_{j+k-1}$ of the length k. Using the same method as in case (2) we obtain

$$\begin{aligned}\sigma_{kB(e)}(P(m-1+n)) &= F_{k,p}(m-k)F_{k,p}(n+1) + F_{k,p}(m-k+1)F_{k,p}(n)\\ &\quad + \cdots + F_{k,p}(m-1)F_{k,p}(n-k+2)\\ &= \sum_{j=0}^{k-1} F_{k,p}(m-k+j)F_{k,p}(n-j+1).\end{aligned}$$

Therefore from possibilities (1), (2) and (3) we have

$$\begin{aligned}\sigma_\alpha(P(m-1+n)) = {} & pF_{k,p}(m-1)F_{k,p}(n+1)\\ & + (p-1)\sum_{i=1}^{k-1} F_{k,p}(m-k+i)F_{k,p}(n-i+2)\\ & + \sum_{j=0}^{k-1} F_{k,p}(m-k+j)F_{k,p}(n-j+1),\end{aligned}$$

which completes the proof. □

Corollary 2.6. *Let $k \geqslant 2$, $m \geqslant k$, $n \geqslant k-2$, $p \geqslant 1$, be integers.*

(1) *If $k=2$, $p=1$, then $F_{m+n} = F_m F_{n+1} + F_{m-1}F_n$.*
(2) *If $k=2$, $p \geqslant 1$, then $F_{2p-1,m+n} = F_{2p-1,m}F_{2p-1,n+1} + F_{2p-1,m-1}F_{2p-1,n}$.*

Now we give another interpretation of the (k,p)-Fibonacci numbers with respect to tilings.

Let $k \geqslant 2$, $n \geqslant 1$, $p \geqslant 1$ be integers. Let consider tilings of $1 \times (n-1)$ boards, called $(n-1)$-boards.

The pieces we are going to use in order to tile our $(n-1)$-boards are: 1×1 red squares (squares), $1 \times (k-1)$ blue rectangles ($(k-1)$-rectangles) and $1 \times k$ white rectangles (k-rectangles). Suppose that we have unlimited resources for these tiles and we distinguish color shades of squares and $(k-1)$-rectangles. Let $\mathcal{R} = \{r_1, r_2, \cdots, r_p\}$ be the set of shades of red squares. Let $\mathcal{B} = \{b_1, b_2, \cdots, b_{p-1}\}$ be the set of shades of blue $(k-1)$-rectangles.

Let $f_{k,p}(n)$ be the number of tilings on an $(n-1)$-board using the mentioned pieces.

Theorem 2.7. *Let $k \geqslant 2$, $n \geqslant 1$, $p \geqslant 1$, be integers. Then $f_{k,p}(n) = F_{k,p}(n)$.*

Proof: We use induction on n. Let k, n, p be as in the statement of the theorem. We consider the following cases:

(1) If $n = 1$ then $f_{k,p}(1)$ counts the empty tiling so $f_{k,p}(1) = 1 = F_{k,p}(1)$.
(2) Let $2 \leqslant n < k$. Then every piece of the $(n-1)$-board can be tiled using only red squares. Since we have p shades of red color so there are $p^{n-1} = F_{k,p}(n)$ possibilities in this case.
(3) Let $n = k$. Then we can use red squares or a blue $(k-1)$-rectangle in order to tile the $(k-1)$-board. Since we have p shades of red color and $(p-1)$ shades of blue color so there are $p^{k-1}+p-1 = F_{k,p}(k)$ possibilities in this case.
(4) Let $n \geqslant k+1$. Assume that for $m < n$ we have $f_{k,p}(m) = F_{k,p}(m)$. We shall show that $f_{k,p}(n) = F_{k,p}(n)$. We consider the following cases:

(a) The $(n-1)$-board ends with the red square in one of the p shades. Then the remaining board can be covered on $f_{k,p}(n-1)$ ways.
(b) The $(n-1)$-board ends with the blue $(k-1)$-rectangle in one of the $p-1$ shades. Then by removing this last piece we are left with $f_{k,p}(n-k+1)$ tilings.
(c) The $(n-1)$-board ends with the white k-rectangle. Then the remaining board can be covered in $f_{k,p}(n-k)$ ways.

Consequently, from above cases we obtain

$$f_{k,p}(n) = pf_{k,p}(n-1) + (p-1)f_{k,p}(n-k+1) + f_{k,p}(n-k).$$

From the above and by the initial conditions we have that $F_{k,p}(n) = f_{k,p}(n)$, which completes the proof. $\square$

Using this interpretation we can prove the following identity.

Theorem 2.8. *Let $k \geqslant 2$, $n \geqslant 2$, $p \geqslant 1$ be integers.*

(1) *If k is an even number then*

$$F_{k,p}(2n) = p \sum_{i=0}^{[(2n-1)/k]} F_{k,p}(2n-1-ki) + (p-1) \sum_{j=0}^{[(2n-k+1)/k]} F_{k,p}(2n-k+1-kj).$$

(2) *If k is an odd number then*

$$F_{k,p}(2n) = p \sum_{i=0}^{[(2n-1)/(k-1)]} (p-1)^i F_{k,p}(2n-1-(k-1)i) + (p-1) \sum_{j=0}^{[(2n-k)/(k-1)]} F_{k,p}(2n-k-(k-1)j).$$

Proof: We prove only case (1) as case (2) can be proved similarly. Suppose that k is an even number. We will show that

$$\begin{aligned} f_{k,p}(2n) = p\Big[f_{k,p}(2n-1) + f_{k,p}(2n-k-1) + \cdots + f_{k,p}\Big(2n-1-\Big[\frac{2n-1}{k}\Big]k\Big)\Big] \\ + (p-1)\Big[f_{k,p}(2n-k+1) + f_{k,p}(2n-2k+1) \\ + f_{k,p}(2n-3k+1) + \cdots + f_{k,p}\Big(2n-k+1-\Big[\frac{2n-k+1}{k}\Big]k\Big)\Big]. \end{aligned}$$

Since $(2n-1)$-board is an odd length so each tiling of this board have to contain at least one square or at least one $(k-1)$-rectangle. Let us consider the location of the last odd length piece. We have the following possibilities

1. The last odd length piece is a square. Of course we have exactly p possibilities to choose a red square. Moreover the last square can occur in cells with number: $(2n-1)$ or $(2n-k-1)$ or $(2n-2k-1)$ $\cdots$ or $(2n-1-[(2n-1)/k]k)$. Then the remaining board can be covered in $f_{k,p}(2n-1)$ or $f_{k,p}(2n-k-1)$ or $f_{k,p}(2n-2k-1)$ $\cdots$ or $f_{k,p}(2n-1-[(2n-1)/k]k)$ ways, respectively. So in that case the number of all possible tilings of the $(2n-1)$-board is equal to

$$pf_{k,p}(2n-1) + pf_{k,p}(2n-k-1) + \cdots + pf_{k,p}\Big(2n-1-\Big[\frac{2n-1}{k}\Big]k\Big).$$

2. The last odd length piece is a blue $(k-1)$-rectangle. Since we have exactly $(p-1)$ shades of the blue $(k-1)$-rectangle, so by considering the location of the last $(k-1)$-rectangle we obtain the following possibilities.
   - If $(k-1)$-rectangle is the last piece of the $(2n-1)$-board, then the remaining board can be covered in $f_{k,p}(2n-k+1)$ ways.
   - If $(k-1)$-rectangle occurs in cells with numbers from $(2n-2k+1)$ to $(2n-k-1)$, then the remaining $(2n-2k)$-board can be covered in $f_{k,p}(2n-2k+1)$ ways.

   $\cdots$

   - If $(k-1)$-rectangle occurs in cells with numbers from $(2n-k+1-[(2n-k+1)/k]k)$ to $(2n-1-[(2n-k+1)/k]k)$, then the remaining board can be covered on $f_{k,p}(2n-k+1-[(2n-k+1)/k]k)$ ways.

   From all above possibilities we obtain that the number of all possible tilings of the $(2n-1)$-board in that case is equal to

$$\begin{aligned}(p-1)f_{k,p}(2n-k+1) + (p-1)f_{k,p}(2n-2k+1) + (p-1)f_{k,p}(2n-3k+1)\\ + \cdots + (p-1)f_{k,p}\Big(2n-k+1-\Big[\frac{2n-k+1}{k}\Big]k\Big).\end{aligned}$$

Finally, we have

$$\begin{aligned}F_{k,p}(2n) = p &\sum_{i=0}^{[(2n-1)/k]} F_{k,p}(2n-1-ki)\\ &+ (p-1)\sum_{j=0}^{[(2n-k+1)/k]} F_{k,p}(2n-k+1-kj).\end{aligned}$$

$\square$

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